EXTREMAL RESULTS CONCERNING THE GENERAL SUM-CONNECTIVITY INDEX IN SOME CLASSES OF CONNECTED GRAPHS

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Abstract

This paper surveys extremal properties of general sum-connectivity index $\chi_\alpha(G)$ in several classes of connected graphs of given order for some values of the parameter $\alpha$: a) trees; b) connected unicyclic or bicyclic graphs; c) graphs of given connectivity.


Keywords: tree, diameter, pendant vertex, unicyclic graph, bicyclic graph, general sum-connectivity index, zeroth-order general Randić index, 2-connected graph, Jensen’s inequality.

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1. INTRODUCTION

Let $G$ be a simple graph having vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $u \in V(G)$ is denoted $d(u)$. If $d(u) = 1$ then $u$ is called pendant; a pendant edge is an edge containing a pendant vertex. The minimum degree of $G$ is denoted $\delta(G)$ and the complement of $G$ is $\overline{G}$. The girth of a graph $G$ containing cycles is the length of a shortest cycle of $G$. The distance between vertices $u$ and $v$ of a connected graph, denoted by $d(u, v)$, is the length of a shortest path between them. The diameter of $G$ is the maximum distance between vertices of $G$. If $A \subset V(G)$ and $u \in V(G)$, the distance between $u$ and $A$ is $d(u, A) = \min_{v \in A} d(u, v)$. If $x \in V(G)$, $G - x$ denotes the subgraph of $G$ obtained by deleting $x$ and its incident edges.

A similar notation is $G - xy$, where $xy \in E(G)$. $K_{p,q}$ will denote the complete bipartite graph, where the partite sets contain $p$ and $q$ vertices, respectively. Given a graph $G$, a subset $S$ of $V(G)$ is said to be an independent set of $G$ if every two vertices of $S$ are not adjacent. The maximum number of vertices in an independent set of $G$ is called the independence number of $G$ and is denoted by $\alpha(G)$. $K_{1,n-1}$ and $P_n$ will denote, respectively, the star and the path on $n$ vertices. For two vertex-disjoint graphs $G$ and $H$, the join $G + H$ is obtained by joining by edges each vertex of $G$ to all vertices of $H$ and the union $G \cup H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

The connectivity of a graph $G$, written $\kappa(G)$, is the minimum size of a vertex set $S$ such that $G - S$ is disconnected or has only one vertex. A graph $G$ is said to be $k$-connected if its connectivity is at least $k$. Similarly, the edge-connectivity of $G$,
written \( \kappa'(G) \), is the minimum size of a disconnecting set of edges. For every graph \( G \) we have \( \kappa(G) \leq \kappa'(G) \). Since a tree on \( n \) vertices is a bipartite graph, at least one partite set, which is an independent set, has at least \( n/2 \) vertices, which implies that for any tree \( T \) we have \( \alpha(T) \geq \lceil n/2 \rceil \) and this bound is reached for paths. Also, \( \alpha(T) \leq n - 1 \) and the equality holds only for the star graph. For every \( n \geq 2 \) and \( n/2 \leq s \leq n - 1 \) the spur \( SP_{n,s} \) [4] is a tree consisting of \( 2s - n + 1 \) edges and \( n - s - 1 \) paths of length 2 having a common endvertex; in other words, it is obtained from a star \( K_{1,s} \) by attaching a pendant edge to \( n - s - 1 \) pendant vertices of \( K_{1,s} \). We have \( \alpha(SP_{n,s}) = s \). A bistar of order \( n \), denoted by \( BS(p,q) \), consists of two vertex disjoint stars, \( K_{1,p} \) and \( K_{1,q} \), where \( p + q = n - 2 \), and a new edge joining the centers of these stars. For \( n \geq 3 \) and \( 0 \leq k \leq n - 3 \), let \( C_{n-k,k} \) denote the unicyclic graph of order \( n \) consisting of a cycle \( C_{n-k} \) and \( k \) pendant edges attached to a unique vertex of \( C_{n-k} \). For other notations in graph theory, we refer [1].

The general sum-connectivity index of graphs was proposed by Zhou and Trinajstić [18]. It is denoted by \( \chi_\alpha(G) \) and defined as

\[
\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha,
\]

where \( \alpha \) is a real number. The sum-connectivity index, previously proposed by the same authors [17] is \( \chi_{-1/2}(G) \). A particular case of the general sum-connectivity index is the harmonic index, denoted by \( H(G) \) and defined as

\[
H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)} = 2\chi_{-1}(G).
\]

The zeroth-order general Randić index, denoted by \( ^0R_\alpha(G) \) is defined as

\[
^0R_\alpha(G) = \sum_{u \in V(G)} d(u)^\alpha,
\]

where \( \alpha \) is a real number. For \( \alpha = 2 \) this index is also known as first Zagreb index (see [7]).

These graph invariants, that are useful for chemical purposes, were named topological indices, or, less confusing, molecular structure-descriptors. Their main use is for designing so-called quantitative structure-property relations, QSPR and quantitative structure-activity relations, QSAR. In this context structure means molecular stucture, property some physical or chemical property, and activity some pharmacologic, biologic, toxicologic, or similar property [7].

In the next sections we shall present some results about graphs having maximum or minimum general sum-connectivity index in various classes of connected graphs for some values of the parameter \( \alpha \).
2. GENERAL SUM-CONNECTIVITY INDEX FOR TREES

In the paper proposing new index $\chi_\alpha(G)$ [18] the following result was deduced:

**Theorem 2.1.** Let $T$ be a tree with $n \geq 4$ vertices. If $\alpha > 0$, then:

$$2 \cdot 3^n + (n-3)4^n \leq \chi_\alpha(T) \leq (n-1)n^n$$

with left (right, respectively) equality if and only if $T = P_n$ ($T = K_{1,n-1}$, respectively). If $\alpha < 0$, then the above inequalities on $\chi_\alpha(T)$ are reversed, where the upper bound holds for $\alpha \geq 1 - \frac{\log 2}{\log(4/3)} \approx -1.4094$.

The maximum value for the general sum-connectivity indices of $n$-vertex trees and the corresponding extremal trees for $\alpha < \gamma_0$, where $\gamma_0 \approx -4.3586$ is the unique root of the equation $(4^x - 5^x)/(5^x - 6^x) = 3$ has been deduced in [6]:

**Theorem 2.2.** In the set of $n$-vertex trees $T$ with $n \geq 4$, if $\alpha < \gamma_0$ maximum of $\chi_\alpha(T)$ if reached if and only if $T$ consists of $(n-1)/2$ copies of $P_3$ having a common endvertex if $n$ is odd and of $(n-4)/2$ copies of $P_3$ and a copy of $P_4$ having a common endvertex if $n$ is even.

For every integers $n, p$ with $2 \leq p \leq n - 1$, let $S_{n,p}$ denote the tree with $p$ pendant vertices formed by attaching $p-1$ pendant vertices to an endvertex of the path $P_{n-p+1}$. In particular $S_{n,2} = P_n$ and $S_{n,n-1} = K_{1,n-1}$. Minimum value of $\chi_\alpha(T)$ for trees of given diameter and $-1 \leq \alpha < 0$ has been deduced in [11] using graph transformations and some parametric inequalities:

**Theorem 2.3.** For every $-1 \leq \alpha < 0$ in the set of trees $T$ having order $n \geq 3$ and diameter equal to $d$ ($2 \leq d \leq n - 1$), $\chi_\alpha(T)$ is minimum if and only if $T = S_{n,n-d+1}$.

An ordering of the trees $T$ having minimum $\chi_\alpha(T)$ was also obtained in [11]:

**Theorem 2.4.** For every $-1 \leq \alpha < 0$ there exists $n_0(\alpha) > 0$ such that for every $n \geq n_0(\alpha)$ the trees $T$ having smallest $\chi_\alpha(T)$ are $K_{1,n-1}, BS(n-3, 1), BS(n-4, 2), S_{n,n-3}$ and $BS(n-5, 3)$ (in this order). Also we have $n_0(-1) = 16$.

If we restrict ourselves to the set of trees with given order and number of pendant vertices, we have the following result [11]:

**Theorem 2.5.** Let $T$ be a tree with $n \geq 5$ vertices and $p$ pendant vertices, where $3 \leq p \leq n-2$ and $-1 \leq \alpha < 0$. Then $\chi_\alpha(T)$ is minimum if and only if $T = S_{n,p}$.

The following theorem characterizes trees $T$ of given order and independence number with maximum $\chi_\alpha(T)$ for $\alpha > 1$ [13]:

**Theorem 2.6.** Let $n \geq 2$, $n/2 \leq s \leq n-1$ and $T$ be a tree of order $n$ with independence number $s$. Then for every $\alpha > 1$, both $\chi_\alpha(T)$ and $R_\alpha(T)$ are maximum if and only if $T = S_{n,s}$, the spur graph.
A connected unicyclic graph of order \( n \) has a unique cycle and \( n \) edges; it can be obtained from a tree \( T \) of order \( n \) by adding a new edge between two nonadjacent vertices of \( T \). Similarly, a connected bicyclic graph has two linear independent cycles. It has \( n + 1 \) edges and can be deduced from a tree \( T \) of order \( n \) by adding two edges between two pairs of nonadjacent vertices of \( T \).

The first results about connected unicyclic graphs of given order having minimum general sum-connectivity index were deduced in [5]. In order to show these results let \( S_n(a, b, c) \) be the \( n \)-vertex graph obtained by attaching \( a - 2 \), \( b - 2 \) and \( c - 2 \) pendant vertices to the three vertices of a triangle, respectively, where \( a + b + c = n + 3 \) and \( a \geq b \geq c \geq 2 \).

**Theorem 3.1.** Among the set of \( n \)-vertex unicyclic graphs with \( n \geq 5 \), for \( \alpha > 0 \), cycle \( C_n \) is the unique graph with the minimum general sum-connectivity index and for \(-1 \leq \alpha < 0\), \( S_n(n - 1, 2, 2) \) and \( S_n(n - 2, 3, 2) \) are respectively the unique graphs with the minimum and the second minimum general sum-connectivity indices.

Note that \( S_n(n - 1, 2, 2) \) consists of a triangle and \( n - 3 \) pendant vertices incident to the same vertex of this triangle. This extremal graph has girth equal to 3. If \( G \) has girth \( k \geq 4 \) extremal graphs are given by the next theorem [12]:

**Theorem 3.2.** Let \( G \) be a connected unicyclic graph of order \( n \) and girth \( k \), where \( n \geq k \geq 4 \) and \(-1 \leq \alpha < 0\). Then \( \chi_\alpha(G) \geq \chi_\alpha(C_{k,n-k}) \), and equality holds if and only if \( G = C_{k,n-k} \), where \( C_{k,n-k} \) denotes the cycle \( C_k \) with \( n-k \) pendant edges attached to a single vertex of \( C_k \).

Let \( A(n, k) \) denote the set of unicyclic graphs of order \( n \) consisting of \( C_k \), \( n - k - 1 \) pendant edges incident to a vertex \( x \in V(C_k) \) and one pendant edge incident to a vertex \( y \in V(C_k) \), such that \( d(x, y) \geq 2 \). In the set of connected unicyclic graphs of order \( n \) and girth \( k \), where \( n \geq k + 2 \geq 6 \) and \(-1 \leq \alpha < 0\) the graphs having the second minimum general sum-connectivity index are graphs in \( A(n, k) \) [12].

A characterization of connected unicyclic graphs of order \( n \) having \( k \) pendant vertices and minimum general sum-connectivity index was done for \(-1 \leq \alpha < 0\) in [14]:

**Theorem 3.3.** Let \( G \) be a connected unicyclic graph of order \( n \geq 3 \) with \( k \) pendant vertices \((0 \leq k \leq n - 3)\). If \(-1 \leq \alpha < 0\) then \( \chi_\alpha(G) \geq f(n, k) = k(k+3)\alpha + 2(k+4)\alpha^2 + (n-k-2)\alpha^3 \). Equality holds if and only if \( G = C_{n-k,k} \), the graph consisting of \( C_{n-k} \) and \( k \) pendant edges incident to a unique vertex of this cycle.

Since function \( f(n, k) \) is increasing in \( k \), we get:

**Corollary 3.1.** If \(-1 \leq \alpha < 0\), in the class of unicyclic connected graphs \( G \) of order \( n \), \( \chi_\alpha(G) \) is minimum if and only if \( G = C_{3,n-3} \).
property also stated by Theorem 3.1. The extremal connected bicyclic graphs of order \( n \geq 1 \) were deduced in [9] and [10] as follows:

**Theorem 3.4.** The unique graph with the largest general sum-connectivity index for \( \alpha \geq 1 \) among all connected bicyclic graphs of order \( n \), consists of two triangles having a common edge and other \( n - 4 \) pendant edges incident to a vertex of degree three of this graph.

**Theorem 3.5.** The set of graphs which minimize the general sum-connectivity index in the set of the connected bicyclic graphs of order \( n \) for \( \alpha \geq 1 \) is \( A \cup B \), where \( A \) is the set of graphs consisting of two vertex disjoint cycles \( C_p \) and \( C_q \), joined by a path \( P_r \) and \( B \) the set of those graphs formed by two cycles \( C_{p+r} \) and \( C_{q+r} \), having in common a path \( P_r \), provided \( r \geq 2 \).

### 4. General Sum-Connectivity Index for \( K \)-Connected Graphs

First we consider the minimum \( \chi_\alpha(G) \) in the class of graphs \( G \) of order \( n \geq 3 \) and minimum degree \( \delta(G) \geq 2 \), when \( -1 \leq \alpha < \alpha_0 \), where \( \alpha_0 \approx -0.866995 \) is the unique root of the equation \( 4(4^x - 5^x) = 6^x \). A similar investigation was done when graphs \( G \) are triangle-free for \( -1 \leq \alpha < \beta_0 \), where \( \beta_0 = \frac{\ln 6}{\ln 4} \approx -0.81706 \). The proofs of the next two theorems use induction, several approximations of exponential functions by polynomials using Taylor formula and some properties of convex functions, like Jensen’s inequality [15].

**Theorem 4.1.** Let \( G \) be a graph of order \( n \geq 3 \) with \( \delta(G) \geq 2 \). If \( -1 \leq \alpha < \alpha_0 \) then \( \chi_\alpha(G) \geq f(n) = 2(n - 2)(n + 1)^\alpha + 2^\alpha(n - 1)^\alpha \). Equality holds if and only if \( G = K_2 + K_{n-2} \).

**Theorem 4.2.** Let \( -1 \leq \alpha < \beta_0 \) and \( G \) be a triangle-free graph of order \( n \geq 4 \) with \( \delta(G) \geq 2 \). Then \( \chi_\alpha(G) \geq g(n) = 2(n - 2)n^\alpha \) and equality is reached if and only if \( G = K_{2,n-2} \).

Since all 2-connected graphs \( G \) have \( \delta(G) \geq 2 \) and both \( K_2 + K_{n-2} \) and \( K_{2,n-2} \) are 2-connected, we deduce the following corollaries.

**Corollary 4.1.** If \( G \) is a 2-connected graph of order \( n \geq 3 \) and \( -1 \leq \alpha < \alpha_0 \), then \( \chi_\alpha(G) \geq f(n) \). The extremal graph is \( K_2 + K_{n-2} \).

**Corollary 4.2.** Let \( G \) be a triangle-free 2-connected graph of order \( n \geq 4 \) and \( -1 \leq \alpha < \beta_0 \). We have \( \chi_\alpha(G) \geq g(n) \) and equality holds if and only if \( G = K_{2,n-2} \).

We proposed the following conjecture [15]:

Let \( n, k \in \mathbb{N}, n \geq 4, k \leq n/2 \) and \( -1 \leq \alpha < \beta_0 \). Then for any triangle-free graph of order \( n \) with \( \delta(G) \geq k \geq 2 \) we have \( \chi_\alpha(G) \geq k(n - k)n^\alpha \) with equality if and only if \( G = K_{k,n-k} \).
This property is true for $\alpha = -1$ [2] and for $k = 2$ and $-1 \leq \alpha < \beta_0$ by Theorem 4.2. If the conjecture is true, then the property also holds for $k$-connected graphs. The following theorem shows extremal graph $G$ of order $n$ with $\kappa(G) = k$ which maximizes $\chi_\alpha(G)$ for $\alpha \geq 1$ [16].

**Theorem 4.3.** Let $G$ be an $n$-vertex graph, $n \geq 3$, with vertex connectivity $k$, $1 \leq k \leq n - 1$ and $\alpha \geq 1$. Then $0R_\alpha(G)$ and $\chi_\alpha(G)$ are maximum if and only if $G = K_k + (K_1 \cup K_{n-k-1})$.

Note that the graph $K_k + (K_1 \cup K_{n-k-1})$ is the graph of order $n$ obtained by joining by edges $k$ vertices of $K_{n-k-1}$ to a new vertex. Since for every graph $G$, $\kappa(G) \leq \kappa'(G)$ holds, we have [16]:

**Corollary 4.3.** Let $G$ be an $n$-vertex graph, $n \geq 3$, with edge connectivity $k$, $1 \leq k \leq n - 1$ and $\alpha \geq 1$. Then $0R_\alpha(G)$ and $\chi_\alpha(G)$ are maximum if and only if $G = K_k + (K_1 \cup K_{n-k-1})$.

For $\alpha > 0$ the 2-connected graph having minimum $\chi_\alpha(G)$ is the cycle $C_n$ [16]:

**Theorem 4.4.** Let $G$ be a 2-(connected or edge-connected) graph with $n \geq 3$ vertices. Then for $\alpha > 0$, $0R_\alpha(G)$ and $\chi_\alpha(G)$ are minimum if and only if $G = C_n$.

**References**


