Evolutionary Monte Carlo methods

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Part IV - Illustrations - Exercises - Solutions

A dozen of examples
- Filtering, HMM, IKF,...
- Importance sampling, rare events,...
- Interacting SA, subset simulation,...
- Confinement, Polymers,...

Laboratory examples

Particle-MCMC methodologies
- Independent Metropolis-Hastings scheme
- Frozen particle path & Particle Gibbs sampler

7 Exercises with solutions

A dozen of examples

- Filtering, HMM, IKF,. . .
- Importance sampling, rare events,. . .
- Interacting SA, subset simulation,. . .
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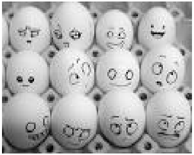
Laboratory examples

Particle-MCMC methodologies

7 Exercises with solutions

A Dozen of Examples



1) Filtering problems (X_n, Y_n)=(Signal, Observation)

FK model for the posterior distributions

$$G_n(x_n) = p(y_n|x_n) \Rightarrow \quad \mathbb{Q}_n(f) = \mathbb{E}\left(f(X_0,\ldots,X_n) \mid Y_p = y_p,\ p < n\right)$$

$$\eta_n(f) = \mathbb{E}\left(f(X_n) \mid Y_p = y_p,\ p < n\right)$$

and

$$\mathcal{Z}_{n+1} = p_n(y_0,\ldots,y_n)$$

2) Filtering the noise ($\mathcal{X}_0 := W_0$)

$$\left\{\begin{array}{lcl} \mathcal{X}_n & := & F_n(\mathcal{X}_{n-1}, W_n) \quad \rightsquigarrow \quad \mathcal{X}_n = \varphi_n(W_0, \ldots, W_n) \\ \mathcal{Y}_n & = & H_n(\mathcal{X}_n, V_n) \overset{\text{ex.}}{=} h_n(\mathcal{X}_n) + V_n \quad \text{with} \quad V_n \sim \mathcal{N}(0,1) \end{array}\right.$$

Bayes' rule

$$p(w_0, \ldots, w_n \mid y_0, \ldots, y_{n-1}) \propto \left\{ \prod_{0 \leq k < n} \underbrace{p(y_k \mid (w_0, \ldots, w_k))}_{:= G_k(w_0, \ldots, w_k)} \right\} p(w_0, \ldots, w_n)$$

$$\Downarrow$$

FK model with the Markov chain $X_n = (W_0, \ldots, W_n)$ and the functions

$$G_n(X_n) = p(y_n \mid (w_0, \ldots, w_n)) \overset{\text{ex.}}{\propto} e^{-\frac{1}{2}(y_n - h_n(\varphi_n(w_0, \ldots, w_n)))^2}$$

$$\Downarrow$$

$$\eta_n(f) \quad = \quad \mathbb{E}\left(f(W_0, \ldots, W_n) \mid Y_p = y_p, \; p < n\right)$$

3) Markov restriction/Confinements

$$G_n(x_n) = 1_{A_n}(x_n) \Rightarrow \begin{array}{rcl} \mathbb{Q}_n(f) & = & \mathbb{E}\left(f(X_0,\ldots,X_n) \mid X_p \in A_p,\ p < n\right) \\ \eta_n(f) & = & \mathbb{E}\left(f(X_n) \mid X_p \in A_p,\ p < n\right) \end{array}$$

and

$$\mathcal{Z}_{n+1} = \mathbb{P}\left(X_p \in A_p,\ p \leq n\right)$$

4) Self-avoiding walks X'_n SRW on $\mathbb{Z}^2$ (starts at the origin)

$$X_n = (X'_0, \ldots, X'_n) \quad \text{and} \quad G_n(X_n) = 1_{\not\in \{X'_0, \ldots, X'_{n-1}\}}(X'_n)$$

$$\Downarrow$$

$$\eta_n(f) = \mathbb{E}\left(f(X'_0, \ldots, X'_n) \mid X'_p \neq X'_q,\ 0 \leq p < q < n\right)$$

and

$$\mathcal{Z}_n = \mathbb{P}\left(X'_p \neq X'_q,\ 0 \leq p < q < n\right) = \frac{1}{4^n} \times \#\{\text{non intersecting } n\text{-paths}\}$$

5) Top-spectrum approximations $E = \{1, \ldots, d\}$

$$Q(i,j) = \sum_j Q(i,j) \times \frac{Q(i,j)}{\sum_j Q(i,j)} := G(i) \times M(i,j)$$

When Q is symmetric

$$Q(i,j) = Q(j,i) \Leftrightarrow \underbrace{\frac{G(i)}{\sum_k G(k)}}_{:=\mu(i)} \; M(i,j) = \underbrace{\frac{G(j)}{\sum_k G(k)}}_{:=\mu(j)} \; M(j,i)$$

$$\Downarrow$$

$\rightsquigarrow$ **Spectral theorem:**

$$Q^n(i,j) = \sum_{0 \leq k < d} \lambda_k^n \, \varphi_k(i)\varphi_k(j)$$

with $\lambda_k \downarrow$, eigen-vectors $Q(\varphi_k) = \lambda_k \varphi_k$ orthonormal basis of $\mathbb{L}_2(\mu)$.

FK interpretation (X_k Markov with transitions M_k)

$$Q^n(\mathbf{i_0}, \mathbf{i_n})$$

$$= \sum_{1 \leq j_0, \ldots, j_n \leq d} \overbrace{1_{\mathbf{i_0}}(j_0)}^{=\eta_0(j_0)} G(j_0) M(j_0, j_1) \ldots G(j_{n-1}) M(j_{n-1}, j_n) \overbrace{1_{\mathbf{i_n}}(j_n)}^{=f(j_n)}$$

$$= \mathbb{E}\left(f(X_n) \ \prod_{0 \leq k < n} G_k(X_k) \right)$$

Long time behavior (power method)

$$Q^n(i_0, i_n) \simeq_{n\uparrow} \ \lambda_0^n \ \varphi_0(i_0)\varphi_0(i_n) \Rightarrow \mathcal{Z}_n = \lambda_0^n \ \times \mathrm{cte} \Rightarrow \frac{1}{n} \log \mathcal{Z}_n \simeq \log \lambda_0$$

and

$$\eta_n(f) = \frac{Q^n(i_0, i_n)}{\sum_{1 \leq j_n \leq d} Q_n(i_0, j_n)} \ \simeq_{n\uparrow} \ \frac{\varphi_0(i_n)}{\sum_{j_n} \varphi_0(j_n)}$$

Particle approximations:
Replace η_n and $\mathcal{Z}_n$ by their approximations $\eta_n^{\mathbf{N}}$ and $\mathcal{Z}_n^{\mathbf{N}}$...

6) Twisted/Importance sampling

$$X_n = (X'_n, X'_{n+1}) \quad \text{and} \quad G_n(X_n) = e^{V_n(X'_n) - V_{n-1}(X'_{n-1})}$$

$$\Downarrow (V_0 = 0)$$

$$\mathbb{Q}_{n+1}(f) \propto \mathbb{E}\left(f((X'_0, X'_1), \ldots, (X'_{n+1}, X'_{n+2})) \underbrace{e^{V_n(X'_n)}}_{\text{favor path with high } V_n(X'_n)} \right)$$

and

$$\mathcal{Z}_{n+1} = \mathbb{E}\left(e^{V_n(X'_n)}\right)$$

Applications

$$f((X'_0, X'_1), \ldots, (X'_{n+1}, X'_{n+2})) = \underbrace{1_{V_n(X'_n) \geq a}}_{= 0 \text{ for paths with small } V_n(X'_n)} e^{-V_n(X'_n)}$$

$\rightsquigarrow$ **Particle algorithm drives paths to V_n-high terminal values!**

7) $\oplus$ 8) = Boltzmann-Gibbs measures

- **7) BG meas.** $\pi_n(dx) = \frac{1}{Z_{\beta_n}}\; e^{-\beta_n V(x)}\; \lambda(dx)$**, with** $\beta_n \uparrow$**.**

$$G_n(x) = e^{-(\beta_{n+1}-\beta_n)V(x)} \quad \text{and} \quad M_n \quad \text{s.t.} \quad \pi_n = \pi_n M_n \quad (\pi_n\text{-Shaker})$$

$$\Downarrow$$

$$\eta_n = \pi_n \quad \text{and} \quad Z_{\beta_n}/Z_{\beta_0} = \mathcal{Z}_n$$

- **8) Tails/Restricted meas.** $\pi_n(dx) = \frac{1}{Z_{A_n}}\; 1_{A_n}(x)\; \lambda(dx)$**, with** $A_n \downarrow$

$$G_n(x) = 1_{A_{n+1}}(x) \quad \text{and} \quad M_n \quad \text{s.t.} \quad \pi_n = \pi_n M_n \quad (A_n\text{-Shaker})$$

$$\Downarrow$$

$$\eta_n = \pi_n \quad \text{and} \quad Z_{A_n}/Z_{A_0} = \mathcal{Z}_n$$

Applications: Ising, Glauber, Potts models, Black-box problems, rare events, tail probabilities,. . .

9) "Product" measures (with any $h_k \geq 0$)

Target product type measures

$$\pi_n(dx) = \frac{1}{Z_n} \left\{ \prod_{0 \leq k \leq n} h_k(x) \right\} \lambda(dx)$$

FK model

$$G_n = h_{n+1} \quad \text{and} \quad M_n \quad \text{s.t.} \quad \pi_n = \pi_n M_n \quad (\pi_n\text{-Shaker})$$

$$\Downarrow$$

$$\eta_n = \pi_n \quad \text{and} \quad Z_n/Z_0 = \mathcal{Z}_n$$

Important observation: "Product" $\supset$ BG $\cup$ Restrictions

$$\prod_{0 \leq k \leq n} 1_{A_k} = 1_{A_n} \quad \text{and} \quad \prod_{0 \leq k \leq n} e^{-(\beta_k - \beta_{k-1})V} = e^{-(\beta_n - \beta_0)V}$$

10) HMM $\Theta \sim p(\theta)d\theta \rightsquigarrow (X_n, Y_n)$ linear-Gauss filtering

$$\underbrace{p(\theta \mid (y_0,\ldots,y_n))d\theta}_{:=\pi_n(d\theta)} = \frac{1}{Z_n}\left\{\prod_{0\leq k\leq n} \underbrace{p(y_k \mid \theta,\ (y_0,\ldots,y_{k-1}))}_{:=h_k(\theta)\ \textbf{(using Kalman)}}\right\} \underbrace{p(\theta)d\theta}_{:=\lambda(d\theta)}$$

FK model

$$G_n = h_{n+1} \quad \text{and} \quad M_n \quad \text{s.t.} \quad \pi_n = \pi_n M_n \quad (\pi_n\text{-Shaker})$$

$$\Downarrow$$

$$\eta_n = \pi_n \quad \text{and} \quad Z_n/Z_0 = \mathcal{Z}_n$$

Important observation to compute $h_k(\theta)$:

$$p(y_k \mid \theta,\ (y_0,\ldots,y_{k-1}))$$

$$= \int \underbrace{p(y_k \mid \theta, x_k\ (y_0,\ldots,y_{k-1}))}_{=\ \textbf{(ex.) Law of } Y_k\ =\ C_k(\theta)x_k + V_k} \underbrace{p(x_k \mid \theta,\ (y_0,\ldots,y_{k-1}))\ dx_k}_{=\textbf{Kalman-predictor}}$$

11) HMM $\Theta \sim p(\theta)d\theta \rightsquigarrow (X_n, Y_n)$ nonlinear filtering

$$p(\theta \mid (y_0, \ldots, y_n)) \propto \left\{ \prod_{0 \leq k \leq n} \underbrace{p(y_k \mid \theta,\ (y_0, \ldots, y_{k-1}))}_{h_k(\theta) \simeq \overline{h}_k(\theta, \xi^\theta)\ \textbf{(using particle filters)}} \right\} p(\theta)$$

Particle filter estimation

$$p(y_k \mid \theta,\ (y_0, \ldots, y_{k-1}))$$

$$= \int\ p(y_k \mid \theta, x_k\ (y_0, \ldots, y_{k-1}))\ \overbrace{p(x_k \mid \theta,\ (y_0, \ldots, y_{k-1}))\ dx_k}^{=\frac{1}{N}\sum_{1 \leq i \leq N} \delta_{\xi_k^{\theta,i}}} = \overline{h}_k(\theta, \xi^\theta)$$

Unbiased estimation

$$\mathbb{E}\left(\prod_{0 \leq k \leq n} \overline{h}_k(\Theta, \xi^\Theta) \mid \Theta = \theta \right) = p((y_0, \ldots, y_n) | \theta) = \prod_{0 \leq k \leq n} h_k(\theta)$$

$\rightsquigarrow$ **Extended target with θ-marginal $= \pi_n$**

$$\overline{\pi}_n(d(\theta, \xi^\theta)) = \left\{ \prod_{0 \leq k \leq n} \overline{h}_k(\theta, \xi^\theta) \right\} \times p(d\theta)\ p(d\xi^\theta \mid \theta)$$

Conclusion

$$\overline{\pi}_n(d(\theta,\xi^\theta)) = \left\{\prod_{0\leq k\leq n} \overline{h}_k(\theta,\xi^\theta)\right\} \times p(d\theta)\ p(d\xi^\theta \mid \theta)$$

Extended "product" measure $\overline{\theta} = (\theta,\xi^\theta)$

$$\overline{\pi}_n(d\overline{\theta}) \propto \overbrace{\left\{\prod_{0\leq k\leq n} \overline{h}_k(\overline{\theta})\right\}}^{:=H_n(\overline{\theta})} \times p(d\overline{\theta})$$

$\Leftrightarrow$ FK model

$$G_n = \overline{h}_{n+1} \quad \text{and} \quad M_n \quad \text{s.t.} \quad \overline{\pi}_n = \overline{\pi}_n M_n \quad (\overline{\pi}_n\text{-Shaker}) \Rightarrow \eta_n = \overline{\pi}_n$$

Example of $\overline{\pi}_n$-Shaker = MH

$$\overline{\theta} = (\theta, \xi^{\theta}) \longrightarrow \overline{\theta}' = (\theta', \xi^{\theta'}) \;\sim\; \overline{q}(d\overline{\theta}'|\overline{\theta}) := q(d\theta'|\theta)\; p(d\xi^{\theta'} \mid \theta')$$

$$\Downarrow$$

$$\begin{aligned}
\frac{\overline{\pi}_n(d\overline{\theta}')\overline{q}(d\overline{\theta}|\overline{\theta}')}{\overline{\pi}_n(d\overline{\theta})\overline{q}(d\overline{\theta}|\overline{\theta}')} &= \frac{H_n(\overline{\theta}')}{H_n(\overline{\theta})} \times \frac{p(d\theta')\; p(d\xi^{\theta'} \mid \theta') q(d\theta|\theta')\; p(d\xi^{\theta} \mid \theta)}{p(d\theta)\; p(d\xi^{\theta} \mid \theta) q(d\theta'|\theta)\; p(d\xi^{\theta'} \mid \theta')} \\
&= \frac{H_n(\overline{\theta}')}{H_n(\overline{\theta})} \times \underbrace{\frac{p(d\theta')q(d\theta|\theta')}{p(d\theta)\; q(d\theta'|\theta)}}_{=1 \text{ if reversible}}
\end{aligned}$$

12) Θ_n Markov $\rightsquigarrow (X_n, Y_n)$ Partial linear-Gauss filtering

$$p((\theta_0,\ldots,\theta_n) \mid (y_0,\ldots,y_n))$$

$$\propto \left\{ \prod_{0\leq k\leq n} \underbrace{p(y_k \mid (\theta_0,\ldots,\theta_{k-1}),\theta_k,\ (y_0,\ldots,y_{k-1}))}_{=G_k(\theta_0,\ldots,\theta_k)} \right\} \times\ p(\theta_0,\ldots,\theta_n)$$

Important observation to compute G_k:

$$G_k(\theta_0,\ldots,\theta_k)$$

$$= \int\ \underbrace{p(y_k \mid (\theta_0,\ldots,\theta_{k-1}),(y_0,\ldots,y_{k-1}),x_k)}_{=\ \textbf{(ex.) Law of } Y_k\ =\ C_k(\theta_k)x_k\ +\ V_k}$$

$$\times\, \underbrace{p(x_k \mid (\theta_0,\ldots,\theta_{k-1}),\ (y_0,\ldots,y_{k-1}))\ dx_k}_{=\textbf{Kalman-predictor}=\eta_{\theta,k}(dx_k)}$$

FK model with $\mathcal{X}_n = (\Theta_0,\ldots,\Theta_n)$ (or $\mathcal{X}_n = (\Theta_n, \eta_{\Theta,n})$)$\Rightarrow$ IKF

LAB-4 (SCILAB)

SCILAB :

- Polymer models.
  Code : Non-intersecting-SRW.sce
- Confinements + top eigen values/vectors
  Code : SRW-Tube.sce
- Interacting Simulated annealing (ISA) on TSP.
  Code : ISA.sce
- Subset simulation (Level restrictions).
  Code : RVLevel-restrictions.sce

The independent Metropolis-Hastings

Boltzmann-Gibbs formulation

$$\mathbb{Q}_n(dx) \propto \mathbb{H}_n(x)\ \mathbb{P}_n(dx)$$

with $\mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n)$ and for any $x = (x_0, \ldots, x_n)$

$$\mathbb{H}_n(x) = \prod_{0 \leq k < n} G_k(x_k)$$

Independent Metropolis-Hastings

$$x \rightsquigarrow y \ \sim \ \mathbb{P}_n \ \rightsquigarrow \ \overline{x} = \begin{cases} y & \text{proba.} \quad a_n(x,y) := \min\left(1, \frac{\mathbb{H}_n(y)}{\mathbb{H}_n(x)}\right) \\ x & \text{proba.} \quad 1 - a_n(x,y) \end{cases}$$

Problem

$$\frac{\mathbb{H}_n(y)}{\mathbb{H}_n(x)} = \prod_{0 \leq k < n} \frac{G_k(y_k)}{G_k(x_k)} \qquad \text{mixing prop/degeneracy w.r.t. } n$$

Island type model

FK model $\sim (\chi_n, \mathcal{G}_n) \rightsquigarrow$ ***N*-IPS = Island particle models**

$$\mathbb{E}\left(\mathcal{F}_n(\chi_n)\prod_{0\leq k<n}\mathcal{G}_k(\chi_k)\right) \stackrel{\forall N}{=\!=} \mathbb{E}\left(f_n(X_n)\prod_{0\leq k<n}G_k(X_k)\right)$$

with

$$\chi_n = \xi_n \qquad \mathcal{F}_n(\chi_n) := \eta_n^N(f_n) \quad \text{and} \quad \mathcal{G}_n(\chi_n) := \eta_n^N(G_n)$$

$\subset$ **FK** $\sim (\chi_n, \mathcal{G}_n)$ **on path space = Many body FK model**

$$\mathcal{Q}_n(f) \propto \mathbb{E}\left(f(\chi_0,\ldots,\chi_n)\left\{\prod_{0\leq k<n}\mathcal{G}_k(\chi_k)\right\}\right)$$

The independent Metropolis-Hastings

Boltzmann-Gibbs formulation

$$\mathcal{Q}_n(d\chi) \propto \mathcal{H}_n(\chi)\ \mathcal{P}_n(d\chi)$$

with $\mathcal{P}_n = \mathrm{Law}(\chi_0, \ldots, \chi_n)$ and

$$\mathcal{H}_n(\chi) = \prod_{0 \le k < n} \mathcal{G}_k(\chi_k) = \prod_{0 \le k < n} \eta_k^N(G_k) \simeq_{N \uparrow \infty} \mathcal{Z}_n$$

Independent Metropolis-Hastings

$$\chi \rightsquigarrow \zeta \ \sim \ \mathcal{P}_n \ \rightsquigarrow \ \overline{\chi} = \begin{cases} \zeta & \text{proba.} \quad a_n(\chi,\zeta) := \min\left(1, \frac{\mathcal{H}_n(\zeta)}{\mathcal{H}_n(\chi)}\right) \\ \chi & \text{proba.} \quad 1 - a_n(\chi,\zeta) \end{cases}$$

Advantage/Drawback

$$\mathcal{H}_n(\zeta)/\mathcal{H}_n(\chi) \simeq_{N \uparrow \infty} 1 \qquad \text{but } O(n)\text{-fluctuations (admitted)}$$

Frozen particle path & Particle Gibbs sampler

Algorithm

1. Run an N-IPS model with a frozen path $\mathbb{X} = x$ on $[0, n]$
2. Select one of the ancestral lines **or** sample a backward path $\overline{\mathbb{X}} = \overline{x}$

Theo

$$\text{Both (Markov) transitions} \quad \mathbb{X} \rightsquigarrow \overline{\mathbb{X}} \quad \text{are } \mathbb{Q}_n\text{-reversible}$$

Advantage

$$\underbrace{\mathbb{X} = \overline{\mathbb{X}}_0 \rightsquigarrow \overline{\mathbb{X}}_1 \rightsquigarrow \ldots \rightsquigarrow \overline{\mathbb{X}}_m}_{m\text{-th step transition}} \Rightarrow \mathrm{Law}(\overline{\mathbb{X}}_m) = \mathrm{O}\left((n/N)^m\right)$$

7 Exercises with solutions

Sol. Ex. 1 - Positioning/Radar obs.

Filtering model

$$\left\{\begin{array}{lcl} X_n^{(1)} & = & X_{n-1}^{(1)} + \epsilon_n \; W_n \\ X_n^{(2)} & = & (1 - \alpha \; \Delta) \; X_{n-1}^{(2)} + \beta \; \Delta \; X_n^{(1)} \\ X_n^{(3)} & = & X_{n-1}^{(3)} + \Delta \; X_n^{(2)} \end{array}\right.$$

Gaussian W_n *and* ϵ_n *i.i.d. Bernoulli* $\sim p = \Delta$

$$Y_n = X_n^{(3)} + \Delta \; V_n \qquad \text{with} \qquad V_n \text{ i.i.d.} \sim \; N(0, \sigma_v^2)$$

Some questions:

- $\mathrm{Law}(X_0, \ldots, X_n \mid (Y_0, \ldots, Y_{n-1} \text{ or } {}_n))$?
- $p(y_0, \ldots, y_n)$?

Solution

FK model:

$$X_n = \left(X_n^{(i)}\right)_{i=1,2,3} \quad \oplus \quad G_n(x_n) = p(y_n|x_n) \propto e^{-\frac{1}{2\Delta^2}(y_n - x_n^{(3)})^2}$$

Solution

FK model:

$$X_n = \left(X_n^{(i)} \right)_{i=1,2,3} \quad \oplus \quad G_n(x_n) = p(y_n|x_n) \propto e^{-\frac{1}{2\Delta^2}(y_n - x_n^{(3)})^2}$$

More developed solution . . .

Solution

FK model:

$$X_n = \left(X_n^{(i)}\right)_{i=1,2,3} \quad \oplus \quad G_n(x_n) = p(y_n|x_n) \propto e^{-\frac{1}{2\Delta^2}(y_n - x_n^{(3)})^2}$$

More developed solution ...

$$\Downarrow$$

$\rightsquigarrow$ **Genetic algo. mutation as $X_{n-1} \rightsquigarrow X_n$, selection fitness G_n:**

$$(\xi_n^i)_{1\leq i\leq N} \xrightarrow{\text{updating/selection}} (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{\text{free exploration}} (\xi_{n+1}^i)_{1\leq i\leq N}$$

Particle estimates

- *After the n-th mutation:*

$$\mathbb{E}(f(X_0,\ldots,X_n)\mid(Y_0,\ldots,Y_{n-1}))\simeq_{N\uparrow\infty}\frac{1}{N}\sum_{1\leq i\leq N} f\,\underbrace{\left(\xi^i_{0,n},\xi^i_{1,n},\ldots,\xi^i_{n,n}\right)}_{i\text{-th ancestral line}}$$

- *After the n-th selection:*

$$\mathbb{E}(f(X_0,\ldots,X_n)\mid(Y_0,\ldots,Y_n))\simeq_{N\uparrow\infty}\frac{1}{N}\sum_{1\leq i\leq N} f\,\underbrace{\left(\widehat{\xi}^i_{0,n},\widehat{\xi}^i_{1,n},\ldots,\widehat{\xi}^i_{n,n}\right)}_{i\text{-th ancestral line}}$$

- **Normalizing constants:**

$$p(y_0,\ldots,y_n)\simeq\prod_{0\leq k\leq n}\frac{1}{N}\sum_{1\leq i\leq N}G_k(\xi^i_k)\quad\textbf{(unbiased)}$$

Sol. Ex. 2 - SRW/Brownian $\in [-1, 1]$

Markov chain model

$$X_n = X_{n-1} + \sqrt{\Delta}\ W_n \quad \simeq \quad dX_t = dW_t$$

Wiener/Brownian process W_t ($X_0 = 0$, $\Delta = 10^{-2}$)

$$\downarrow$$

Some questions:

- $\mathrm{Law}(X_0, \ldots, X_n \mid \forall 0 \leq k < n\ X_k \in [-1, 1])$ (two ways)?
- $\mathbb{P}\left(\forall 0 \leq k \leq n\ X_k \in [-1, 1]\right)$?

Solution = FK: $G_n = 1_{[-1,1]}$

Solution = FK: $G_n = 1_{[-1,1]}$

More developed solution . . .

Solution = FK: $G_n = 1_{[-1,1]}$

More developed solution ...

$\rightsquigarrow$ **Genetic algo. mutation as $X_{n-1} \rightsquigarrow X_n$, selection fitness G_n:**

$$(\xi_n^i)_{1\leq i\leq N} \xrightarrow{\textbf{updating/selection}} (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{\textbf{free exploration}} (\xi_{n+1}^i)_{1\leq i\leq N}$$

Solution = FK: $G_n = 1_{[-1,1]}$

More developed solution ...

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Particle estimates

- *After the n-th mutation:*

$$\mathbb{E}(f(X_0,\ldots,X_n) \mid \forall 0\leq k<n\ X_k\in[-1,1])$$

$$\simeq_{N\uparrow\infty} \frac{1}{N}\sum_{1\leq i\leq N} f\underbrace{\left(\xi_{0,n}^i,\xi_{1,n}^i,\ldots,\xi_{n,n}^i\right)}_{i\text{-th ancestral line}}$$

- *Normalizing constants (unbiased):*

$$\mathbb{P}\left(\forall 0\leq k\leq n\ X_k\in[-1,1]\right)\simeq\prod_{0\leq k\leq n}\frac{1}{N}\sum_{1\leq i\leq N}1_{[-1,1]}(\xi_k^i)$$

Sol. Ex. 3 - Top spectrum

Matrix with positive entries

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}$$

$$\downarrow$$

Some questions:

- To of the spectrum of $Q = P'P$?
- Corresponding eigenvector?

Solution $=$ FK (G, M) (cf. example (5))

$$Q(i,j) = G(i) \times M(i,j) \quad \text{with} \quad G(i) = \sum_{1 \leq j \leq 4} Q(i,j)$$

FK ($\mu = \eta_0 = 1_{i_0}$, $f = 1_{i_n}$, and X_n Markov transition M)

$$Q^n(i_0, i_n) = \mathbb{E}\left(f(X_n) \prod_{0 \leq k < n} G_k(X_k) \right) \propto \eta_n(f)$$

Spectral theo $\rightsquigarrow$ long time approx./power method

$$Q^n(i_0, i_n) \quad = \quad \sum_{0 \leq k < 4} \lambda_k^n \, \varphi_k(i_0)\varphi_k(i_n) \simeq_{n\uparrow\infty} \lambda_0^n \, \varphi_0(i_0)\varphi_0(i_n)$$

$$\sum_{1 \leq i_n \leq 4} Q^n(i_0, i_n) \quad = \quad \mathbb{E}\left(\prod_{0 \leq k < n} G_k(X_k) \right) = \mathcal{Z}_n \simeq_{n\uparrow\infty} \text{cte } \lambda_0^n$$

$$\Downarrow$$

$$\frac{1}{n} \log \mathcal{Z}_n \simeq_{n\uparrow\infty} \log \lambda_0 \quad \text{and} \quad \eta_n(f) = \frac{Q^n(i_0, i_n)}{\sum_{1 \leq j_n \leq 4} Q_n(i_0, j_n)} \simeq_{n\uparrow\infty} \frac{\varphi_0(i_n)}{\sum_{j_n} \varphi_0(j_n)}$$

Sol. Ex. 4 - Black-box/inverse pb

Input(X)/Output(Y) model

$$X = (X^1, X^2) \sim \text{uniform in the unit disk} \ \mapsto \ Y = V(X) = |X^1| + |X^2|$$

$$\downarrow$$

Some questions:

- $\text{Law}(X \mid V(X) \leq 1/10)$ (MH+SMC)?
- $\mathbb{P}(V(X) \leq 1/10)$?
- When X is Gaussian?

FK-Proba. restrictions (example (8))

Notation

$$\text{Law}(X) = \lambda \text{ and } A_n = \{V(X) \leq \epsilon_n\} \text{ with } \epsilon_n \downarrow$$

$$\Downarrow$$

$$\begin{aligned} \mathbb{P}(X \in dx \mid V(X) \leq \epsilon_n) &= Z_{A_n}^{-1} \; 1_{A_n}(x) \; \lambda(dx) := \pi_n(dx) \\ Z_{A_n} &= \mathbb{P}(V(X) \leq \epsilon_n) \end{aligned}$$

FK model

$$G_n = 1_{A_{n+1}} \quad \text{and} \quad M_n \quad \text{s.t.} \quad \pi_n = \pi_n M_n \quad (A_n\text{-Shaker})$$

$$\Downarrow$$

$$\eta_n = \pi_n \quad \text{and} \quad Z_{A_n}/Z_{A_0} = \mathcal{Z}_n$$

Initialisation:

$$\epsilon_0 = \sqrt{2} \Rightarrow \{|x_1|^2 + |x_2|^2 \leq 1\} \subset \{|x_1| + |x_2| \leq \sqrt{2}\} \Rightarrow Z_{A_0} = \pi$$

On the shakers $\pi_n = \pi_n M_n$

Case λ = Uniform in the unit disk $D(\supset A_n)$

$$1) \quad x(\in A_n) \xrightarrow{\text{Gibbs move/proposal in } D} y \xrightarrow{\text{MH-accept/reject}} \overline{x} = \begin{cases} y & \text{if} \quad y \in A_n \\ x & \text{if} \quad y \notin A_n \end{cases}$$

$$2) \quad x(\in A_n) \xrightarrow{\text{local gauss. } p_\epsilon(x,y)} y_i = x_i + \epsilon\, \mathcal{N}(0,1)\ , i = 1,2$$

$$\downarrow \text{MH-accept/reject}$$

$$\overline{x} = \begin{cases} y & \text{proba} \quad \min\left(1, \frac{1_{A_n}(y) p_\epsilon(y,x)}{1_{A_n}(x) p_\epsilon(x,y)}\right) = 1_{A_n}(y) \\ x & \text{otherwise} \quad x \end{cases}$$

On the shakers $\pi_n = \pi_n M_n$

Case λ = Gaussian $\mathcal{N}(0,1)$

$$x(\in A_n) \xrightarrow{\text{local reversible gauss. } p_\epsilon(x,y)} \quad y_i = \sqrt{1-\epsilon^2}\, x_i + \epsilon\, \mathcal{N}(0,1), \;\; i = 1,2$$

$$\downarrow \text{MH-accept/reject}$$

$$\overline{x} = \begin{cases} y & \text{proba} \quad \min\left(1, \frac{1_{A_n}(y)\lambda(dy)p_\epsilon(y,x)}{1_{A_n}(x)\lambda(dx)p_\epsilon(x,y)}\right) = 1_{A_n}(y) \\ x & \text{otherwise} \quad x \end{cases}$$

Sol. Ex. 5 - Polymer models

Toy polymer model X_n SRW on $\mathbb{Z}^2$ starting at the origin

$$\downarrow$$

Some questions:

- Mean displacement given self-repulsion?

$$\mathbb{E}\left(\frac{1}{n}\sum_{1\leq k\leq n} X_k^2 \mid \forall 0\leq k\neq l\leq n,\ X_k\neq X_l\right)$$

- Probability of self-avoiding paths?

Solution $=$ Example (4) with

$$f(X_0,\ldots,X_n)=\frac{1}{n}\sum_{1\leq k\leq n} X_k^2$$

Ex. 6 - Partially linear [HMM (X, W, V) Gauss]

$$(a) \quad \begin{cases} \Theta & \sim & N(0,1) \\ X_n & = & \Theta\ X_{n-1} + W_n \\ Y_n & = & X_n + V_n \end{cases}$$

Unknown parameter
Signal
Observation

$$(b) \qquad X_n = X_{n-1} + W_n \quad \text{with} \quad W_n \sim \mathrm{N}(0, \Theta^2)$$

$$(c) \qquad Y_n = X_n + V_n \quad \text{with} \quad V_n \sim \mathrm{N}(0, \Theta^2)$$

IN ALL CASES $=$ HMM lin-Gauss $\rightsquigarrow$ Sol. $=$ Ex. (10)

Addition:

$$\mathbb{P}(X_n \in dx \mid (Y_0, \ldots, Y_n))$$

$$= \int \underbrace{\mathbb{P}(X_n \in dx \mid (Y_0, \ldots, Y_n),\ \Theta = \theta)}_{\text{given by Kalman!}} \quad \underbrace{p(d\theta \mid (y_0, \ldots, y_n))}_{= \pi_n(d\theta) \text{ approx. particles ex. (10)}}$$

Ex. 7 - general HMM

HMM model = positioning Radar problem

$$\left\{\begin{array}{rcl} \Theta & = & (\Theta_1, \Theta_2, \Theta_3) \sim N(0, I_{3\times 3}) \\ \\ dX_t^{(1)} & = & \Theta_1 W_t \ dN_t \\ \dfrac{dX_t^{(2)}}{dt} & = & \Theta_2 \ X_t^{(2)} + \Theta_3 \ X_t^{(1)} \\ \dfrac{dX_t^{(3)}}{dt} & = & X_t^{(2)} \end{array}\right.$$

$=$ **Nonlinear/Non Gaussian HMM** $\rightsquigarrow$ **Sol.** $=$ **Ex. (11)**

As above (same addition):

$$\mathbb{P}(X_n \in dx \mid (Y_0, \ldots, Y_n))$$

$$= \int \underbrace{\mathbb{P}(X_n \in dx \mid (Y_0, \ldots, Y_n), \ \Theta = \theta)}_{\text{given by the particle filter!}} \quad \underbrace{p(d\theta \mid (y_0, \ldots, y_n))}_{= \pi_n(d\theta) \text{ approx. particles ex. (10)}}$$