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ON SOME GENERALIZED I-CONVERGENT DOUBLE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

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Abstract In this article we introduce the sequence spaces ${}_2c_0^I(F, p)$, ${}_2c^I(F, p)$ and ${}_2l_\infty^I(F, p)$ for $F = (f_{ij})$ a sequence of moduli and $p = (p_{ij})$ sequence of positive reals and study some of the properties and inclusion relations on these spaces.

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1. INTRODUCTION AND PRELIMINARIES

The notion of ideal convergence was first introduced by Kostyrko et al. [1], as a generalization of statistical convergence [2], which was further studied in topological spaces by Das et al [3].
A family I$\subseteq 2^X$ of subsets of a non empty set X is said to be *an ideal in X* if
(1) $\phi \in I$,
(2) I is additive i.e A, B $\in$ I $\Rightarrow$A$\cup$ B$\in$I,
(3) I is hereditary i.e A$\in$ I, B$\subseteq$A$\Rightarrow$B$\in$I.

A non-empty family of sets £ $\subseteq 2^X$ is said to be *a filter* on X if and only if $\Phi \notin$ £(I), for A, B$\in$ £(I) we have A$\cap$B$\in$ £(I) and for each A$\in$ £(I)and A$\subseteq$B implies B$\in$ £(I).
An ideal I$\subseteq 2^X$ is called *non-trivial* if I$\neq 2^X$.
A non-trivial ideal I$\subseteq 2^X$ is called *admissible* if $\{\{x\} : x \in X\}$ $\subseteq$I.
A non-trivial ideal I is *maximal* if there cannot exist any non-trivial ideal J$\neq$I containing I as a subset.
For each ideal I, there is a filter £(I) corresponding to I i.e £(I) = $\{K \subseteq X : K^c \in I\}$, where K^c = X-K.

Example 1. Let $I_2(P)$ be the class of all subsets of $\mathbb{N} \times \mathbb{N}$ such that $D \in I_2(P)$ implies that there exists $n_0, k_0 \in \mathbb{N}$ such that $D \subset \mathbb{N}\times\mathbb{N}-\{(n, k) \in \mathbb{N}\times\mathbb{N} : n \geq n_0, k \geq k_0\}$.

Then $I_2(P)$ is an ideal of $\mathbb{N} \times \mathbb{N}$ in the usual Pringsheim's sense of convergence of double sequences. If $I_2(P)$ is replaced by $I(f)$, the class of finite subsets of $\mathbb{N}$, then we get the usual regular convergence of double sequences [4].

A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ [5, 6, 7]. A number $a \in \mathbb{C}$ is called *a double limit* of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$|x_{ij} - a| < \epsilon, \qquad \forall\ i, j \geq N.$$

Throughout the article $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and ω denotes the set of natural, real, complex numbers and the class of all double sequences respectively.

The idea of modulus was structured in 1953 by Nakano [8].
A function $f : [0,\infty) \longrightarrow [0,\infty)$ is called *a modulus* if
(1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t+u) \leq f(t) + f(u)$ for all $t,u \geq 0$,
(3) f is increasing, and
(4) f is continuous from the right at zero.

Example 2. Define $f : [0,\infty) \to [0,\infty)$ then if we take $f(x) = \frac{x}{x+1}$, $f(x)$ is a bounded modulus function and if we take $f(x) = x^p, 0 < p < 1$, then $f(x)$ is an unbounded modulus function.

Ruckle [9] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}.$$

This space is an FK space, and Ruckle [10] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.
The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [11] proved that $X(f) \subset l_1$ and $X(f)^{\alpha} = l_{\infty}$. The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.$$

After that E. Kolk [12, 13] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) < \infty\}.$$

1.1. SOME BASIC DEFINITIONS.

A sequence space E is said to be *solid or normal* if $(x_{ij}) \in$ E implies $(\alpha_{ij}x_{ij}) \in$ E for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all i,j $\in \mathbb{N}$ [8].

A sequence $(x_{ij}) \in \omega$ is said to be *I-convergent to a number L* if for every $\epsilon > 0$, $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in$ I. In this case we write I-$lim x_{ij}$ = L.

The space c^I of all I-convergent sequences to L is given by

$$c^I = \{(x_{ij}) \in \omega : \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I, \text{for some L } \in \mathbb{C} \}.$$

A sequence$(x_{ij}) \in \omega$ is said to be I-*null* if L = 0. In this case we write I-$lim x_{ij}$ = 0.

A double sequence (a_{nk}) is said to be I-*Cauchy* if for every $\epsilon > 0$ there exist $s = s(\epsilon), t = t(\epsilon)) \in \mathbb{N}$ such that

$$\{(n, k) \in \mathbb{N} \times \mathbb{N} : |a_{nk} - a_{st}| \geq \epsilon\} \in I_2.$$

A sequence $(x_{ij}) \in \omega$ is said to be I-*bounded* if there exists M >0 such that $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\} \in I.$

Let X be a linear metric space. A function $g : X \longrightarrow R$ is called *paranorm*, if for all $x, y, \in X$,
(1) $g(x) = 0 \; if \; x = \theta$,
(2) $g(-x) = g(x)$,
(3) $g(x + y) \leq g(x) + g(y)$,
(4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$, in the sense that $g(x_n - a) \to 0 \; (n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.
A paranorm g for which $g(x) = 0$ implies $x = 0$ is called a *total paranorm* on X, and the pair (X, g) is called a *total paranormed space* [14].
A *convergence field of I-convergence* is a set

$$F(\text{I}) = \{x = (x_k) \in l_\infty : \text{there exists I-lim } x \in \mathbb{R}\}.$$

The convergence field F(I) is a closed linear subspace of l_∞ with respect to the supremum norm, therefore $F(\text{I}) = l_\infty \cap c^I$ ([15], [16], [17], [18], [19]).

Throughout the article ${}_2l_\infty$, ${}_2c^I$, ${}_2c^I_0$, ${}_2m^I$ and ${}_2m^I_0$ represent the bounded , I-convergent, I-null, bounded I-convergent and bounded I-null double sequence spaces respectively.

1.2. SOME USEFUL LEMMAS

Lemma 1.1. [22]. *Let* $h = \inf_k p_k$ *and* $H = \sup_k p_k$. *Then the following conditions are equivalent:*
(a) $H < \infty$ *and* $h > 0$*;*
(b) $c_0(p) = c_0$ *or* $l_\infty(p) = l_\infty$*;*
(c) $l_\infty\{p\} = l_\infty(p)$*;*
(d) $c_0\{p\} = c_0(p)$*;*
(e) $l\{p\} = l(p)$.

Lemma 1.2. [18, 19]. *Let* $K \in £(I)$ *and* $M \subseteq N$. *If* $M \notin I$, *then* $M \cap K \notin I$.

2. MAIN RESULTS

In this article we introduce the following classes of sequence space:

$$ {}_2c^I(F,p) = \{(x_{ij}) \in \omega : f_{ij}(|x_{ij} - L|^{p_{ij}}) \geq \epsilon \text{ for some L} \in \mathbb{C}\} \in I, $$

$$ {}_2c_0^I(F,p) = \{(x_{ij}) \in \omega : f_{ij}(|x_{ij}|^{p_{ij}}) \geq \epsilon\} \in I, $$

$$ {}_2l_\infty^I(F,p) = \{(x_{ij}) \in \omega : \sup_{i,j} f_{ij}(|x_{ij}|^{p_{ij}}) < \infty\} \in I. $$

Also we denote

$$ {}_2m^I(F,p) = {}_2c^I(F,p) \cap_2 l_\infty(F,p) $$

and

$$ {}_2m_0^I(F,p) = {}_2c_0^I(F,p) \cap {}_2l_\infty(F,p). $$

Theorem 2.1. *Let* $p = (p_{ij}) \in {}_2l_\infty$. *Then* ${}_2c^I(F,p)$, ${}_2c_0^I(F,p)$, ${}_2m^I(F,p)$ *and* ${}_2m_0^I(F,p)$ *are linear spaces.*

Proof. Let $(x_{ij}), (y_{ij}) \in {}_2c^I(F,p)$ and α,β be two scalars. Then, for a given $\epsilon > 0$, we have

$$ \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - L_1|^{p_{ij}}) \geq \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathrm{C}\} \in I, $$

$$ \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|y_{ij} - L_2|^{p_{ij}}) \geq \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathrm{C}\} \in I, $$

where

$$ M_1 = D \cdot max\{1, \sup_{ij} |\alpha|^{p_{ij}}\} \quad and \quad M_2 = D \cdot max\{1, \sup_{ij} |\beta|^{p_{ij}}\} $$

and $D = max\{1, 2^{H-1}\}$ where $H = \sup_{ij} p_{ij} \geq 0$. Let

$$ A_1 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - L_1|^{p_{ij}}) < \frac{\epsilon}{2M_1}, \text{ for some } L_1 \in \mathrm{C}\} \in I $$

$$A_2 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|y_{ij} - L_2|^{p_{ij}}) < \frac{\epsilon}{2M_2}, \text{ for some } L_2 \in \mathbb{C}\} \in I$$

be such that $A_1^c, A_2^c \in I$. Then

$$A_3 = \{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|(\alpha x_{ij} + \beta y_{ij}) - f_{ij}(\alpha L_1 + \beta L_2)|^{p_{ij}}) < \epsilon\}$$

$$\supseteq \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : |\alpha|^{p_{ij}} f_{ij}(|x_{ij} - L_1|^{p_{ij}}) < \frac{\epsilon}{2M_1}|\alpha|^{p_{ij}} \cdot D\right\}$$

$$\bigcap \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : |\beta|^{p_{ij}} f_{ij}(|y_{ij} - L_2|^{p_{ij}}) < \frac{\epsilon}{2M_2}|\beta|^{p_{ij}} \cdot D\right\}$$

Thus $A_3^c = A_1^c \cap A_2^c \in I$. Hence $(\alpha x_{ij} + \beta y_{ij}) \in_2 c^I(F,p)$. Therefore ${}_2c^I(F,p)$ is a linear space. The rest of the result follows similarly. ∎

Theorem 2.2. *Let $(p_{ij}) \in {}_2l_\infty$. Then ${}_2m^I(F,p)$ and ${}_2m_0^I(F,p)$ are paranormed spaces, paranormed by $g(x_{ij}) = \sup_{ij} f_{ij}(|x_{ij}|^{\frac{p_{ij}}{M}})$ where $M = max\{1, \sup_{ij} p_{ij}\}$.*

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in {}_2m^I(F,p)$.
(1) Clearly, $g(x) = 0$ if and only if $x = 0$.
(2) $g(x) = g(-x)$ is obvious.
(3) Since $\frac{p_{ij}}{M} \leq 1$ and $M > 1$, using Minkowski's inequality and the definition of f we have

$$\sup_{i,j} f_{ij}(|x_{ij} + y_{ij}|^{\frac{p_{ij}}{M}}) \leq \sup_{i,j} f_{ij}(|x_{ij}|^{\frac{p_{ij}}{M}}) + \sup_{i,j} f(|y_{ij}|^{\frac{p_{ij}}{M}})$$

(4) Now for any complex λ we have (λ_k) such that $\lambda_k \to \lambda$, $(k \to \infty)$.
Let $x_{ij} \in_2 m^I(f,p)$ such that $f_{ij}(|x_{ij} - L|^{p_{ij}}) \geq \epsilon$.
Therefore, $g(x_{ij} - L) = \sup_{ij} f_{ij}(|x_{ij} - L|^{\frac{p_{ij}}{M}}) \leq \sup_{i,j} f_{ij}(|x_{ij}|^{\frac{p_{ij}}{M}}) + \sup_{i,j} f_{ij}(|L|^{\frac{p_{ij}}{M}})$.
Hence $g(\lambda_n x_{ij} - \lambda L) \leq g(\lambda_n x_{ij}) + g(\lambda L) = \lambda_n g(x_{ij}) + \lambda g(L)$ as $((i,j) \to \infty)$.
Hence ${}_2m^I(F,p)$ is a paranormed space.
The rest of the result follows similarly. ∎

Theorem 2.3. *A sequence $x = (x_{ij}) \in {}_2m^I(F,p)$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that*

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|^{p_{ij}}) < \epsilon\} \in {}_2m^I(F,p). \qquad (1)$$

Proof. Suppose that $L =$I-lim x. Then

$$B_\epsilon = \{(i,j) \in \mathbb{N}\times\mathbb{N} : |x_{ij} - L|^{p_{ij}} < \frac{\epsilon}{2}\} \in {}_2m^I(F,p), \quad \text{for all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have for all $(i,j) \in B_\epsilon$

$$|x_{N_\epsilon} - x_{ij}|^{p_{ij}} \leq |x_{N_\epsilon} - L|^{p_{ij}} + |L - x_{ij}|^{p_{ij}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|^{p_ij}) < \epsilon\} \in {}_2m^I(F,p)$.

Conversely, suppose that $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - x_{N_\epsilon}|^{p_{ij}}) < \epsilon\} \in {}_2m^I(F,p)$. That is $\{(i,j) \in \mathbb{N}\times\mathbb{N} : (|x_{ij} - x_{N_\epsilon}|^{p_{ij}}) < \epsilon\} \in {}_2m^I(F,p)$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in {}_2m^I(F,p) \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in {}_2m^I(F,p)$ as well as $C_{\frac{\epsilon}{2}} \in {}_2m^I(f,p)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in {}_2m^I(F,p)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in J\} \in {}_2m^I(F,p)$$

that is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J.
In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \in I_k\} \in_2 m^I(F,p)$ for (k=1,2,3,4,......).
Then there exists a $\xi \in \cap I_k$ such that ξ = I-lim x. So that $f_{ij}(\xi)$ =I-lim $f_{ij}(x)$, that is L =I-lim $f_{ij}(x)$. ∎

Theorem 2.4. *Let* $H = \sup_{i,j} p_{ij} < \infty$ *and I an admissible ideal. Then the following are equivalent.*
(a) $(x_{ij}) \in {}_2c^I(F,p)$;
(b) there exists$(y_{ij}) \in {}_2c(F,p)$ *such that* $x_{ij} = y_{ij}$,
(c) there exists$(y_{ij}) \in {}_2c(F,p)$ *and* $(x_{ij}) \in {}_2c_0^I(F,p)$ *such that* $x_{ij} = y_{ij} + z_{ij}$ *for all* $(i,j) \in \mathbb{N}\times\mathbb{N}$ *and* $\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|y_{ij} - L|^{p_{ij}}) \geq \epsilon\} \in I$;
(d) there exists a subset $K = \{k_1 < k_2....\}$ *of* $\mathbb{N}$ *such that* $K \in \pounds(I)$ *and* $\lim_{n\to\infty} f_{ij}(|x_{k_ik_j} - L|^{p_{k_ik_j}}) = 0$.

Proof. (a) ⇒ (b). Let $(x_{ij}) \in {}_2c^I(F,p)$. Then there exists $L \in \mathbb{C}$ such that

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : f_{ij}(|x_{ij} - L|^{p_ij}) \geq \epsilon\} \in I.$$

Let $(m_t, n_t) \in \mathbb{N}\times\mathbb{N}$ be an increasing sequence such that,

$$\{(i,j) \leq (m_t, n_t) : f_{ij}(|x_{ij} - L|^{p_{ij}}) \geq \epsilon\} \in I.$$

Define a sequence (y_{ij}) as

$$y_{ij} = x_{ij}, \text{ for all } (i, j) \leq (m_1, n_1).$$

For $(m_t, n_t) \leq (i, j) \leq (m_{t+1}, n_{t+1}), \quad t \in \mathbb{N}$.

$$y_{ij} = \begin{cases} x_{ij}, & \text{if } |x_{ij} - L|^{p_{ij}} < t^{-1}, \\ L, & otherwise. \end{cases}$$

Then $(y_{ij}) \in {}_2c(F, p)$ and form the following inclusion

$$\{(i, j) \leq (m_t, n_t) : x_{ij} \neq y_{ij}\} \subseteq \{k \leq (m_t, n_t) : f_{ij}(|x_{ij} - L|^{p_{ij}}) \geq \epsilon\} \in I.$$

We get $x_{ij} = y_{ij}$.

(b) $\Rightarrow$ (c). For $(x_{ij}) \in {}_2c^I(F, p)$. Then there exists $(y_{ij}) \in {}_2c(F, p)$ such that $x_{ij} = y_{ij}$. Let $K = \{(i, j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \neq y_{ij}\}$, then $(i, j) \in I$.
Define a sequence (z_{ij}) as

$$z_{ij} = \begin{cases} x_{ij} - y_{ij}, & \text{if } (i, j) \in K, \\ 0, otherwise. \end{cases}$$

Then $z_{ij} \in {}_2c_0^I(F, p)$ and $y_{ij} \in {}_2c(F, p)$.

(c) $\Rightarrow$ (d). Let $P_1 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}(|x_{ij}|^{p_{ij}}) \geq \epsilon\} \in I$ and

$$K = P_1^c = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < ...\} \in £(I).$$

Then we have $\lim_{n\to\infty} f_{ij}(|x_{i_n,j_n} - L|^{p_{i_n,j_n}}) = 0$.

(d) $\Rightarrow$ (a). Let
$K = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < ...\} \in £(I)$ and $\lim_{n\to\infty} f_{ij}(|x_{i_n,j_n} - L|^{p_{i_n,j_n}}) = 0$.
Then for any $\epsilon > 0$, and Lemma 1.9, we have

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}(|x_{ij} - L|^{p_{ij}}) \geq \epsilon\} \subseteq K^c \cup \{(i, j) \in K : f_{ij}(|x_{ij} - L|^{p_{ij}}) \geq \epsilon\}.$$

Thus $(x_{ij}) \in {}_2c^I(F, p)$. ∎

Theorem 2.5. *Let* (p_{ij}) *and* (q_{ij}) *be two sequences of positive real numbers. Then* ${}_2m_0^I(F, p) \supseteq {}_2m_0^I(F, q)$ *if and only if* $\lim_{(i,j)\in K} \inf \frac{p_{ij}}{q_{ij}} > 0$, *where* $K^c \subseteq \mathbb{N} \times \mathbb{N}$ *such that* $K \in I$.

Proof. Let $\lim_{(i,j)\in K} \inf \frac{p_{ij}}{q_{ij}} > 0$. and $(x_{ij}) \in {}_2m_0^I(F, q)$. Then there exists $\beta > 0$ such that $p_{ij} > \beta q_{ij}$, for all sufficiently large $(i, j) \in K$.
Since $(x_{ij}) \in {}_2m_0^I(F, q)$, for a given $\epsilon > 0$, we have

$$B_0 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}(|x_{ij}|^{q_{ij}}) \geq \epsilon\} \in I.$$

Let $G_0 = K^c \cup B_0$ Then $G_0 \in I$, and for all sufficiently large $(i, j) \in G_0$,

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}(|x_{ij}|^{p_{ij}}) \geq \epsilon\} \subseteq \{(i, j) \in \mathbb{N} \times \mathbb{N} : f_{ij}(|x_{ij}|^{\beta q_{ij}}) \geq \epsilon\} \in I.$$

Therefore $(x_{ij}) \in {}_2m_0^I(F, p)$. ∎

Theorem 2.6. *Let (p_{ij}) and (q_{ij}) be two sequences of positive real numbers. Then ${}_2m_0^I(F, q) \supseteq {}_2m_0^I(F, p)$ if and only if $\lim_{(i,j)\in K} \inf \frac{q_{ij}}{p_{ij}} > 0$, where $K^c \subseteq \mathbb{N} \times \mathbb{N}$ such that $K \in I$.*

Proof. The proof is similar to Theorem 2.5. ∎

Corollary 2.7 Let (p_{ij}) and (q_{ij}) be two sequences of positive real numbers. Then ${}_2m_0^I(F, q) = {}_2m_0^I(F, p)$ if and only if $\lim_{(i,j)\in K} \inf \frac{p_{ij}}{q_{ij}} > 0$, and $\lim_{(i,j)\in K} \inf \frac{q_{ij}}{p_{ij}} > 0$, where $K \subseteq \mathbb{N}$ such that $K^c \in I$.

Proof. On combining Theorem 2.5 and 2.6 we get the required result. ∎

Theorem 2.7. *Let $h = \inf_{(i,j)} p_{ij}$ and $H = \sup_{(i,j)} p_{ij}$.
Then the following results are equivalent.
(a) $H < \infty$ and $h > 0$.
(b) ${}_2c_0^I(F, p) = {}_2c_0^I$.*

Proof. Suppose that $H < \infty$ and $h > 0$, then the inequalities $min\{1, s^h\} \leq s^{p_{ij}} \leq max\{1, s^H\}$ hold for any $s > 0$ and for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.
Therefore the equivalent of (a) and (b) is obvious. ∎

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