COURANT ALGEBROID STRUCTURES ON
BANACH VECTOR BUNDLES
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Abstract The known three definitions of a Courant algebroid structure are extended to Banach vector bundles and their equivalence is discussed. Using a definition with three axioms we prove that the direct sum $E \oplus E^*$ of a Lie algebroid $E$ and its dual $E^*$, without any additional structure, has a Courant algebroid structure.

Keywords: Lie Algebroids, Courant Algebroids.

1. INTRODUCTION

The notion of Courant algebroid structure was introduced in [7], using five axioms. The authors, Zhang-Ju Liu, Alan Weinstein and Ping Xu searched for a correct definition of double for a Lie bialgebroid. An alternative definition was given in [8], again with five axioms. Later, it was shown in [9] that in both definitions two of the five axioms are consequences of the other three. For a short history of Courant algebroid structures we refer to [6]; for applications to Physics see [10] and for the foliated manifolds see [11].

In both definitions of a Courant algebroid structure the basic ingredients are the neutral metric and brackets of Courant type. The skew-symmetric bracket used in [7] is a generalization of that of T. Courant [4], while in the paper [5], a definition with three axioms involving its non skew-symmetric version is used. We shall reformulate the definition from [7] in one with three axioms and skew-symmetric bracket and we shall remark that it is equivalent to that from [5] if the brackets switch places.

Recently, the Lie algebroids, the Dirac structures and the Courant algebroid structures were considered and studied in the category of Banach vector bundles over Banach manifolds, [1], [2], [3].

In the paper [5] one proves that the big tangent bundle $TM \oplus T^*M$ of a smooth manifold $M$ with the neutral metric and the non skew-symmetric Courant bracket is a Courant algebroid. On the other hand, in [7] one shows that if one uses the skew-symmetric Courant bracket then $TM \oplus T^*M$ is a Courant algebroid only if $T^*M$ is endowed with a Lie algebroid structure with null anchor and null Courant bracket.
In this paper we show that the direct sum $E \oplus E^*$ has a Courant algebroid structure for any Lie algebroid $E$ and $E^*$ its dual, without any additional structure on $E^*$. To this aim we adapt the neutral metric and the non skew-symmetric bracket from [5] to any Lie algebroid.

The structure of the paper is as follows.

In Section 2 we recall without any proofs the results necessary for defining a non skew-symmetric bracket, similar to that from [5].

In Section 3 we discuss the various definitions of a Courant algebroid and prove that $E \oplus E^*$ is a Courant algebroid, without any additional structure on $E^*$.

Throughout the vector bundles will be Banach vector bundles over Banach manifolds.

2. PRELIMINARIES

Let $M$ be a smooth i.e. a $C^\infty$ Banach manifold modeled on a Banach space $M$ and let $\pi : E \to M$ be a Banach vector bundle whose type fiber is a Banach space $E$. The spaces $M$ and $E$ will be supposed reflexive Banach spaces. We denote by $\tau : TM \to M$ the tangent bundle of $M$.

Let $\mathcal{F}(M)$ denote the ring of smooth real functions on $M$, $\Gamma(E)$ the $\mathcal{F}(M)$-module of smooth sections in the vector bundle $(E, \pi, M)$ and $\mathcal{X}(M)$ the $\mathcal{F}(M)$-module of smooth sections in the tangent bundle of $M$ (vector fields on $M$).

Definition 2.1. Let $\pi : E \to M$ be a Banach vector bundle. We say that it has a Banach Lie algebroid structure if the following conditions hold good:

1 There is a morphism $\rho_E : E \to TM$ called anchor. Thus $(E, \rho_E)$ is an anchored vector bundle. The anchor $\rho_E$ induces a morphism $\Gamma(E) \to \mathcal{X}(M)$, denoted also by $\rho_E$.

2 There is a bracket $[,]_E$ defined on the sections of $E$, such that $(\Gamma(E), [,]_E)$ is a real Lie algebra.

3 The followings hold:
   
   (i) $\rho_E : (\Gamma(E), [,]_E) \to (\mathcal{X}(M), [,])$ is a Lie algebra homomorphism and 
   
   (ii) $[s_1, f s_2]_E = f[s_1, s_2]_E + \rho_E(s_1)(f)s_2$, for every $f \in \mathcal{F}(M)$ and $s_1, s_2 \in \Gamma(E)$.

Examples

1 Let $M$ be a smooth Banach manifold. The tangent bundle $\tau : TM \to M$ together with the anchor the identity map and the usual Lie bracket of vector fields on $M$ is a Lie algebroid.
2 For any submersion $\pi : F \to M$, the vertical bundle $VF$ over $F$ is an anchored Banach vector bundle. As the Lie bracket of two vertical vector fields is again a vertical vector field it follows that $(VF, i, [\cdot, \cdot]_{VF})$, where $i : VF \to TF$ is the inclusion map, is a Lie algebroid. This applies, in particular, to any vector bundle $\pi : E \to M$.

Let $L^q_s(E)$ be the vector bundle of alternating $q$-linear forms, $q = 1, 2, 3, \ldots$, on $E$. The fiber at each point is the space $L^q_s(E_x)$ consisting of all $q$-linear alternating continuous maps on the fiber $E_x$, $x \in M$. The sections $\Omega^q(E) := \Gamma(L^q_s(E))$ will be called differential forms of degree $q$. The set $\Omega^q(E)$ is an $\mathcal{F}(M)$-module. In particular, $\Omega^0(TM)$ will be denoted by $\Omega^0(M)$.

The exterior differential operator $d_E : \Omega^q(E) \to \Omega^{q+1}(E)$ is given by formula

$$
(d_E \omega)(s_0, \ldots, s_q) = \sum_{i=0}^{q} (-1)^i \rho_E(s_i) \omega(s_0, \ldots, \widehat{s_i}, \ldots, s_q) + \sum_{0 \leq i < j \leq q} (-1)^{i+j} \omega([s_i, s_j]_E, s_0, \ldots, \widehat{s_i}, \ldots, \widehat{s_j}, \ldots, s_q),
$$

(2.1)

where $s_0, s_1, \ldots, s_q \in \Gamma(E)$, and $\omega \in \Omega^q(E)$, for $q = 0, 1, 2, \ldots$. The set $\Omega^0(E)$ is defined as the set of functions $\{f \circ \pi \mid f \in \mathcal{F}(M)\}$. The hat over a symbol means that symbol under it must be deleted.

It is well known that $d_E$ has the usual properties of the exterior differential operator.

Let $s$ be a section of $E$. We define the interior product (contraction) with respect to $s$ as a $\mathcal{F}(M)$-linear map $i_s : \Omega^q(E) \to \Omega^{q-1}(E)$, given by formula

$$i_s(\omega)(s_1, s_2, \ldots, s_{q-1}) = \omega(s, s_1, s_2, \ldots, s_{q-1}),
$$

(2.2)

if $q > 1$ and $i_s f = 0$, for $f \in \Omega^0(E)$, where $s_1, \ldots, s_q \in \Gamma(E)$. We extend the contraction with respect to a given section of $\Gamma(E)$ to a linear map $i : \Gamma(E) \to L(\Omega^q(E), \Omega^{q-1}(E))$, given by $i(s)(\omega) = i_s(\omega)$, for $s \in \Gamma(E)$ and $\omega \in \Omega^q(E)$.

**Lemma 2.1.** The linear operator $i$ has the following properties:

1. $i_s \circ i_r + i_r \circ i_s = 0$,
2. $i^2_s = 0$,
3. $i_s \circ d_E \circ i_r + d_E \circ i_s \circ i_r = i_r \circ i_s \circ d_E + i_r \circ d_E \circ i_s + [i_s, i_r]_E$,

for any $s, r \in \Gamma(E)$.

**Definition 2.2.** Let $s$ be a section of $E$. We define the Lie derivative with respect to $s$, denoted by $L_s$, as follows
1. $L_s f = \rho_E (s) f$, for any $f \in \mathcal{F}(M)$,
2. $L_s r = [s, r]_E$, for any $r \in \Gamma (E)$,
3. $L_s \omega = (d_E \circ i_s + i_s \circ d_E) (\omega)$, for any $\omega \in \Omega^q (E)$, where $d_E$ is the exterior differential and $i_s$ is the interior product with respect to $s$.

For a $q$-form $\omega$, the Lie derivative $L_s \omega$ is explicitly given by

$$(L_s \omega) (s_1, ..., s_q) = \rho_E (s) \left( \omega (s_1, ..., s_q) \right)$$

$$- \sum_{k=1}^{q} \omega \left( s_1, ..., [s, s_k]_E, ..., s_q \right).$$

**Lemma 2.2.** Let $L_s : \Omega^q (E) \to \Omega^q (E)$ denote the Lie derivative with respect to $s \in \Gamma (E)$. The following properties hold good:

1. $d_E \circ L_s = L_s \circ d_E$,
2. $i_s \circ L_s = L_s \circ i_s$,
3. $L_{[s, r]_E} = L_s \circ L_r - L_r \circ L_s$,
4. $L_s \circ i_v - i_v \circ L_s = i_{(s, r)]_E}$.

### 3. COURANT ALGEBROIDS

In the paper [7], a Courant algebroid is defined as a finite dimensional vector bundle $\pi : E \to M$ endowed with a bilinear symmetric and nondegenerate map $(\cdot, \cdot)$, a skew-symmetric bracket $[,]$ and a morphism (anchor) $\rho_E : E \to TM$ such that the following conditions are satisfied:

(C1) $ [[e_1, e_2], e_3] + c. p. (\text{circular permutation}) = DT (e_1, e_2, e_3)$,

(C2) $\rho_E ([e_1, e_2]) = [\rho_E (e_1), \rho_E (e_2)]_TM$,

(C3) $[e_1, fe_2] = f [e_1, e_2] + \rho_E (e_1) f - \langle e_1, e_2 \rangle Df$,

(C4) $\rho_E \circ D = 0$, i.e. for any $f, g \in \mathcal{F} (M)$, $\langle Df, Dg \rangle = 0$,

(C5) $\rho (e_1) (\langle e_2, e_3 \rangle) = \langle [e_1, e_2] + D (\langle e_1, e_2 \rangle), e_3 \rangle + \langle e_2, [e_1, e_3] + D (\langle e_1, e_3 \rangle) \rangle$, where

$$T (e_1, e_2, e_3) = \frac{1}{3} \langle [e_1, e_2], e_3 \rangle + c. p., \quad (T)$$

and $D : C^\infty (M) \to \Gamma (E)$, is given by

$$\langle Df, e \rangle = \frac{1}{2} \rho_E (e) (f) \quad (D)$$

for any $e_1, e_2, e_3 \in \Gamma (E)$ and $f, g \in C^\infty (M)$. 


**Definition 3.1.** A Courant algebroid is a vector bundle $\pi: E \to M$ together with a bilinear, symmetric and nondegenerate map $h : \Gamma(E) \times \Gamma(E) \to \mathbb{R}$, and non skew-symmetric bilinear map $\circ : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$, and a bundle morphism $\rho_E : E \to TM$ such that

1. for any $e_1, e_2, e_3 \in \Gamma(E)$ the following identity holds
$$e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3),$$  

2. for any $e_1, e_2 \in \Gamma(E)$ the mapping $\circ$ satisfies
$$e_1 \circ e_2 + e_2 \circ e_1 = D(h(e_1, e_2)),$$  

where $D : C^\infty(M) \to \Gamma(E)$ is given by
$$h(Df, e) = (\rho_E (e)) f,$$  

for any $e \in \Gamma(E)$ and $f \in \mathcal{F}(M)$,

3. for any $e_1, e_2, e_3 \in \Gamma(E)$ the following identity holds:
$$(\rho_E (e_1))(h(e_2, e_3)) = h(e_1 \circ e_2, e_3) + h(e_2, e_1 \circ e_3).$$

We have

**Proposition 3.1.** In any Courant algebroid $(E, \pi, M, h, \circ, \rho_E)$,

(i) the map $D$ given by (3.2') is a derivation, i.e.,
$$D(fg) = fDg + gDf,$$  

for all $f, g \in \mathcal{F}(M)$.

(ii) the Leibnitz rule holds good, i.e.,
$$e_1 \circ (fe_2) = f(e_1 \circ e_2) + (\rho_E (e_1)) fe_2,$$  

for all $e_1, e_2 \in \Gamma(E)$ and for all $f \in \mathcal{F}(M)$.

(iii) the map $\rho_E$ induces an algebra morphism from $\Gamma(E)$ to $\Gamma(TM)$, i.e., it satisfies
$$\rho_E (e_1 \circ e_2) = [\rho_E (e_1), \rho_E (e_2)].$$

**Proof.** The fact that $D$ is a derivation follows from the nondegeneracy of $h$ and the properties of the action of vector fields on smooth functions, i.e.,
$$h(D(fg), e) = \rho_E (e)(fg) = f\rho_E (e)(g) + g\rho_E (e)(f)$$
$$= fh(Dg, e) + gh(Df, e) = h(fDg + gDf, e)$$  

(D.1.1)
In order to prove (3.4) we evaluate $\rho_E(e_1)(h(e_2, f e_2))$ in two ways. Firstly, by using (3.3) we obtain

$$\rho_E(e_1)(h(e_2, f e_2)) = f h(e_1 \circ e_2, e_2) + h(e_2, e_1 \circ (f e_2)).$$  \hspace{1cm} (3.4.1)

Secondly, by using the Leibnitz rule for vector fields we obtain

$$\rho_E(e_1)(h(e_2, f e_2)) = h(e_2, e_1 \circ (f e_2)),$$

for all $e_1, e_2 \in \Gamma(E)$ and for all $f \in \mathcal{F}(M)$ and (3.4) follows from (3.4.1), (3.4.2) and the nondegeneracy of $h$.

In order to prove (3.5) we evaluate $e_1 \circ (e_2 \circ (f e_3))$ in two ways as well, using (3.4) and (3.1).

Now we consider the skew-symmetric map $[,]$ given by

$$e_1 \circ e_2 = [e_1, e_2] + \frac{1}{2}D(h(e_1, e_2)).$$ \hspace{1cm} (3.6)

for all $e_1, e_2 \in \Gamma(E)$. Identifying $h$ with $\langle \cdot, \cdot \rangle$ and replacing (3.6) in (3.4) and (3.3), we find the axioms (C3) and (C5), respectively, of a Courant algebroid as given in [7]. Substituting (3.6) in (3.1) we obtain the axiom (C1) of the same definition, while axiom (C2) follows from (3.6) and axiom (C4).

These remarks suggest us to reformulate the definition from [7] of a Courant algebroid structure, with three axioms only, as follows.

**Definition 3.2.** A Courant algebroid is a Banach vector bundle $\pi : E \to M$ together with a bilinear, symmetric and nondegenerate map $h : E \times E \to \mathbb{R}$, a skew-symmetric bilinear map $[,] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and a bundle morphism $\rho_E : E \to TM$ such that

(i) for all $e_1, e_2, e_3 \in \Gamma(E)$ the following identity holds

$$[[e_1, e_2], e_3] + c.p. = DT(e_1, e_2, e_3),$$ \hspace{1cm} (3.7)

(ii) $\rho_E \circ D = 0$, where the $D$ is given by (3.2'),

(iii) for all $e_1, e_2, e_3 \in \Gamma(E)$ we have

$$\rho_E(e_1) h(e_2, e_3) = h([e_1, e_2] + \frac{1}{2}D(h(e_1, e_2)), e_3) + h(e_2, [e_1, e_3] + \frac{1}{2}D(h(e_1, e_3))).$$ \hspace{1cm} (3.8)

The remarks made before this definition together with the Proposition 3.1 shows that the following Proposition holds.

**Proposition 3.2.** Let $(E, \pi, M, h, \circ, \rho, D)$ be a Courant algebroid in the sense of Definition 3.1. Let $[,]$ be given as in (3.6). Then $(E, \pi, M, h, [,], \rho, D)$ is a Courant algebroid in the sense of Definition 3.2, and conversely.
Now we prove the main result of this paper.

**Theorem 3.1.** Let \( \pi : E \to M \) be a Banach Lie algebroid and \( E^* \) its dual, without any additional structure. Then the Banach vector bundle \( \pi \oplus \pi^* : E \oplus E^* \to M \) is a Courant algebroid.

**Proof.** We endow the Banach vector bundle \( \pi \oplus \pi^* : E \oplus E^* \to M \) with a bilinear symmetric nondegenerate map \( \langle , \rangle \) given by

\[
\langle (s, \alpha), (v, \beta) \rangle = i_* \beta + i_! \alpha,
\]

for any \( (s, \alpha), (v, \beta) \in \Gamma(E \oplus E^*) \), a map \([, ]\) given by

\[
[(s, \alpha), (v, \beta)] = ([s, v]_E, L_s \beta - i_! d_E \alpha),
\]

for any \( (s, \alpha), (v, \beta) \in \Gamma(E \oplus E^*) \), and a map \( f \to Df = (0, d_E f) \),

and prove that (3.1), (3.2) and (3.3) are satisfied.

In order to prove that (3.1) holds i.e.,

\[
[(s, \alpha), [(v, \beta), (w, \gamma)]] = [[(s, \alpha), (v, \beta)], (w, \gamma)] + [(v, \beta), [(s, \alpha), (w, \gamma)]]
\]

(3.1a)
we will evaluate the both sides of (3.1a) for any arbitrary sections \( e_1 = (s, \alpha), e_2 = (v, \beta), e_3 = (w, \gamma) \in \Gamma(E \oplus E^*) \) and compare them. Each one has two components.

Since \( (\Gamma, [\cdot, \cdot])_E \) is a real Lie algebra, by the Jacobi identity we have

\[
[s, [v, w]_E]_E + [v, [s, w]_E]_E + [w, [s, v]_E]_E = 0,
\]

(3.12)
for any \( s, v, w \in \Gamma(E) \). From the antisymmetry of \([\cdot, \cdot]_E \) we get

\[
[s, [v, w]_E]_E = [[s, v]_E, w]_E + [v, [s, w]_E]_E,
\]

(3.13)
Using (3.12) and (3.13), by a direct computation one obtains the equality of the first components of the both sides of (3.1a).

In order to prove that the equality between the second components of (3.1a) holds we need the following result.

**Lemma 3.1.** Let \( s, v \) and \( w \) be sections of the Lie algebroid \( (E, \rho_E) \) and \( \alpha, \beta \) and \( \gamma \) be 1-forms defined on \( E \). Then the following identity holds good:

\[
L_s (i_w (d_E \beta)) + i_{[v, w]_E} (d_E \alpha) = i_w (d_E (L_s \beta - i_! d_E \alpha))
\]

\[
+ L_v (i_w (d_E \alpha)) + i_{[s, w]_E} (d_E \beta).
\]

(3.14)

**Proof.** In order to prove the identity above we will compute both of its sides using the definition and properties of the Lie derivative. On one hand, we have
we obtain that the third axiom of Definition 3.1 holds good, therefore Courant algebroid, q.e.d.

for all \((s, \alpha), (\nu, \beta), (w, \gamma) \in \Gamma(E \oplus E^*)\).

The proof of (3.18) is straightforward. The third axiom of the Courant algebroid is proved in a similar way, by evaluating both sides of (3.4), i.e.,

\[
\rho(s)(\langle (v, \beta) , (z, \gamma) \rangle)_+ = \rho(s)(\beta(z) + \gamma(v)),
\]

\[
\langle (s, \alpha) , (v, \beta) , (w, \gamma) \rangle_+ = \langle [s, v]_E, L_s \beta - i_w d_E \alpha , (w, \gamma) \rangle_+.
\]

\[
(\langle v, \beta \rangle, \langle (s, \alpha), (w, \gamma) \rangle)_+ = \langle (v, \beta), (\langle (s, w)_E, L_s \gamma - i_w d_E \alpha \rangle)_+ = \beta(\langle [s, w]_E, L_s \gamma - i_w d_E \alpha \rangle (v)),
\]

for all \((s, \alpha), (v, \beta), (w, \gamma) \in \Gamma(E \oplus E^*)\). Replacing (3.19), (3.20) and (3.21) in (3.4), we obtain that the third axiom of Definiton 3.1 holds good, therefore \(E \oplus E^*\) is a Courant algebroid, q.e.d.

A remarkable example of Courant algebroid is \(T^{big} M = TM \oplus T^* M\), where \(M\) is a smooth Banach manifold.

We notice the following corollary of the Proposition 3.1 and the Theorem 3.1.

**Corollary 3.2.** Let \(E\) be a Banach Lie algebroid and \(E^*\) its dual. Consider the Courant algebroid \((E \oplus E^*, \pi, M, \langle \cdot, \cdot \rangle, [\cdot, \cdot], D)\). The following identities hold good:
(i) for all \( (s, \alpha), (v, \beta) \in \Gamma(E \oplus E^*) \) and all \( f \in \mathcal{F}(M) \) the Leibnitz rule holds, i.e,
\[
[(s, \alpha), f(v, \beta)] = f[(s, \alpha), (v, \beta)] + (\rho(s)(f))(v, \beta),
\]
(3.22)

(ii) The map \( \rho_{E \oplus E^*} \) induces a morphism of algebras, i.e.,
\[
\rho_{E \oplus E^*} \left( [(s, \alpha), (v, \beta)] \right) = \left[ \rho_E(s), \rho_E(v) \right]_{TM},
\]
(3.23)

for all \( (s, \alpha), (v, \beta) \in \Gamma(E \oplus E^*) \).

Proof. By the Theorem 3.1 we have that \((E \oplus E^*, \pi, M, \langle , \rangle, [ , ], D)\) is a Courant algebroid. But in any Courant algebroid the identities (3.4) and (3.5) from the Proposition 3.1 hold good. For the Courant algebroid \((E \oplus E^*, \pi, M, \langle , \rangle, [ , ], D)\), these reduce to (3.22) and (3.23), respectively, q.e.d.

We remark that the Courant algebroid structure of the Lie bialgebroid \( E \oplus E^* \) provided by the Theorem 3.1 does not coincide with that discovered by Liu, Weinstein and Xu in [7]. They differ by brackets and anchors and coincide if the bracket and anchor of \( E^* \) are null. But in this case \( E \oplus E^* \) is no more a Lie bialgebroid.

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References
MODIFIED HAN ALGORITHM FOR
MARITIME CONTAINERS
TRANSPORTATION PROBLEM

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Abstract The most important particularity of the maritime container transportation is the fact that
the containers have different capacities, so the transportation cost of a unit of cargo
is not the same for each container for a particular route. If we consider the cost of
transport of a Twenty-foot Equivalent Unit (TEU, for short) between the supplier to the
warehouse, the difference consists in that the costs are for TEU and not for each con-
tainer. The mathematical model of a real world context will provide us an inconsistent
transportation problem. In this respect, we reformulate the problem as an inconsistent
(incompatible) system of linear inequalities, for which we propose a modified version
of Han’s iterative algorithm. Numerical experiments and comparisons with the classical
Simplex algorithm are presented.

Keywords: inconsistent linear inequalities; least squares solutions; Han-type algorithms; maritime
containers transportation problem.

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1. INTRODUCTION

The classical transportation problem involves sources \((S_i)_{i \in \{1, \ldots, n\}},\) where supplies
\((s_i)_{i = 1, \ldots, n}\) of some goods are available, and destinations \((D_j)_{j \in \{1, \ldots, m\}},\) where
some demands \((d_j)_{j = 1, \ldots, m}\) are requested (see for details [7]). The costs of shipping
\((c_{ij})_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}}\) for the transportation of one unit from source \(S_i\) to destination \(D_j\)
become the entries of the \(C : n \times m\) cost matrix (see Table 1.1).

If we denote by \(x_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\) the number of units transported from
source \(S_i\) to destination \(D_j,\) we get the following mathematical model of the (classi-
cal) transportation problem:
Table 1: The classical transportation problem

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>...</th>
<th>$D_m$</th>
<th>Supply(s)</th>
</tr>
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<td>$c_{12}$</td>
<td>$c_{13}$</td>
<td>...</td>
<td>$c_{1m}$</td>
<td>$s_1$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$c_{21}$</td>
<td>$c_{22}$</td>
<td>$c_{23}$</td>
<td>...</td>
<td>$c_{2m}$</td>
<td>$s_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$S_n$</td>
<td>$c_{n1}$</td>
<td>$c_{n2}$</td>
<td>$c_{n3}$</td>
<td>...</td>
<td>$c_{nm}$</td>
<td>$s_n$</td>
</tr>
<tr>
<td>Demand(d)</td>
<td>$d_1$</td>
<td>$d_2$</td>
<td>$d_3$</td>
<td>...</td>
<td>$d_m$</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{min} & \quad \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{i=1}^{n} x_{ij} \geq d_j, j = 1, \ldots, m \quad (\ast) \\
& \quad \sum_{j=1}^{m} x_{ij} = s_i, i = 1, \ldots, n \quad (\ast\ast) \\
& \quad x_{ij} \geq 0, i = 1, \ldots, n, j = 1, \ldots, m,
\end{align*}
\]

Remark 1. Some arguments for the relations (\ast)-(\ast\ast) are as follows:
- for (\ast): at each destination, the demand has to be "at least" satisfied (e.g. the construction of a building will not be started if we do not have at least a minimal amount of materials)
- for (\ast\ast): all available units must be supplied.

The problem is called balanced if the total supply equals the total demand (i.e. $\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} d_j$), and unbalanced otherwise. In the balanced case or the unbalanced one with $\sum_{i=1}^{n} s_i \geq \sum_{j=1}^{m} d_j$, the linear program (1) is consistent and well known methods (including Simplex-type algorithms) are available (see [5, 9, 10]). We will consider in this paper the unbalanced case

\[
\sum_{i=1}^{n} s_i < \sum_{j=1}^{m} d_j \quad (2)
\]

for which the linear program (1) becomes inconsistent (i.e. the set of feasible solutions is empty).
2. MARITIME CONTAINER TRANSPORTATION PROBLEM

2.1. TYPES OF CONTAINERS

The maritime container transportation is a transportation problem subject to the following additional hypothesis (see e.g. [8]):

(M1) the unit of cargo has different capacities, so the cost of a unit of transport is different for a particular route. The most common unit used in cargo transportation is TEU (Twenty-foot Equivalent Unit), corresponding to a 20-foot-long (6.1 m) intermodal container. Other types of containers (expressed in TEU) are shown in Table 2.1.

(M2) the destinations become in this case warehouses with specific dimensions; hence, the number of containers transported to a given warehouse will be restricted by these dimensions.

Details for (M2): we will assume that all the warehouses are rectangular buildings, with length (L), width (W) and height (H). Thus, the containers will be stored on superposed rows in each warehouse. In the next section we will present on a model problem, a mathematical model for the problem of containers’ storage.

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Length</th>
<th>Width</th>
<th>Height</th>
<th>Volume</th>
<th>TEU</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type(1)</td>
<td>20ft(6.1m)</td>
<td>8 ft</td>
<td>8 ft 6 in(2.59m)</td>
<td>1,360 cu ft (38.5 m³)</td>
<td>1</td>
</tr>
<tr>
<td>Type(2)</td>
<td>40 ft(12.2m)</td>
<td>8 ft</td>
<td>8 ft 6 in(2.59m)</td>
<td>2,720 cu ft (77 m³)</td>
<td>2</td>
</tr>
<tr>
<td>Type(3)</td>
<td>45 ft(13.7m)</td>
<td>8 ft</td>
<td>8 ft 6 in(2.59m)</td>
<td>3,060 cu ft (86.6 m³)</td>
<td>2 or 2.25</td>
</tr>
<tr>
<td>Type(4)</td>
<td>48 ft(14.6m)</td>
<td>8 ft</td>
<td>8 ft 6 in(2.59m)</td>
<td>3,264 cu ft (92.4 m³)</td>
<td>2.4</td>
</tr>
<tr>
<td>Type(5)</td>
<td>53 ft(16.2m)</td>
<td>8 ft</td>
<td>8 ft 6 in(2.59m)</td>
<td>3,604 cu ft (102.1 m³)</td>
<td>2.65</td>
</tr>
<tr>
<td>Type(6) (High cube)</td>
<td>20 ft(6.1m)</td>
<td>8 ft</td>
<td>9 ft 6 in(2.90 m)</td>
<td>1,520 cu ft (43 m³)</td>
<td>1</td>
</tr>
<tr>
<td>Type(7) (Half height)</td>
<td>20 ft(6.1m)</td>
<td>8 ft</td>
<td>4 ft 3 in(1.30 m)</td>
<td>680 cu ft (19.3 m³)</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Types of containers

2.2. A MODEL PROBLEM

Since the standard width and height of the containers used in practice are 2.44 and 2.59, respectively (see e.g. [8, 10]), and taking into consideration the handling procedures in the warehouses, we will use for the number of rows \( R(j) \) and columns \( C(j) \) of containers stored in a warehouse the formulas

\[
R(j) = \left\lfloor \frac{W_j}{3} \right\rfloor, \quad C(j) = \left\lfloor \frac{H_j}{3} \right\rfloor, \quad (3)
\]

where \( W_j \) and \( H_j \) are the width and height of the warehouse \( D_j \), \( j = 1, \ldots, m \), and \( \lfloor z \rfloor \) denotes the integer part of the real number \( z \). Then, for each warehouse denoted by \( D_j \) we can compute a specific parameter called Total_TEU_Length \( (T(j)) \) given by

\[
T(j) = L(j) \cdot C(j) \cdot R(j) \quad (4)
\]

where \( L(j) \) is the length of the warehouse \( D_j \), and \( R(j), C(j) \) are computed as in (3). More clearly - \( T(j) \) represents the maximum length of a series of containers that can be stored in \( D_j \), expressed in TEU, assuming that they were placed in a straight line.

Let us suppose that there exist 7 sources of containers \( S_1, \ldots, S_7 \), each source \( S_i \) providing only containers of Type \( (i) \), \( i = 1, \ldots, 7 \) (see Table 2.1). We will denote by \( x_{ij} \) the number of containers of Type \( (i) \) (i.e. from source \( S_i \)) which will be transported to the warehouse \( D_j \). Let us suppose that the dimensions of the warehouses \( D_1, \ldots, D_7 \) are those from Table 2.3 below.

Hence, we can compute \( R(j) \) and \( C(j) \) according to (3) and get the values from Table 3. Then, according to the specific lengths of the containers from each source \( S_1, \ldots, S_7 \) (see column 2 on Table 2.1), the restrictions (inequalities) imposed by the dimensions of each warehouse can be written as follows

\[
\sum_{i=1}^{7} l_i x_{ij} \leq T(j), \quad j = 1, \ldots, 7, \quad (5)
\]

where \( l = (6.1, 12.2, 13.7, 14.6, 16.2, 6.1, 6.1) \) is a vector containing the lengths in meters from the second column of Table 2.1.
To these inequalities we must add the supply/demand restrictions from (1). For this, we will consider the unbalanced and inconsistent transport problem described in Table 2.2 for which the inequalities from (1) are the following.

**Table 5: The unbalanced transportation problem**

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>$D_6$</th>
<th>$D_7$</th>
<th>Supply(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>12</td>
<td>20</td>
<td>5</td>
<td>9</td>
<td>1050</td>
</tr>
<tr>
<td>$S_2$</td>
<td>7</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td>4</td>
<td>350</td>
</tr>
<tr>
<td>$S_3$</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>6</td>
<td>12</td>
<td>3</td>
<td>12</td>
<td>470</td>
</tr>
<tr>
<td>$S_4$</td>
<td>4</td>
<td>5</td>
<td>14</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>600</td>
</tr>
<tr>
<td>$S_5$</td>
<td>8</td>
<td>2</td>
<td>12</td>
<td>9</td>
<td>8</td>
<td>4</td>
<td>2</td>
<td>600</td>
</tr>
<tr>
<td>$S_6$</td>
<td>6</td>
<td>1</td>
<td>8</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>480</td>
</tr>
<tr>
<td>$S_7$</td>
<td>9</td>
<td>10</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>450</td>
</tr>
<tr>
<td>Demand(d)</td>
<td>455</td>
<td>320</td>
<td>540</td>
<td>460</td>
<td>760</td>
<td>830</td>
<td>780</td>
<td></td>
</tr>
</tbody>
</table>

\[
\sum_{i=1}^{7} x_{ij} \geq d_j, \quad j = 1, \ldots, 7, \quad (6)
\]

\[
\sum_{j=1}^{7} x_{ij} = s_i, \quad i = 1, \ldots, 7. \quad (7)
\]

But, in our maritime container transportation model problem $x_{ij}$ represents the number of containers of $Type(i)$ transported to the warehouse $D_j$, and not the numbers of TEU’s. Hence, we have to introduce in the relations (6)-(7) the weights.
$T_w = (1, 2, 2.25, 2.4, 2.65, 1, 1)$ from the last column of Table 2.1 and we obtain

$$\sum_{i=1}^{7} T_{w_i} x_{ij} \geq d_j, \ j = 1, \ldots, 7,$$  \hfill (8)

$$T_{w_i} \sum_{j=1}^{7} x_{ij} = s_i, \ i = 1, \ldots, 7.$$  \hfill (9)

Likewise, we have to remodel the matrix of shipping cost $C$ from Table 2.2, by multiplying its entries with the corresponding weights $T_{w_i}, i = 1, \ldots, 7$ from the last column of Table 2.1, and we get the values from Table 6.

Table 6: Matrix $C$ of shipping costs for containers transportation problem

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
<th>12</th>
<th>20</th>
<th>5</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>2</td>
<td>10</td>
<td>6</td>
<td>12</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>11.25</td>
<td>9</td>
<td>15.75</td>
<td>13.5</td>
<td>11.25</td>
<td>27</td>
<td>6.75</td>
</tr>
<tr>
<td>9.6</td>
<td>12</td>
<td>33.6</td>
<td>24</td>
<td>21.6</td>
<td>19.2</td>
<td>16.8</td>
</tr>
<tr>
<td>21.2</td>
<td>5.3</td>
<td>31.8</td>
<td>23.85</td>
<td>21.2</td>
<td>10.6</td>
<td>5.3</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>8</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>6</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

3. NUMERICAL EXPERIMENTS

We will present in this section some numerical experiments on inconsistent transportation problems as those described in section 2. They are performed with the Matlab R2010a linprog implementation of Simplex solver and the Modified Han algorithm proposed by us in [2]. In this respect we will prove in what follows a result mentioned (without proof) by Han in his original paper [4]. This result gives us the possibility to express in an equivalent way the primal-dual pair of linear programs

$$\min \langle c, y \rangle \text{ subject to } By \geq d, \ y \geq 0,$$  \hfill (10)

$$\max \langle d, u \rangle \text{ subject to } B^T u \leq c, \ u \geq 0,$$  \hfill (11)

$B: m \times n, c, y \in \mathbb{R}^n, d, u \in \mathbb{R}^m$, as a linear system of inequalities ($\langle \cdot, \cdot \rangle$, $\| \cdot \|$ denote the Euclidean scalar product and norm, respectively). We will denote by $\mathcal{P}$, $\mathcal{D}$ the set of feasible solutions of the primal (10) and dual (11) problem, respectively.
Proposition 1. Let us suppose that both problems (10)-(11) have feasible solutions, i.e. $\mathcal{P} \neq \emptyset$, $\mathcal{D} \neq \emptyset$. Then the following assumptions are equivalent:

(i) $\hat{y} \in \mathcal{P}$, $\hat{u} \in \mathcal{D}$ are optimal solutions for problems $\mathcal{P}$ and $\mathcal{D}$, respectively.

(ii) the vector $x = [\hat{y}^T, \hat{u}^T]^T \in \mathbb{R}^{m+n}$ is a solution of the system of linear inequalities

$$Ax \leq b,$$

where

$$A = \begin{bmatrix} c^T & -d^T \\ -B & 0 \\ 0 & B^T \\ -I & 0 \\ 0 & -I \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -d \\ c \\ 0 \\ 0 \end{bmatrix}$$

Proof. (i) $\Rightarrow$ (ii) If $\hat{y}, \hat{u}$ are optimal solutions for (10) and (11), respectively, then $\hat{y} \in \mathcal{P}$, $\hat{u} \in \mathcal{D}$ and (see [1], Corollary 5.1, page 182)

$$\langle c, \hat{y} \rangle = \langle d, \hat{u} \rangle.$$  

All these give us the fact that $x = [\hat{y}^T, \hat{u}^T]^T$ satisfies (12) - (13).

(ii) $\Rightarrow$ (i) If $x = [\hat{y}^T, \hat{u}^T]^T$ is a solution of (12) - (13), by writing in details the inequality $Ax - b \leq 0$ we obtain that $\hat{y} \in \mathcal{P}$, $\hat{u} \in \mathcal{D}$ and

$$\langle c, \hat{y} \rangle - \langle d, \hat{u} \rangle \leq 0.$$  

But, because $\hat{y}^T, \hat{u}^T$ are feasible solutions, from [1], Proposition 5.1, page 180 it results $\langle c, \hat{y} \rangle \geq \langle d, \hat{u} \rangle$, which together with (15) gives us (14). Then, according to Proposition 5.2, page 181 in [1] $\hat{y}^T, \hat{u}^T$ are optimal solutions for (10) and (11), respectively, and the proof is complete.

Hence, from the above result it holds that solving a pair of feasible primal-dual linear programs is equivalent with solving a certain system of linear inequalities. According to Theorem 5.1, page 181 in [1], there are two more possible cases that can occur beside the feasible one: one of the problem has feasible solutions and the other does not, and both problems do not have feasible solutions. In these cases, as Han himself mentioned in [4], the system (12) - (13) provides a kind of least squares solution for one or both linear programs, respectively. Such a situation will be considered in the rest of the section.

According to the considerations and constructions in section 2 above, we will analyse in our numerical experiments the following two problems:

**Problem P1** - the unbalanced inconsistent problem (1), with the costs $c_{ij}$ from Table 2.2, and the restrictions $(\ast) - (\ast\ast)$ corresponding to the relations (6)-(7).

**Problem P2** - the inconsistent maritime container transportation problem of the form (1), with the costs $\hat{c}_{ij}$ from Table 6, the restrictions $(\ast)$ corresponding to the relations...
In this respect, we first renumber the unknowns as

\[ x_{ij} \rightarrow y_l, \quad i \in \{1, \ldots, 7\}, \quad j \in \{1, \ldots, 7\}, \quad l \in \{1, \ldots, 49\} \]  

(16)

Let \( c, \hat{c} \in \mathbb{R}^{49} \) be the cost vectors of the above problems, \( B_1, B_2 \) the \( 7 \times 49 \) matrices corresponding to the 7 inequalities from (6) and the 7 equalities from (7), respectively, and \( \hat{B}_0, \hat{B}_1, \hat{B}_2 \) the \( 7 \times 49 \) matrices corresponding to the 7 inequalities in (5), the 7 inequalities in (8) and the 7 equalities in (9), respectively, (all of them constructed according to the renumbering (16)). Moreover, let \( T, d, s \in \mathbb{R}^7 \) be defined by (see (4) and Table 2.2):

\[
T = (8125, 3120, 3828, 8550, 8640, 6825, 12240)^T,
\]

\[
d = (455, 320, 540, 460, 760, 830, 780)^T,
\]

\[
s = (1050, 350, 470, 600, 600, 480, 450)^T.
\]

(17)

Then, the above problems \( P_1, P_2 \) can be written as follows

**Problem P1**

\[
\begin{align*}
\min & \langle c, y \rangle \\
\text{s.t.} & \quad B_1 y \geq d, \quad B_2 y = s, \quad y \geq 0.
\end{align*}
\]

(18)

**Problem P2**

\[
\begin{align*}
\min & \langle \hat{c}, y \rangle \\
\text{s.t.} & \quad \hat{B}_0 y \leq T, \quad \hat{B}_1 y \geq d, \quad \hat{B}_2 y = s, \quad y \geq 0.
\end{align*}
\]

(19)

If we define the matrices \( B : 21 \times 49, \hat{B} : 28 \times 49 \) by

\[
B = \begin{bmatrix} B_1^T & B_2^T \end{bmatrix}^T, \quad \hat{B} = \begin{bmatrix} -\hat{B}_0^T & \hat{B}_1^T & \hat{B}_2^T \end{bmatrix}^T
\]

(20)

and the vectors \( d \in \mathbb{R}^{21}, \hat{d} \in \mathbb{R}^{28} \) by

\[
d = [d^T \ s^T - s^T], \quad \hat{d} = [-T^T d^T \ s^T - s^T]
\]

(21)

the problems \( P_1 \) (18), and \( P_2 \) (19) can be written as

\[
\begin{align*}
\min & \langle c, y \rangle \quad \text{s.t.} \quad By \geq d, \quad y \geq 0,
\end{align*}
\]

(22)

\[
\begin{align*}
\min & \langle \hat{c}, y \rangle \quad \text{s.t.} \quad \hat{B} y \geq \hat{d}, \quad y \geq 0,
\end{align*}
\]

(23)

with the corresponding dual problems given by (see e.g. [9])

\[
\begin{align*}
\max & \langle d, u \rangle \quad \text{s.t.} \quad B^T u \leq c, \quad u \geq 0,
\end{align*}
\]

(24)

\[
\begin{align*}
\max & \langle \hat{d}, u \rangle \quad \text{s.t.} \quad \hat{B}^T u \leq \hat{c}, \quad u \geq 0,
\end{align*}
\]

(25)

respectively.

According to the discussion from the beginning of this section, solving the pair of dual problems (22)-(24) and (23)-(25) is equivalent with solving the systems of inequalities \( Ax \leq b \) and \( \hat{A} \hat{x} \leq \hat{b} \), respectively, where \( x = [y^T, u^T]^T \in \mathbb{R}^{49} \times \mathbb{R}^{21} \) and
The expressions from (28) are related to some theoretical properties of \(y\) can be found in [3].

Our transportation problems \(\textbf{P}_1\) and \(\textbf{P}_2\), together with their duals, are inconsistent, so will be the systems of linear inequalities \(Ax \leq b\) and \(\hat{A}\hat{x} \leq \hat{b}\). The problems were solved with \textit{linprog} Matlab implementation of Simplex algorithm, whereas the two associated systems with Modified Han (MH) algorithm presented below (see for details [2]):

Let \(\text{ALG}\) be an iterative algorithm which approximates the minimal norm solution of a linear least squares problem of the form

\[
\| Bx - c \| = \text{min!}
\]

where \(B\) is an arbitrary rectangular matrix and \(c\) an appropriate vector.

**Algorithm MH.** Let \(x^0 \in \mathbb{R}^n\) be arbitrary fixed; for \(k = 0, 1, \ldots\) do:

**Step 1.** Find \(I_k = I(x^k)\) and compute an approximation \(d^{k,j} \in \mathbb{R}^n\) of the minimal norm solution of the linear equalities least squares problem

\[
\| A_{I_k}d - (b_{I_k} - A_{I_k}x^k) \| = \text{min!}
\]

by performing \(j \geq 1\) iterations of the algorithm \(\text{ALG}\), with 0 as initial approximation on (27).

**Step 2.** Compute \(\lambda^{k,j} \in \mathbb{R}\) as the smallest minimizer of

\[
\theta(\lambda) = f(x^k + \lambda d^{k,j}), \lambda \in \mathbb{R}.
\]

**Step 3.** Set \(x^{k+1} = x^k + \lambda^{k,j}d^{k,j}\).

As \(\text{ALG}\) in Step 1 of the algorithm MH we used the Kaczmarz Extended (KE) algorithm from [6].

In our computations implemented in Matlab R2010a, all runs with respect to MH algorithm are started with the initial approximations \(x_0 = (y_0^T, 0)^T\), \(\hat{x}_0 = (\hat{y}_0^T, 0)^T\), with \(y_0 \geq 0, \hat{y}_0 \geq 0\), and are terminated if at the current iterations \(x^k, \hat{x}_k\) satisfy

\[
\| \hat{A}^T(Ax^k - b)_+ \| \leq 10^{-14}, \quad \| \hat{A}^T(\hat{A}\hat{x}_k - \hat{b})_+ \| \leq 10^{-14},
\]

where \(z \in \mathbb{R}^n, z_+ \in \mathbb{R}^n\) denote the vector with components \((z_+)_i = z_i\), if \(z_i \geq 0\) and \((z_+)_i = 0\), if \(z_i < 0\). Results are presented in Tables 7 and 8.

**Remark 2.** The expressions from (28) are related to some theoretical properties of original Han’s algorithm from [4] when solving a system of linear inequalities of the form (12). More clear, Han proves that \(x^* \in \mathbb{R}^n\) is a solution of (12) if and only if it satisfies the normal equation in the inequality case \(A^T(Ax - b)_+ = 0\) (some details can be found in [3]).

where * denotes that the Simplex algorithm failed to solve the problem, returning instead a result that minimizes the worst case constraint violation (see [9]).

Let \(w = Fy - f \in \mathbb{R}^{14}\) where \(F = [\hat{B}_1^T \hat{B}_2^T]^T\), \(f = [d_1^T d_2^T]^T\) and \(\hat{w} = \hat{F}\hat{y} - \hat{f} \in \mathbb{R}^{21}\) where \(\hat{F} = [\hat{B}_0^T \hat{B}_1^T \hat{B}_2^T]^T\), \(\hat{f} = [\hat{T}^T d_1^T s_1^T]^T\) be the vectors of the unmet inequalities
<table>
<thead>
<tr>
<th>Algorithm</th>
<th>cost</th>
<th>$| (Ax - b)_+ |$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MH</td>
<td>15336</td>
<td>38.7529</td>
</tr>
<tr>
<td>Simplex</td>
<td>31235*</td>
<td>145.0241</td>
</tr>
</tbody>
</table>

Table 7: Results for problem P1

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>cost</th>
<th>$| (\hat{A}x - \hat{b})_+ |$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MH</td>
<td>15336</td>
<td>38.7529</td>
</tr>
<tr>
<td>Simplex</td>
<td>29070*</td>
<td>145.0626</td>
</tr>
</tbody>
</table>

Table 8: Results for problem P2

<table>
<thead>
<tr>
<th>$w$</th>
<th>MH</th>
<th>Simplex</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$w_2$</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$w_3$</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$w_4$</td>
<td>-10</td>
<td>-145</td>
</tr>
<tr>
<td>$w_5$</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$w_6$</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$w_7$</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>$w_8$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$w_9$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$w_{10}$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$w_{11}$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$w_{12}$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$w_{13}$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$w_{14}$</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 9: The values of unmet inequalities and equalities for problem P1
Table 10: The values of unmet inequalities and equalities for problem P2

<table>
<thead>
<tr>
<th>( \hat{w}_i )</th>
<th>MH</th>
<th>Simplex</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{w}_1 )</td>
<td>-5420</td>
<td>-5349</td>
</tr>
<tr>
<td>( \hat{w}_2 )</td>
<td>-1233</td>
<td>-1168</td>
</tr>
<tr>
<td>( \hat{w}_3 )</td>
<td>-597</td>
<td>-534</td>
</tr>
<tr>
<td>( \hat{w}_4 )</td>
<td>-5808</td>
<td>-5744</td>
</tr>
<tr>
<td>( \hat{w}_5 )</td>
<td>-4069</td>
<td>-4006</td>
</tr>
<tr>
<td>( \hat{w}_6 )</td>
<td>-1825</td>
<td>-1774</td>
</tr>
<tr>
<td>( \hat{w}_7 )</td>
<td>-5498</td>
<td>-6319</td>
</tr>
<tr>
<td>( \hat{w}_8 )</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_9 )</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{10} )</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{11} )</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{12} )</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{13} )</td>
<td>-10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{14} )</td>
<td>-10</td>
<td>-145</td>
</tr>
<tr>
<td>( \hat{w}_{15} )</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{16} )</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{17} )</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{18} )</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{19} )</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{20} )</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{w}_{21} )</td>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

and equalities (6)-(7) and (5), (8)-(9), respectively. The values of the components of vectors \( w \) and \( \hat{w} \) for Simplex and MH algorithms are presented in Tables 9 and 10. Tables 11 and 12 indicate the solutions obtained for the inconsistent transportation problem P1. We observe that MH algorithm solution is more reliable (for a practical view point).

4. CONCLUSIONS

In this paper we first considered an inconsistent version of the classical transportation problem (Problem P1). Based on its transportation (inconsistent) assumption, we derive the maritime container transportation problem (Problem P2). We propose
Table 11: The values $x_{ij}, i = 1, \ldots, 7, j = 1, \ldots, 7$ for the solution of problem $P_1$ with MH algorithm.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>144</td>
<td>530</td>
<td>0</td>
<td>0</td>
<td>387</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>360</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>89</td>
<td>232</td>
<td>0</td>
<td>159</td>
</tr>
<tr>
<td>4</td>
<td>445</td>
<td>166</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>610</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>490</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>28</td>
<td>433</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 12: The values $x_{ij}, i = 1, \ldots, 7, j = 1, \ldots, 7$ for the solution of problem $P_1$ with Simplex algorithm.

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>415</td>
<td>635</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>350</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>405</td>
<td>65</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>245</td>
<td>355</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>385</td>
<td>215</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>320</td>
<td>155</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>450</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

As a further step in our research, we are interested in eliminating the assumption (see Section 2.2) constraining each source $S_i$ to provide only one category of containers ($Type(i)$). Work is in project on this subject.

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References


ON THE EXISTENCE OF BERGE EQUILIBRIUM WITH PSEUDOCONTINUOUS PAYOFFS

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Abstract
In this paper, we generalize the existence results of Berge equilibrium in [16], [13], [1], [2], [9] and [15] in the case where the payoff function of each player is pseudocontinuous (see [14]). Moreover, we study essential games (see [10] and [20]).

Keywords: Berge equilibrium, essential games, pseudocontinuity, Nash equilibrium.

1. INTRODUCTION

The concept of Berge equilibrium goes back to the book of Berge [5] and was later formalized in [21] for differential games. For a non cooperative game with finite number of persons, this equilibrium means that if each person plays his strategy at a Berge equilibrium, then he obtains the maximum payoff if all the remaining players play their strategy in the Berge equilibrium. It is worth noticing that the Berge equilibrium is totally different from the Nash equilibrium since the Nash equilibrium is stable with respect the deviation of any unique player. For the concept of Nash equilibrium, we refer the reader to the paper of [17]. The existence of Berge equilibrium has been studied in [16], [13], [1, 2] and [9]. More recently, [15] have established the existence of Berge equilibrium without using Nash equilibrium. Previously mentioned works, the authors have assumed that payoffs of persons are continuous. However, many games as the oligopolies defined in [6] and [12] have discontinuous payoffs. Several authors have studied the existence of Nash equilibrium where payoffs are not necessarily continuous. Let us quote for example, [14]. In that paper, the authors have proved the existence of Nash equilibrium with pseudocontinuous payoffs. In this paper, we prove the existence and the essential stability [20] and [10] of the Berge equilibrium with pseudocontinuous payoffs.

This paper is organized as follows: In section 2, we define the Berge equilibrium. In section 3, we give definitions of pseudocontinuous functions and better reply secure games. Moreover, we prove the existence of Berge equilibrium. In section 4, we study the essential stability of Berge equilibrium (called essential games) for two models. In the first model, we consider games parametrized by payoff profiles. In the second model, games are parametrized by payoff profiles and strategies sets.
2. DEFINITION AND EXISTENCE OF BERGE EQUILIBRIUM

2.1. BERGE EQUILIBRIUM OF NONCOOPERATIVE GAME

Let us consider the following noncooperative game in normal form:

\[ G = (I, (X_i, u_i)_{i \in I}) \]

where \( I = \{1, ..., n\} \) is a finite set of players, \( X_i \) is a set of strategies of player \( i \), \( X = \prod_{i=1}^{n} X_i \) is a set of issues (joint strategy) of the game \( G \) and \( u_i : X \to \mathbb{R} \) is a payoff function of the player \( i \). For each player \( i \), we let \( I \setminus \{i\} = \{1, ..., i-1, i+1, ..., n\} \) and we denote \( X_{-i} = \prod_{j \neq i} X_j = \prod_{j \in I \setminus \{i\}} X_j \) and if \( x \in X = \prod_{i=1}^{n} X_i \), then \( x_{-i} = (x_j)_{j \neq i} \in X_{-i} \). Choosing a strategy \( x_i \in X_i \), the aim of each player in the game \( G \) is to maximize his payoff function. Recall that \( \bar{x} \in X \) is a Nash equilibrium of the game \( G \) if for every \( i \in I \), for all \( x_i \in X_i, u_i(\bar{x}) \geq u_i(x_i, x_{-i}) \). The following definition is due to ([5]).

Definition 2.1. A Berge equilibrium of the game \( G \) is an \( n \)-tuple of strategies \( \bar{x} \in X \) such that \( \forall i \in I, \forall y_{-i} \in X_{-i}, u_i(\bar{x}) \geq u_i(x_i, y_{-i}) \).

3. PSEUDOCONTINUOUS FUNCTIONS AND BETTER REPLY SECURE GAMES

In this section, we introduce the definitions of pseudocontinuous functions and better reply secure games.

Definition 3.1. ([14]). Let \( f \) be a real valued function defined on a topological vector space \( E \). The function \( f \) is said to be upper pseudocontinuous at \( x_0 \in E \) if for all \( x \in E \) such that \( f(x_0) < f(x) \), it follows that:

\[ \lim_{y \to x_0} \sup_{y \to x_0} f(y) < f(x) \]

The function \( f \) is said to be upper pseudocontinuous on \( E \) if it is upper pseudocontinuous at all \( x_0 \in E \).

Definition 3.2. Let \( f \) be a real valued function defined on a topological vector space \( E \). The function \( f \) is said to be lower pseudocontinuous at \( x_0 \in E \) if \( -f \) is upper pseudocontinuous at \( x_0 \) and the function \( f \) is said to be lower pseudocontinuous on \( E \) if it is lower pseudocontinuous at all \( x_0 \in E \).

Definition 3.3. Let \( f \) be a real valued function defined on a topological vector space \( E \). The function \( f \) is said to be pseudocontinuous if it is both upper and lower pseudocontinuous.
Definition 3.4. ([18]). The game \( G = (I, (X_i, u_i)_{i \in I}) \) is called better reply secure if for every non Nash’s equilibrium \( z \) and every vector \( v \) such that \((z, v)\) belongs to the closure of the graph of \((u_1, ..., u_n)\), then there exists some player \( i \) with strategy \( x_i \) and \( u_i(x_i, t_{-i}) > v_i + \epsilon \) for all \( t_{-i} \) in some neighborhood of \( z_{-i} \). and suitable \( \epsilon \).

Remark 3.1. It is worth noticing that payoff continuity assumption is stronger than pseudocontinuity (see Example 4, [14]) and every game with pseudocontinuous payoffs is better reply secure (see Proposition 4.1, [14]).

In the following, we introduce the analogous definition of the better reply secure games given in the Definition 3.4 to the context of the Berge equilibrium.

Definition 3.5. The game \( G = (I, (X_i, u_i)_{i \in I}) \) is called better reply secure if for every non Berge equilibrium \( z \) and every vector \( v \) such that \((z, v)\) belongs to the closure of the graph of \((u_1, ..., u_n)\), then there exists some player \( i \) and a strategy \( x_{-i} \in X_{-i} \) such that \( u_i(t_i, x_{-i}) > v_i + \epsilon \) for all \( t_i \) in some neighborhood of \( z_i \) and suitable \( \epsilon \).

Next, in the proposition below, we prove that every game \( G \) where each player payoff function is pseudocontinuous verifies the better reply secure games of the Definition 3.5. The steps are similar to that of the proof in the Proposition 4.1 of [14] except for some variables permutations in payoffs functions of players. This proposition will be used later-on to prove the Theorem 4.1 of subsection 4.1.

Proposition 3.1. Let \( G = (I, (X_i, u_i)_{i \in I}) \) be a game. Suppose that for each \( i \in I = \{1, ..., n\} \), the payoff function \( u_i \) of the player \( i \) is pseudocontinuous, then the game \( G \) verifies the better reply secure of the Definition 3.5.

Proof. Let \( z \) be a non Berge equilibrium for the game \( G \) and \( v \) a vector such that \((z, v)\) belongs to the closure of the graph of \((u_1, ..., u_n)\). So, there exists some player \( i \) and a strategy \( x_{-i} \in X_{-i} \) such that \( u_i(z_i, z_{-i}) < u_i(z_i, x_{-i}) \).

We consider two cases:

In the first case, we suppose that there exists \((t_i, t_{-i}) \in X \) such that:

\[
    u_i(z_i, z_{-i}) < u_i(t_i, t_{-i}) < u_i(z_i, x_{-i}).
\]

Since the function \( u_i \) is upper pseudocontinuous at the point \((z_i, z_{-i})\) one has:

\[
    \limsup_{(w_i, w_{-i}) \rightarrow (z_i, z_{-i})} u_i(w_i, w_{-i}) < u_i(t_i, t_{-i}) .
\]

Using that the point \((z, v)\) belongs to the closure of the graph of \((u_1, ..., u_n)\) and the inequation (1), we obtain: \( v_i < u_i(t_i, t_{-i}) \). Now, the function \( u_i \) is lower pseudocontinuous at the point \((z_i, x_{-i})\), so:

\[
    u_i(t_i, t_{-i}) < \liminf_{(w_i, w_{-i}) \rightarrow (z_i, x_{-i})} u_i(w_i, w_{-i}) .
\]
It follows that:
\[ v_i < u_i(t_i, t_{-i}) < \liminf_{(w_i, w_{-i}) \to (z_i, x_{-i})} u_i(w_i, w_{-i}) . \]

Since,
\[ \liminf_{(w_i, w_{-i}) \to (z_i, x_{-i})} u_i(w_i, w_{-i}) < u_i(t_i, x_{-i}) \]
for all \( t_i \) in some neighborhood of \( z_i \), we deduce:
\[ u_i(t_i, x_{-i}) > v_i + \epsilon. \]

It follows that the game \( G \) verifies the Definition 3.5.

In the second case, we suppose that:
\[ u_i(X) \cap [u_i(z_i, z_{-i}), u_i(z_i, x_{-i})] = \emptyset. \]

Since the function \( u_i \) is lower semicontinuous at the point \( (z_i, z_{-i}) \), then:
\[ u_i(z_i, z_{-i}) < \liminf_{(w_i, w_{-i}) \to (z_i, x_{-i})} u_i(w_i, w_{-i}) . \]

Since there are not values of the payoff function \( u_i \) between \( u_i(z_i, z_{-i}) \) and \( u_i(z_i, x_{-i}) \), it follows that:
\[ u_i(z_i, z_{-i}) < u_i(z_i, x_{-i}) < \liminf_{(w_i, w_{-i}) \to (z_i, x_{-i})} u_i(w_i, w_{-i}) . \]

Using the same arguments as in the first case, we deduce that \( u_i(t_i, x_{-i}) > v_i + \epsilon \) for all \( t_i \) in some neighborhood of \( z_i \). Then, the game \( G \) verifies also the Definition 3.5.

Next, we give sufficient conditions for the existence of the Berge equilibrium. From now, we assume that each strategy set \( X_i \) is a subset of a locally convex topological vector space \( E_i \). For each \( i \), we call the best reply correspondence for the player \( i \), the correspondence \( \Gamma_i : X \to X \) defined by:
\[ \Gamma_i(x) = \{ y \in X : u_i(x, y_{-i}) \geq u_i(t_i, x_{-i}) \ \forall t_{-i} \in X_{-i} \}. \]

For each \( x \in X \), we set
\[ \Gamma(x) = \bigcap_{i \in I} \Gamma_i(x) . \]

With these notations, a Berge equilibrium is a fixed point of the correspondence \( \Gamma \), that it is an \( n \)-tuple \( \tilde{x} \in \Gamma(\tilde{x}) \).

**Theorem 3.1.** Assume the following assumptions on the game \( G \):

1. \( \forall i \in I, X_i \) is a nonempty, compact and convex subset of \( E_i \);
2. \( \forall i \in I, \forall x_i \in X_i, \) the function \( y_{-i} \to u_i(x_i, y_{-i}) \) is quasi-concave on \( X_{-i} \);
3 \( \forall i \in I \), the function \( u_i \) is pseudocontinuous on \( X_i \times X_{-i} \);

4 \( \forall x \in X, \Gamma (x) \neq \emptyset \).

Then there exists a Berge equilibrium.

Proof. Let us assume that each strategy space \( E_i \) is locally convex. We remark that under the assumptions (1)-(3) of the Theorem 3.1, each correspondence \( \Gamma_i \) is convex valued but may have empty values. Under these assumptions, the correspondence \( \Gamma \) is closed (for definition and properties of the different continuity concepts for correspondences, we refer the reader to [3], [4] or to the Appendix of [11]) while for statements of classical fixed point theorems, we refer to [3], [11]. Thus \( \Gamma \) is upper semicontinuous with compact values and the set \( F = \{ x \in X : \Gamma (x) \neq \emptyset \} \) is closed in \( X \). Indeed, the correspondence \( x_i \to X_{-i} \) is obviously continuous and an easy adaptation of the proof of the Berge maximum theorem (see Theorem 3.1, [14]) shows that the correspondence:

\[
x_i \to \left\{ y : u_i(x_i, y_{-i}) = \max_{t_{-i} \in X_{-i}} u_i(x_i, t_{-i}) \right\},
\]

has a closed graph. As intersection of closed correspondences, each correspondence \( \Gamma_i \) is closed, thus upper semicontinuous with compact values. Consequently, if \( x^\nu \to x \) with, for each \( \nu \) of a directed set, \( x^\nu \in F \), that is, \( y^\nu \in \Gamma (x^\nu) \), in view of the compactness of \( X \), one can assume without loss of generality that \( y^\nu \to y \in \Gamma (y) \). It now follows from the Kakutani-Fan theorem that \( \Gamma \) has a fixed point. Let \( x \) denote a fixed point of the correspondence \( \Gamma : X \to X \) defined as previously by:

\[
\forall x \in X, \Gamma (x) = \bigcap_{i \in I} \Gamma_i (x),
\]

where \( \forall i \in I \), the correspondence \( \Gamma_i : X \to X \) is the best reply correspondence of the player \( i \). Then, \( \forall i \in I, x \in \Gamma_i (x) \). From the definition of the best reply correspondence \( \Gamma_i \), we deduce:

\[
\forall i \in I, \forall t_{-i} \in X_{-i} : u_i (x_i, x_{-i}) \geq u_i (x_i, t_{-i}).
\]

It follows from the above inequation that the fixed point \( x \) of the correspondence \( \Gamma \) is a Berge equilibrium (see Definition 1.1) for the game \( G \).}

3.1. BERGE EQUILIBRIUM OF AN ABSTRACT ECONOMY

We now consider the following generalized game that will be called abstract economy. For this definition, we refer the reader to [7]:

\[
H = (I, (X_i, F_i, u_i)_{i \in I}),
\]
where \( I = \{1, \ldots, n\} \) is a (finite set) of agents, \( X_i \) is the strategy set of the agent \( i \), and if \( X = \prod_{i \in I} X_i \), then \( u_i : X \to \mathbb{R} \) is the payoff function of the player \( i \), while \( F_i : X \to X_i \) denotes a feasibility correspondence for the player \( i \) given the strategies of the other agents. The following definition extends the Definition 2.1 to abstract economies.

**Definition 3.6.** A Berge equilibrium of \( H \) is an \( n \)-tuple of strategies \( \bar{x} \) such that

\[
\forall i \in I, \bar{x}_i \in F_i(\bar{x}),
\]

and

\[
\forall i \in I, \forall y \in \prod_{j \in I \setminus \{i\}} F_j(x), \ u_i(x_i, y) \geq u_i(x_i, y - i).
\]

The first condition guarantees that \( \bar{x} \) is a vector of feasible strategies. If, as previously, we call for each \( i \in I \), the best reply correspondence for the player \( i \), the correspondence \( \Gamma_i : X \to X \) defined by:

\[
\Gamma_i(x) = \left\{ y \in X : y_{-i} \in \prod_{j \in I \setminus \{i\}} F_j(x), \ u_i(x_i, y_{-i}) \geq u_i(x_i, y - i) \land \right. \ \
\forall t_{-i} \in \prod_{j \in I \setminus \{i\}} F_j(x) \left. \right\},
\]

both the above conditions jointly express that

\[
\bar{x} \in \bigcap_{i \in I} \Gamma_i(\bar{x}).
\]

**Theorem 3.2.** Assume the following assumptions on the game \( H \):

1. \( \forall i \in I, E_i \) is a locally convex topological vector space and \( X_i \) is a nonempty, compact and convex subset of \( E_i \);
2. \( \forall i \in I, \forall x_i \in X_i \), the function \( y_{-i} \to u_i(x_i, y_{-i}) \) is quasi-concave on \( X_{-i} \);
3. \( \forall i \in I \), the function \( u_i \) is pseudocontinuous on \( X_i \times X_{-i} \);
4. \( \forall i \in I \), the correspondence \( F_i : X \to X_i \) is continuous with nonempty, convex and compact values;
5. \( \forall x \in X, \bigcap_{i \in I} \Gamma_i(x) \neq \emptyset. \)

Then \( H \) has a Berge equilibrium.

**Proof.** The proof is a slight modification of the proof given for the Theorem 3.1 in the case where each strategy space is locally convex topological vector space. For every \( i \in I \), let us denote by \( F_{-i} : X \to \prod_{j \in I \setminus \{i\}} X_j \) the correspondence defined by:

\[
F_{-i}(x) = \prod_{j \in I \setminus \{i\}} F_j(x).
\]
On the existence of Berge equilibrium with pseudocontinuous payoffs

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Since $F_j$ is continuous with compact values, we have the Berge maximum theorem:

$$x_i \rightarrow \left\{ y : u_i(x_i, y_{-i}) = \max_{t_{-i} \in F_{-i}(x)} u_i(x_i, t_{-i}) \right\},$$

has a closed graph. Once again, as intersection of closed correspondences, each correspondence $\Gamma_i$ is closed, thus upper semicontinuous with compact values. The same is true for the correspondence $\Gamma : X \rightarrow X$ defined by $\Gamma(x) = \bigcap_{i \in I} \Gamma_i(x)$. As by assumptions (2), (4) and (5), $\Gamma$ is nonempty and convex valued, the existence of the Berge equilibrium of $H$ follows from the Kakutani-Fan theorem.

4. ESSENTIAL GAMES

4.1. GAMES PARAMETRIZED BY PAYOFF PROFILES

Let us assume that, defined on the same strategy spaces the games:

$$\left((\mathcal{I}, (X_i)_{i=1}^n, (u_i)^n_{i=1})\right)$$

are parametrized by the payoff function profiles $g = (u_1, ..., u_n)$. More precisely, let $U$ be the set of payoff function profiles $g = (u_1, ..., u_n)$ that satisfy the assumptions of Theorem 3.1 and verify $\sum_{i \in I, x \in X} |u_i(x)| < \infty$, endowed with the distance $\rho$ defined as follows. For each $g^1 = (u_1^1, ..., u_n^1)$ and $g^2 = (u_1^2, ..., u_n^2) \in U$

$$\rho(g^1, g^2) = \sum_{i \in I} \sup_{x \in X} |u_i^1(x) - u_i^2(x)|.$$

It is easy to see that $U$ endowed with the distance $\rho$ is a complete metric space. Now, define the Berge equilibria correspondence $J : U \rightarrow X$ where for each $g \in U$, $J(g) \subset X$ is the set of Berge equilibria of the game $g$.

In the following theorem, we give some properties of the correspondence $J$ which require the study of essential games.

**Theorem 4.1.** The Berge equilibria correspondence $J$ is upper semicontinuous with nonempty and compact values.

**Proof.** It follows from the Theorem 3.1 that the correspondence $J$ has nonempty values. We now prove that the graph of $G$ is closed. Since the correspondence $J$ has values in a space $X$ without a countable basis of neighborhoods, then we have to use a net instead of a sequence. By contradiction, assume that there is a net $g^\alpha \rightarrow g$ and a net $x^\alpha \rightarrow x \notin J(g), x^\alpha \in J(g^\alpha)$. Let $u_i = \limsup_n u_i(x^\alpha)$. Then, $(x, u = (u_1, ..., u_n)) \in cl\text{graph}(u_1, ..., u_n)$. It follows from the Proposition 3.1 that:
\[ \exists i \in I, \exists x^i \in X, u_i(t, x^i) > u_i + \epsilon, \forall t_i \in V(x_i), \epsilon > 0. \]

Since \( \lim \rho(g, g^\alpha) = 0 \), we have \( u_i(t, x^i_i) > u_i + \rho(g, g^\alpha) + \epsilon \).

We obtain:
\[ u_i(t, x^i_i) > u_i + u_i^\alpha(x^i_i, x^i_i) - u_i(x^i_i, x^i_i) + \epsilon \]

Since \( x^i \in J(g^\alpha) \), then:
\[ u_i^\alpha(x^i_i, x^i_i) \geq u_i^\alpha(x^i_i, x^i_i). \]

So,
\[ u_i(t, x^i_i) > u_i + u_i^\alpha(x^i_i, x^i_i) - u_i(x^i_i, x^i_i) + \epsilon \]

We have
\[ \rho(g, g^\alpha) \geq u_i(t, x^i_i) - u_i^\alpha(t, x^i_i) \]

and
\[ u_i(t, x^i_i) - u_i^\alpha(t, x^i_i) > u_i + u_i^\alpha(x^i_i, x^i_i) - u_i(x^i_i, x^i_i) - u_i^\alpha(t, x^i_i) + \epsilon. \]

Since \( x^i_i \rightarrow x_i \) and \( t_i \in V(x_i) \), then
\[ \rho(g, g^\alpha) > u_i - u_i(x^i_i, x^i_i) + \epsilon > u_i - u_i(x^i_i, x^i_i) + \epsilon > \epsilon \]

Contradiction. \( \blacksquare \)

**4.2. GAMES PARAMETRIZED BY PAYOFF PROFILES AND THE STRATEGIES SETS**

For any \( i \in I = \{1, \ldots, n\} \), let \( X_i \) be a closed subset of a Hausdorff locally convex topological vector space \( E_i \) and let \( CK(X_i) \) be the set of all non empty, convex and compact subsets \( S_i \) of \( X_i \) equipped with the Vietoris topology (see ([20])). We endowed \( \prod_{i=1}^{n} CK(X_i) \) with the topology product of Vietoris topologies on each \( X_i \). Let
\[ C = \{ g = (u_1, \ldots, u_n): \sup_\times x \in X | u_i(x) < \infty \} \]

where \( X = \prod_{i=1}^{n} X_i \) is the set profiles such that for each \( i \in I = \{1, \ldots, n\} \), the payoff function \( u_i \) of the player \( i \) verifies the hypothesis in the Theorem 3.1 and it is endowed with the metric:
\[ \rho(g^1, g^2) = \sum_{i \in I} \sup_{x \in X} | u_i^1(x) - u_i^2(x) | \]
Let $Y = C \times \prod_{i=1}^{n} CK(X_i)$ be the space of games $g = (u_1, \ldots, u_n, S_1, \ldots, S_n) \in Y$, parametrized by payoff function profiles and strategy sets.

Recall that $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \prod_{i=1}^{n} S_i$ is a Berge equilibrium for the game $g = (u_1, \ldots, u_n, S_1, \ldots, S_n) \in Y$ if:

\[ \forall i \in I, \quad \text{max}_{u \in S_{-i}} u_i(x, u_{-i}) = u_i(\bar{x}, \bar{u}_{-i}). \]

Let:

\[ K : Y \to \prod_{i=1}^{n} S_i \]

\[ g = (u_1, \ldots, u_n, S_1, \ldots, S_n) \to K(g) \]

the correspondence of the Berge equilibria. As in subsection 4.1, we shall prove in the following theorem that the correspondence of the Berge equilibria $K$ is upper semicontinuous with nonempty and compact values.

**Theorem 4.2.** Assume the following assumptions on $H$:

1. \( \forall i \in I, E_i \) is a Hausdorff locally convex topological vector space and $X_i$ is a nonempty, closed and convex subset of $E_i$;
2. \( \forall i \in I, \forall x_i \in X_i, \text{ the function } y_{-i} \to u_i(x_i, y_{-i}) \) is quasi-concave on $X_{-i}$;
3. \( \forall i \in I, \text{ the function } u_i \text{ is pseudocontinuous on } X_i \times X_{-i} \);
4. \( \forall i \in I, \text{ the correspondence } F_i : X \to X_i \text{ is continuous with nonempty, convex and compact values;} \)
5. \( \forall x \in X, \bigcap_{i \notin I} \Gamma_i (x) \neq \emptyset. \)

Then the Berge equilibria correspondence $K$ is upper semicontinuous with nonempty and compact values.

**Proof.** Let us consider the following noncooperative games parametrized by payoff function profiles and strategy sets $\{I, (S_i)_{i=1}^{n}, (u_i)_{i=1}^{n}\}$. From the Theorem 3.1 or the Theorem 3.2, $K$ has non empty values. As we have noticed in the proof of the Theorem 4.1, we have to take a net instead of sequence.

Let $x^\alpha \in K(g)$ be a net where $g = (u_1, \ldots, u_n, S_1, \ldots, S_n) \in Y$ and $\lim_{\alpha} x^\alpha = x$. Then, for each $i \in I$, $u_i(x^0_i, x_{-i}^0) = \max_{t \in S_{-i}} u_i(x^0_i, t_{-i})$. We prove that $K$ has closed values in a compact set $\prod_{i=1}^{n} S_i$. Assume that $x \notin K(g)$, then there exists $i_0$ such that

\[ u_{i_0}(x_{i_0}, x_{-i_0}) < u_{i_0}(x_{i_0}, u_{-i_0}) \]
where $u^0_{-t_0} \in S_{-t_0}$. Since the function $u_{t_0}$ is pseudocontinuous, we have:

$$
\forall (z_{t_0}, z_{-t_0}) \in V(x_{t_0}) \times V(x_{-t_0}), \forall (t_{-t_0}, \rho_{-t_0}) \in V'(x_{t_0}) \times V'(u^0_{-t_0})
$$

$$
u_{t_0}(z_{t_0}, z_{-t_0}) < \nu_{t_0}(t_{-t_0}, \rho_{-t_0})
$$

Let $W(x_{t_0}) = V(x_{t_0}) \cap V'(x_{t_0})$, then $\nu_{t_0}(z_{t_0}, z_{-t_0}) < \nu_{t_0}(z_{t_0}, \rho_{-t_0})$ for all $(z_{t_0}, z_{-t_0}) \in W(x_{t_0}) \times V(x_{-t_0})$.

It follows that, $(W(x_{t_0}) \times V(x_{-t_0})) \cap K(g) = \emptyset$. Then $K$ has closed values in a compact set $\prod_{i=1}^n S_i$. Now, we prove that $K$ is upper semicontinuous. If it is not true at a point $y \in Y$, then there exists an open set $O$ of $X$ and a net $g^\alpha \in Y$ such that $O \supset K(g)$, $\lim \nu g^\alpha = g$ and $x^\alpha \in K(g^\alpha)$, $x^\alpha \notin O$. Thus $\lim \nu \rho(g^\alpha, g) = 0$ and for each $i \in I$, $\lim_i S_i^\alpha = S_i$ for the Vietoris topology on $CK(X)$. In view of Lemma 2.3 in ([20]), let $x$ be a cluster point of $x^\alpha$. It is obvious that $x \notin O \supset K(g)$ and hence there exists $i_0 \in I$, such that

$$
u_{t_0}(x_{t_0}, x_{-t_0}) < \nu_{t_0}(x_{t_0}, u^0_{-t_0})
$$

where $u^0_{-t_0} \in S_{-t_0}$. Let $X$ be a topological space and $t \in X$, we denote by $O(t)$ any open set of $X$ which contains a point $t$. By the pseudocontinuity of the function $u_{t_0}$ (see Proposition 2.2, [19]), there exists $x^0, x^1$ such that:

$$
u_{t_0}(x_{t_0}, x_{-t_0}) < \nu_{t_0}(x^0_{t_0}, x^0_{-t_0}) < \nu_{t_0}(x^1_{t_0}, x^1_{-t_0}) < \nu_{t_0}(x_{t_0}, u^0_{-t_0}).
$$

Using the upper pseudocontinuous of the function $u_{t_0}$ at the point $(x_{t_0}, x_{-t_0})$, we obtain:

$$
u(x^0_{t_0}, x^0_{-t_0}) < \nu(x^1_{t_0}, x^1_{-t_0}) < \nu(x_{t_0}, u^0_{-t_0}).
$$

Let $V_{t_0}(u^0_{t_0}) \subset X_{t_0}$ be an open set such that $u^0_{t_0} \in S_{t_0}$ and $\prod_{j \in I \backslash I(t_0)} V_j(u^0_{j(t_0)}) \subset O(u^0_{-t_0})$.

Since $V_{t_0}(u^0_{t_0}) \cap S_{t_0} \neq \emptyset$, then for $\alpha \geq \alpha_0, \rho(g^\alpha, g)$ converges to 0, $V_{t_0}(u^0_{t_0}) \cap S_{t_0}^\alpha \neq \emptyset$ and $x^\alpha \in O(x_{t_0}) \times O(x_{-t_0})$. Take $u^{\alpha \beta}_{t_0} \in O(u^0_{t_0}) \cap \prod_{j \in I \backslash I(t_0)} S_{t_0}^\alpha$. It is obvious that

$$
u_{t_0}(x^\alpha_{t_0}, x^\beta_{t_0}) + \rho(g^\alpha, g) < \nu_{t_0}(x^1_{t_0}, x^1_{-t_0}) < \nu_{t_0}(x_{t_0}, u^\alpha_{-t_0}).
$$

Then,

$$u_{t_0}(x^\alpha_{t_0}, x^\beta_{t_0}) + u^\alpha_{t_0}(x^\alpha_{t_0}, x^\beta_{t_0}) - u_{t_0}(x^\alpha_{t_0}, x^\beta_{t_0}) < u_{t_0}(x^1_{t_0}, x^1_{-t_0}) < u_{t_0}(x_{t_0}, u^\alpha_{-t_0}),
$$

we get:

$$u^\alpha_{t_0}(x^\alpha_{t_0}, x^\beta_{t_0}) < u_{t_0}(x^1_{t_0}, x^1_{-t_0}) < u_{t_0}(x_{t_0}, u^\alpha_{-t_0}) < u_{t_0}(x^\alpha_{t_0}, u^\alpha_{-t_0}).
$$

From the Proposition 2.3 (see [19]), we have:
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\[ u_{\alpha i}^{\alpha 0}(x_{\alpha i}^{\alpha 0}, x_{\alpha i}^{\alpha -i_0}) < u_{\alpha i}^{\alpha 0}(x_{\alpha i}^{\alpha 0}, u_{\alpha i}^{\alpha 0}), \]

Since \( u_{\alpha i}^{\alpha 0} \in S_{\alpha i}^{\alpha 0} \), we obtain a contradiction with \( x^\alpha \) is a net of the Berge equilibria. ■

**Remark 4.1.** The pseudocontinuity in Theorem 4.2 cannot be relaxed by the better reply secure games given in the Definition 3.5 (see Example 3.1, [19]). However, this latter has been used in the Theorem 4.1 of Subsection 4.1.

Now, we introduce the definitions of essential equilibrium and essential games.

**Definition 4.1.** ([20]). Let \( M \) be a nonempty and closed subset of \( U \) defined in subsection 4.1 or \( Y \) defined in Subsection 4.2 and \( g \in M \). An element \( x \in J(g) \) or \( x \in K(g) \) is called **essential equilibrium** of the game \( g \) relative to \( M \) if for any \( O \in V(x) \) there exists \( W \in W(g) \) such that for each \( g_1 \in M \cap W \), there exists \( x_1 \in J(g_1) \) or \( x_1 \in K(g_1) \) with \( x_1 \in O \).

**Definition 4.2.** ([20]). Let \( M \) be a nonempty and closed subset of \( U \) defined in subsection 4.1 or \( Y \) defined in Subsection 4.2 and \( g \in M \). The game \( g \in M \) is said to be **essential** relative to \( M \) if all its equilibria are essential relative to \( M \).

It follows from the Definition 4.2 that the game \( g \in M \) is essential if and only if the correspondence \( J: M \rightarrow K(X) \) or \( K: M \rightarrow K(X) \) is lower semicontinuous at \( g \) (see Theorem 4.1, [20]), where \( K(X) \) denotes the space of all nonempty compact subsets of \( X \).

In the following theorem, we establish that most of games in subsections 4.1 and 4.2 are essential in the sense of Baire category.

**Theorem 4.3.** Assume that for each \( i \in I \), the strategy space \( E_i \) is a normed space. Then most of the games in Subsections 4.1 and 4.2 are essential in the sense of Baire category.

**Proof.** The proof is similar as in [10]. ■

5. CONCLUSION

In this paper, we have proved in subsection 4.1 that games in normal form having essential Berge equilibria are the generic case in the space of discontinuous games. And in Subsection 4.2, we have proved that abstract economies having essential Berge equilibria are also the generic case in the space of discontinuous games. We have used weakening of continuity called pseudocontinuity in [14] in Subsection 4.1, we have showed in Theorem 4.1 that the pseudocontinuity could be weakened by the better reply secure games given in the Definition 3.5 to the context of the Berge equilibrium. However, for games parametrized by payoff function profiles and strategy sets, the same assumption could not be relaxed in the Theorem 4.2 of subsection 4.2.
using the Definition 3.5, because the correspondence of the Berge equilibria is defined on an abstract space endowed with a product of metric and a Vietoris topology and has values in a non-fixed compact space. The maximum theorem in the setting of the pseudocontinuity functions given in the paper of [14], the fixed point theorem of Kakutani and the Baire theorem have played a central role in the main results of this paper.

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On the existence of Berge equilibrium with pseudocontinuous payoffs

ON EXTENSIONS OF MAPPINGS INTO COMPLETE METRIZABLE SPACES
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Abstract
For a subspace $Y$ of a space $X$ and a space $E$ are determined the conditions for which any continuous mapping $f : Y \rightarrow E$ is continuous extendable on $X$ (Theorem 2.1, Theorem 2.2, Theorem 3.2). Some results proved by Kulesza, Levy, Nyikos[4] are generalized.

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1. INTRODUCTION

Given a topological space $X$ we will denote by $cl_XA$ the closure of any set $A$ from $X$. Also a subset $B$ of $X$ is clopen if it is simultaneously closed and open.

All spaces considered are assumed to be completely regular. A regular space $X$ is said to be zero-dimensional if it is of small inductive dimension zero ($indX = 0$), i.e., $X$ has a base of clopen sets. A normal space $X$ has large inductive dimension zero ($IndX = 0$) if and only if for any two disjoint closed subsets $A$ and $B$ of $X$ there is a clopen set $C$ such that $A \subseteq C$ and $B \subseteq (X \setminus C)$. A normal space $X$ has Lebesgue covering dimension zero ($dimX = 0$) if any finite open cover of $X$ can be refined to a partition of $X$ into clopen sets. It is well known that for any metric space $X$, $IndX = dimX$. Also if $X$ is Lindelöf then $indX = 0$ if and only if $IndX = 0$ [2, Theorem 1.6.5] and if $X$ is normal then $IndX = 0$ if and only if $dimX = 0$ [2, Theorem 1.6.11].

2. ON EXTENSION OF DISCRETE-VALUED MAPPINGS

All spaces considered in this section are assumed to be zero-dimensional.

We are going to use the symbol $\mathcal{D}_\tau$ ($\tau$ is any cardinal number) for the discrete space consisting of $\tau$ elements. As usual, we write $\mathcal{D}$ instead of $\mathcal{D}_2$ and is very convenient to view $\mathcal{D}$ as being $\{0, 1\}$ endowed with the discrete topology. Also $C(X, \mathcal{D}_\tau)$ is the set of all continuous functions defined on the topological space $X$ and with values in $\mathcal{D}_\tau$ (discrete-valued continues functions). If $Y \subseteq X$ and $f \in C(Y, \mathcal{D}_\tau)$ then we say that $f$ extends to a function $g \in C(X, \mathcal{D}_\tau)$ if $g(x) = f(x)$, for every $x \in Y$. 

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Throughout this section we will use, frequently, the fact that every discrete-valued continuous function \( f \) on a space \( X \) generates a clopen partition of the space \( X \) and conversely every clopen partition of \( X \) induces a continuous discrete-valued function.

**Theorem 2.1.** Let \( Y \subseteq X \), \( X \) normal and \( \dim X = 0 \), then the following assertions are equivalent:

(i) For every clopen subset \( U \) of \( Y \) the set \( \text{cl}_X U \) is clopen in \( \text{cl}_X Y \).

(ii) For every clopen partition \( \gamma = \{U, V\} \) of \( Y \) there exists a clopen partition \( \gamma' = \{U', V'\} \) of \( X \) such that \( U = U' \cap Y \) and \( V = V' \cap Y \).

(iii) Every function \( f \in C(Y, D) \) extends to a function in \( C(X, D) \).

**Proof.** (i) \( \rightarrow \) (ii) Let \( X \) be normal and \( \dim X = 0 \). Then \( \text{Ind} X = 0 \), which means that any two disjoint closed subsets of \( X \) can be separated by disjoint clopen subsets of \( X \), i.e., \( X \) is ultranormal [1]. Let \( \gamma = \{U, V\} \) a clopen partition of \( Y \). Then, by assumption, \( \text{cl}_X U \) is clopen in \( \text{cl}_X Y \). As \( \text{cl}_X Y \) is closed subset of \( X \), applying [1, Lemma 1.1] we can find a clopen subset \( U' \) of \( X \), such that \( \text{cl}_X U = U' \cap \text{cl}_X Y \). On the other hand, \( U \subseteq \text{cl}_X U \) and \( U \) is clopen in \( Y \), therefore \( U' \cap Y = U \).

The collection \( \{U', X \setminus U'\} \) is the desired partition.

(ii) \( \rightarrow \) (i) Let \( U \) be a clopen subset of \( Y \). Then the collection \( \{U, V = Y \setminus U\} \) is a clopen partition of \( Y \). Therefore, by assumption, we can find a clopen partition \( \{U', V'\} \) of \( X \) such that \( U = U' \cap Y \) and \( V = V' \cap Y \). Now, as \( \{U, V\} \) is partition of \( Y \) we have that \( \text{cl}_X Y = \text{cl}_X U \cup \text{cl}_X V \). On the other hand, \( U \subseteq \text{cl}_X U \subseteq U' \), therefore \( \text{cl}_X U = U' \cap \text{cl}_X Y \), i.e., \( \text{cl}_X U \) is clopen in \( \text{cl}_X Y \).

(ii) \( \leftrightarrow \) (iii) This is obvious. \( \blacksquare \)

**Remark 2.1.** The above Theorem holds if we change the condition "\( \dim X = 0 \)" to "\( X \) is Lindelöf and zero-dimensional (\( \text{ind} X = 0 \))".

**Proof.** It is enough to recall that if \( X \) is Lindelöf then \( \text{ind} X = 0 \) iff \( \text{Ind} X = 0 \) [2, Theorem 1.6.5]. \( \blacksquare \)

**Remark 2.2.** The equivalence (ii) \( \leftrightarrow \) (iii) from Theorem 2.1 is true for any space \( X \).

**Example 2.1.** Let \( Y = \mathbb{N} \) with the discrete topology and \( X = \beta \mathbb{N} \). Then \( X \) is normal and \( \dim X = 0 \). Let \( \tau < \omega \). Then every continuous function from \( Y \) into \( D_\tau \) extends to a continuous function on \( X \). But if \( \tau \geq \omega \) then, since a continuous function on a compact space must be bounded, not every continuous function from \( Y \) into \( D_\tau \) extends to a continuous function on \( X \). Thus, in case of continuous functions into a infinite discrete space, the conditions for \( X \) to be normal and \( \dim X = 0 \) are not enough.

**Theorem 2.2.** Let \( Y \subseteq X \), \( X \) \( \tau \)-collectionwise normal, \( \tau \geq \omega \) and \( \dim X = 0 \). Then the following assertions are equivalent:
Let \( X \) be essential in Theorem 2.2.

(i) For every discrete collection \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) of clopen subsets of \( Y \) the collection \( \{ \text{cl}_X U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is discrete in \( X \).

(ii) For every clopen subset \( U \) of \( Y \) the set \( \text{cl}_X U \) is clopen in \( X \) and every discrete collection \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) of clopen subsets of \( Y \) is locally finite in \( X \).

(iii) Every function \( f \in C(Y, \mathcal{D}_\tau) \) extends to a function in \( C(X, \mathcal{D}_\tau) \).

**Proof.** (i)\(\rightarrow\)(ii) Let \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) be a discrete collection of clopen subsets of \( Y \). By assumption, \( \{ \text{cl}_X U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is discrete in \( X \), but \( U_\alpha \subseteq \text{cl}_X U_\alpha \) for every \( \alpha \in \mathcal{D}_\tau \), therefore \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is locally finite in \( X \).

Now, we may assume that \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is a discrete cover of \( Y \) with clopen subsets. Using this assumption we get that \( \text{cl}_X U_\alpha = \text{cl}_X Y \setminus \cup \{ U_\beta : \beta \in \mathcal{D}_\tau \setminus \{ \alpha \} \} \) which means that \( \text{cl}_X U_\alpha \) is clopen in \( Y \) for every \( \alpha \in \mathcal{D}_\tau \).

Now, let \( U \) be a clopen subset of \( Y \). Then \( \{ U, Y \setminus U \} \) is a discrete collection of clopen subsets of \( Y \). Applying previous results, \( \text{cl}_X U \) is clopen in \( \text{cl}_X Y \).

(ii)\(\rightarrow\)(i) Let \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) be a discrete collection of clopen subsets of \( Y \). By assumption, the family \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is locally finite in \( X \) and \( \text{cl}_X U_\alpha \) is clopen in \( \text{cl}_X Y \) for every \( \alpha \in \mathcal{D}_\tau \). As, for every \( \alpha \in \mathcal{D}_\tau \), \( U_\alpha \) are pairwise disjoint and \( \text{cl}_X U_\alpha \) are clopen in \( \text{cl}_X Y \) this implies that \( \text{cl}_X U_\alpha \) are also pairwise disjoint. Thus \( \{ \text{cl}_X U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is discrete in \( X \).

(i)\(\rightarrow\)(iii) Let \( f \in C(Y, \mathcal{D}_\tau) \). Then \( \{ U_\alpha = f^{-1}(\alpha) : \alpha \in \mathcal{D}_\tau \} \) be a discrete collection of clopen subsets of \( Y \). By assumption \( \{ \text{cl}_X U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is discrete in \( X \) and since \( X \) is \( \tau \)-collectionwise normal and \( \dim X = 0 \), there exists a discrete collection \( \{ V_\alpha : \alpha \leq \tau \} \) of clopen subsets of \( X \) such that \( \text{cl}_X U_\alpha \subseteq V_\alpha \) for every \( \alpha \in \mathcal{D}_\tau \). We can assume that \( \{ V_\alpha : \alpha \leq \tau \} \) is a cover of \( X \). Let \( g \in C(X, \mathcal{D}_\tau) \), \( g^{-1}(\alpha) = V_\alpha \) for every \( \alpha \in \mathcal{D}_\tau \).

The function \( g \) is the desired extension.

(iii)\(\rightarrow\)(i) Let \( \{ U_\alpha : \alpha \in \mathcal{D}_\tau \} \) be a discrete collection of clopen subsets of \( Y \). Let \( f \in C(Y, \mathcal{D}_\tau) \) such that \( f^{-1}(\alpha) = U_\alpha \) for every \( \alpha \in \mathcal{D}_\tau \). Then, by assumption, there exists \( g \in C(X, \mathcal{D}_\tau) \) such that \( g(x) = f(x) \) for every \( x \in Y \). Therefore \( \{ V_\alpha = g^{-1}(\alpha) : \alpha \in \mathcal{D}_\tau \} \) is a clopen partition of \( X \) such that \( U_\alpha = V_\alpha \cap Y \) for every \( \alpha \in \mathcal{D}_\tau \). Then \( \text{cl}_X U_\alpha = V_\alpha \cap \text{cl}_X Y \) and \( \{ \text{cl}_X U_\alpha : \alpha \in \mathcal{D}_\tau \} \) is a clopen partition of \( \text{cl}_X Y \).

**Example 2.2.** Let \( X = \{(x, y) : x^2 + y^2 \leq 1\} \) be a subspace of the plane with the distance: \( d((0, 0), (x, y)) = |x| + |y| \) for any \((x, y) \in X\); if \((x, y), (u, v) \in X \) and \( x : y = u : v \) (lie on the same straight line which crosses the origin), then \( d((x, y), (u, v)) = |x-u|+|y-v|\); if \((x, y), (u, v) \in X \) and \( x \neq u : v \), then \( d((x, y), (u, v)) = |x|+|y|+|u|+|v|\).

The space \( X \) is a metrizable Hedgehog [3]. Let \( Y = \{(x, y) \in X : \frac{1}{4} < x^2 + y^2\} \). For any \( \alpha \in [0, 2\pi] \) we put \( I_\alpha = \{(\cos \alpha, \sin \alpha) : 0 \leq t \leq 1\} \) and \( F_\alpha = I_\alpha \cap Y \). Then any clopen subset \( U \) of \( Y \) is a union of some \( F_\alpha \). For any clopen subset \( H \) of \( Y \) the set \( \text{cl}_X H \) is clopen in \( \text{cl}_X Y \).

Let \( f : Y \to \mathcal{D}_\tau \) be a continuous mapping of \( Y \) in some \( \mathcal{D}_\tau \). If \( |f(Y)| \geq 2 \), the mapping \( f \) do not admit a continuous extension on \( X \). Hence the condition \( \dim X = 0 \) is essential in Theorem 2.2.
3. EXTENSION OF MAPPINGS INTO METRIC SPACES

A topological space $X$ is Dieudonné complete if there exists a complete uniformity on the space $X$. A space $X$ is topologically complete if $X$ is homeomorphic to a closed subspace of a product of metrizable spaces [2]. The Dieudonné completion $\mu X$ of a space $X$ is a topological complete space for which $X$ is a dense subspace of $\mu X$ and each continuous mapping $g$ from $X$ into a topologically complete space $Y$ admits a continuous extension $\mu g$ over $\mu X$.

Let $\tau$ be an infinite cardinal number. Denote by $\mu \tau X$ a topological complete space for which $X$ is a dense subspace of $\mu \tau X$ and each continuous mapping $g$ from $X$ into a metrizable space $Y$ of the weight $\leq \tau$ admits a continuous extension $\mu \tau g$ over $\mu \tau X$.

The infinite cardinal $c(X) = \sup\{|\gamma| : \gamma \subseteq t^*(X) \text{ is disjoint}\}$ is called the Souslin number of $X$.

If $\tau$ and $\kappa$ are infinite cardinal numbers and $\tau \leq \kappa$, then $\mu \tau X \subseteq \mu \kappa X$. The extension $\nu X = \mu_{\aleph_0} X$ is called the Hewitt-Nachbin completion of a space $X$. If $\tau \geq c(X)$, then $\mu \tau X = \mu X$.

**Theorem 3.1.** Let $Y$ be a subspace of the space $X$, $E$ be a topological complete space and for each closed subspace $Z$ of $X$ and any continuous mapping $g : Z \to E$ there exists a continuous extension $\bar{g} : X \to E$. If $\mu \tau Y = \text{cl}\, \mu \tau X Y$, then for each continuous mapping $g : Y \to E$ into a Banach space $E$ of the weight $\leq \tau$ there exists a continuous extension $\bar{g} : X \to E$.

**Proof.** Fix a continuous mapping $g : Y \to E$. This mapping admits a continuous extension $\mu \tau g$ over $\mu \tau Y$ i.e., $\mu \tau g : \mu \tau Y \to E$.

But $\mu \tau Y = \text{cl}\, \mu \tau X Y$, because $X$ is embedded in $\mu \tau X$ and $Y \subseteq X$ we can view $\mu \tau Y$ as a closed subspace of $\mu \tau X$. Now we put $Z = \mu \tau Y \cap X$ and $\varphi = \mu \tau g|Z$.

By the condition of the theorem every continuous mapping over $Z$ into $E$ admits a continuous extensions over $X$. Thus the continuous extension of $g$, $\varphi$ admits a continuous extension $\bar{g} : X \to E$.

We say that the family $\{F_\alpha : \alpha \in A\}$ of subsets of a space $X$ is functionally discrete if there exists a continuous mapping $g : X \to Y$ in some metrizable space $Y$ such that $\{g(F_\alpha) : \alpha \in A\}$ is a discrete family of subsets of the space $Y$.

**Theorem 3.2.** Let $Y$ be a subspace of the space $X$, $\tau$ be an infinite cardinal number, and for any continuous mapping $g : Z \to E$ of a closed subspace $Z$ of $X$ into a Banach space $E$ of the weight $\leq \tau$ there exists a continuous extension $\bar{g} : X \to E$. Then the following assertions are equivalent:

(i) $\mu \tau Y = \text{cl}\, \mu \tau X Y$,

(ii) For each continuous mapping $g : Y \to E$ into a Banach space $E$ of the weight $\leq \tau$ there exists a continuous extension $\bar{g} : X \to E$.

(iii) For each continuous mapping $g : Y \to E$ into a Fréchet space $E$ of the weight $\leq \tau$ there exists a continuous extension $\bar{g} : X \to E$. 
We establish firstly that the space $U \in \alpha \text{cl} \mu$ is an injection. Consider the mapping $H$ space with weight such that $\alpha \mid [5]$. $g \colon H$ is embeddable in $\tau$:

(iii) Follows from the fact that $\bar{g} : Y \rightarrow H$ is a continuous extension $\bar{g}$ of $g$. Then $V = \bar{g}^{-1}(U) \cap \alpha \subseteq A$ is a discrete family of open subsets of $X$ and $F_{\alpha} \subseteq V_{\alpha}$ for each $\alpha \in A$. Thus $X$ is $\tau$-collectionwise normal.

(i)→(ii) Follows from Theorem 3.1.

(i)→(iv) Let $E$ be a metrizable space of the weight less than or equal to $\tau$ and $g : Y \rightarrow E$ be a continuous mapping. There exists a continuous extension $\mu, g : \mu, Y \rightarrow E$ of $g$. Since $\mu, g \subseteq \mu, X$, we have that $\text{cl}_{\mu} Y = \mu, Y \cap X$. Thus $\mu, \bar{g}|_{\text{cl}_{\mu} Y} : \text{cl}_{\mu} Y \rightarrow E$ is a continuous extension of $g$.

(ii)→(i) Any metrizable space with weight $\leq \tau$ can be embedded in a Hilbert space with with weight $\leq \tau$. Let $H$ be the Hilbert space with $w(H) \leq \tau$ such that $E$ is embeddable in $H$.

Let $g : Y \rightarrow E \subseteq H$ be a continuous mapping and $\bar{g} : X \rightarrow E \subseteq H$ a continuous extension of $g$. Then $\bar{g}$ can be extended to $\bar{g} : \mu, X \rightarrow H$, but $\varphi = \bar{g}|_{\text{cl}_{\mu} X} Y$ is a continuous extension of $g$. Thus $\mu, Y = \text{cl}_{\mu, X} Y$.

(iii)→(v) It is obviously because every Fréchet space is a Banach space.

(iv)→(v) Let $\{F_{\alpha} : \alpha \in A\}$ be a functionally discrete family of the space $Y$ and $|A| \leq \tau$. Then there exists a metrizable space $Y_{1}$ and a continuous mapping $g : Y \rightarrow Y_{1}$ such that $g(F_{\alpha}) \subseteq \alpha \subseteq Y_{1}$ is a discrete family of subsets of $Y_{1}$. Then there exists a discrete family $\{U_{\alpha} : \alpha \in A\}$ of open subsets of $Y_{1}$ such that $\text{cl}_{Y_{1}} g(F_{\alpha}) \subseteq U_{\alpha}$ for each $\alpha \in A$. Thus for each $\alpha \in A$ there exists a continuous function $h_{\alpha} : Y \rightarrow l = [0, 1]$ such that $Y \setminus g^{-1}(U_{\alpha}) \subseteq h_{\alpha}^{-1}(0)$ and $F_{\alpha} \subseteq h_{\alpha}^{-1}(1)$.

Let $H_{1}$ be the Hilbert space of all function $f : A \rightarrow \mathbb{R}$ such that the set $f(A)$ is finite and $< f_{1}, f_{2} >= \sum[f_{1}(x) \cdot f_{2}(x) : x \in A]$. The completion $H$ of $H_{1}$ is a Hilbert (and a Banach) space of the weight $|A| \leq \tau$. For any $y \in Y$ consider the mapping $f_{y} : A \rightarrow \mathbb{R}$, where $f_{y}(\alpha) = h_{\alpha}(y)$ for any $\alpha \in A$. Then $\varphi : Y \rightarrow H_{1}$, where $\varphi(y) = f_{y}$, is a continuous mapping. Thus there exists a continuous extension $\mu, \varphi : \text{cl}_{\mu} Y \rightarrow E$.

If $y \in F_{\alpha}$ then $f_{\alpha}(\alpha) = 1$ and $f_{\alpha}(\beta) = 0$ for any $\beta \in A$ and $\alpha \neq \beta$. Thus $f_{y} = f_{\alpha}$ for all $y, z \in F_{\alpha}$ and $\alpha \in A$. We put $l_{\alpha} = f_{\alpha}$ for some $y \in F_{\alpha}$. Then $||l_{\alpha} - l_{\beta}|| = 2$ and $||l_{\alpha}|| = 1$ for all $\alpha, \beta \in A$ and $\alpha \neq \beta$. Hence $\{\varphi(F_{\alpha}) : \alpha \in A\}$ is a discrete family of points of
Let $Y$ be a subspace of the following assertions are equivalent:

1. $\sigma$ locally finite
2. $z = z$
3. $\phi$
4. $\tau$ less than or equal to $\beta$

Since $F \subseteq \Phi$, the implication (iv) $\rightarrow$ (v) is proved.

(v) $\rightarrow$ (iv) Let $g : Y \rightarrow E$ be a continuous mapping into a metric space $E$ of weight less than or equal to $\tau$. From condition (v) it follows that $cl_b \beta Y = \beta Y$, i.e., $\beta Y \subseteq \beta X$.

Let $Z = cl Y$. Then $Y \subseteq Z \subseteq X$ and $\beta Y = \beta Z \subseteq \beta X$.

Consider the continuous extension $\varphi : \beta Z \rightarrow \beta E$ of the function $g$. We affirm that $\varphi(Z) \subseteq E$.

Assume that $z_0 \in Z$ and $\varphi(z_0) \in \beta E \setminus E$. We firstly affirm that $z_0 \in \nu Y \subseteq \beta Z$.

Assume that $z_0 \in \beta Z \setminus \nu Z$.

In this case there exists a continuous function $h : \beta Z \rightarrow [0, 1]$ such that $h(z_0) = 0$ and $h(y) > 0$ for each $y \in Y [2, \text{Theorem 3.11.10}]$. We put $U_n = \{y \in Y : \frac{1}{n+1} < h(y) < \frac{1}{n}\}$. By construction, $U = \bigcup \{U_n : n \in \mathbb{N}\} = \{y \in Y : 0 < h(y) < 1\}$. Hence $z_0 \in cl_{\beta U} U$.

Then the families $\gamma_1 = \{cl_{\beta U} V_n : n \in \mathbb{N}\}$, $\gamma_2 = \{cl_{\beta U} V_{n-1} : n \in \mathbb{N}\}$, $\gamma_3 = \{cl_{\beta U} V_{n-1} : n \in \mathbb{N}\}$ and $\gamma_4 = \{cl_{\beta U} V_n : n \in \mathbb{N}\}$ are discrete in $Z$, i.e., the family $g = \{cl_{\beta U} V_n : n \in \mathbb{N}\}$ is locally finite. By construction, $z_0 \not\in cl_{\beta U} U_n$ for each $n \in \mathbb{N}$.

Hence, $z_0 \not\in cl_{\beta U} U_n$ for each $n \in \mathbb{N}$. Then $z_0 \not\in cl Y$. A contradiction, such that $z_0 \in \nu Y$.

A set $F \subseteq Y$ is a zero-set if $F = h^{-1}(0)$ for some continuous function $h$ on $Y$. In this case $Y \setminus E$ is called a co-zero-set of $Y$.

We assume that $\varphi(z_0) \in \beta E \setminus E$. For each $x \in E$ there exists an open subset $V_x$ of $E$ such that $x \in V_x$ and $\varphi(z_0) \not\in cl_{\beta E} V_x$. In this case $z_0 \not\in cl_x (g^{-1}(V_x))$. There exists a locally finite $\sigma$-discrete cover $\{W_\beta : \beta \in B\}$ such that:

- $|B| \leq \tau$ and for each $\beta \in B$ there exists $\chi(\beta) \in E$ such that $W_\beta \subseteq \nu(\beta)$;
- $B = \bigcup \{B_n : n \in \mathbb{N}\}$ and each family $\{W_\beta : \beta \in B_n\}$ is discrete in $E$.

Then $z_0 \not\in cl_x g^{-1}(W_\beta)$ for each $\beta \in B$. By construction, the family $\{g^{-1}(W_\beta) : \beta \in B_n\}$ is functionally discrete in $Y$. Hence, $\{cl_x g^{-1}(W_\beta) : \beta \in B_n\}$ is a discrete family of $X$ and $Z_n = \bigcup \{cl_x g^{-1}(W_\beta) : \beta \in B_n\}$ is a closed subset of $X$. By construction, $z_0 \in Z_n = \bigcup \{Z_n : n \in \mathbb{N}\}$. Since $Z$ is a normal space, there exists a continuous function $h : Z \rightarrow [0, 1]$ such that $h(z_0) = 0$ and $h(z) \geq 2^{-n}$ for all $z \in Z_n$ and $n \in \mathbb{N}$. Then $z_0 \not\in \nu Y$, a contradiction.

**Corollary 3.1.** Let $Y$ be a subspace of the $\tau$-collectionwise normal space $X$. Then the following assertions are equivalent:

(i) $\mu Y = cl_x Y$;
(ii) For each continuous mapping $g : Y \rightarrow E$ into a Banach space $E$ there exists a continuous extension $\tilde{g} : \nu Y \rightarrow E$.
(iii) For each continuous mapping $g : Y \rightarrow E$ into a Fréchet space $E$ there exists a continuous extension $\tilde{g} : Y \rightarrow E$.
(iv) For each continuous mapping $g : Y \rightarrow E$ into a complete metrizable space $E$ there exists a continuous extension $\mu Y \rightarrow E$.
(v) For each functionally discrete family $\{F_\alpha : \alpha \in A\}$ of the space $Y$ the family $\{cl X F_\alpha : \alpha \in A\}$ is discrete in $X$.
Example 3.1. Assume that $Y$ is a discrete space which is not a Hewitt-Nachbin complete space. Fix a countable non-empty subset $H \subseteq \nu \setminus Y$. We put $Z = Y \cup H$. Let $S = \{0\} \cup \{n^{-1} : n \in \mathbb{N}\}$ be the subspace of reals and $X = Y \times (H \times S)$ be a subspace of the space $Z \times S$. Then $X$ is a paracompact topologically complete space which is not a Hewitt-Nachbin complete space. By construction, $\dim X = 0$ and $Y \subseteq Z \subseteq \text{cl}_\nu X \subseteq \nu X$. For each continuous mapping $g : Y \to E$ into a complete separable metrizable space $E$ there exists a continuous extension $\bar{g} : X \to E$. If $Y$ is a closed subspace of the Banach space $E$ then the identical mapping $g : Y \to Y \subseteq E$ do not admits a continuous extension over $Z$ and $X$.

References


THE SPLITTING OPERATOR METHOD FOR CONVECTION-DIFFUSION
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Abstract This paper shows the solution of a convection-diffusion problem which is considered on a composite material. We apply a very good numerical method to approximate the limit problem: the splitting of the operator, more precisely, the Glowinski’s scheme of the fractional step. A problem of this type is often met to the components of some electronic devices.

Keywords: perforated domain with small holes, homogenization, operator splitting, finite element method.


1. INTRODUCTION
We consider a non-stationary diffusion problem on a domain which is perforated by holes with the diameter much smaller than the period, that are distributed in the domain. Then, we discuss the homogenization of the non-stationary diffusion problem with Robin conditions. The limit problem obtained is a non-stationary convection-diffusion problem considered on the cylindrical domain $Q = \Omega \times (0, T)$, where $\Omega$ is the initial fixed domain without holes.

The novelty of the present article is the approximation of the limit problem on $Q = \Omega \times (0, T)$. The fifth section presents the spatial discretization of the locale problems which were obtained after the homogenization process from the fourth section, using the finite element method. In the sixth section we approximate the limit problem of the section four, by combining the operator splitting - the Glowinski’s scheme of the fractional step for the temporal discretization (decomposition of the convection diffusion operator) - with the finite element method for the spatial discretization. The last section presents the result concerning the convergence of the approximation method.

2. THE PERFORATED DOMAIN
We consider the open and bounded domain $\Omega \subset \mathbb{R}^n$, with the Lipschitz border $\partial \Omega$, the reference cell $Y = (0, l_1) \times (0, l_2) \times \cdots \times (0, l_n)$ and an open domain $S \subset Y$ so that $\bar{S} \subset Y$ with smooth border $\partial S$. We take $r_\varepsilon << \varepsilon$ so that $\lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} = 0$ and $\lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon^2} = 0$. 

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We consider $\mathcal{T}(r_{e}S)$ the translated of $r_{e}\bar{S}$ with the form $(ekl + r_{e}\bar{S})$, where $k \in \mathbb{Z}^{n}$, $kl = (k_{1}l_{1}, k_{2}l_{2}, ..., k_{n}l_{n})$, representing the micro holes from $\mathbb{R}^{n}$. Let us denote by $S_{e} = \cup_{k \in \mathcal{K}_{e}}(ekl + r_{e}\bar{S})$, where $\mathcal{K}_{e} = \left\{ k \in \mathbb{Z}^{n} \big| (ekl + r_{e}\bar{S}) \cap \Omega \neq \emptyset \right\}$, $S_{e}$ the finite reunion of the holes from $\Omega$, which can intersect $\partial \Omega$.

We define now the perforated domain $\Omega_{e} = \Omega \setminus \bar{S}_{e}$ where the holes are distributed with period $\varepsilon$ and the diameter $r_{e}$ is much smaller than $\varepsilon$.

### 3. THE STATEMENT OF THE PROBLEM

We consider the following non-stationary diffusion problem in the perforated domain $\Omega_{e}$.

\[
\begin{aligned}
\frac{\partial u_{e}}{\partial t} - \text{div}(A_{e}\nabla u_{e}) + \mu_{e}u_{e} &= f_{e} \quad \text{in} \quad \Omega_{e} \times (0, T) \\
A_{e}\nabla u_{e} + \alpha_{e}u_{e} &= g_{e} \quad \text{on} \quad \Sigma_{e} \times (0, T) \\
u_{e} &= 0 \quad \text{on} \quad \partial \Omega \times (0, T) \\
u_{e}(0) &= u_{0}^{e} \quad \text{on} \quad \Omega_{e}
\end{aligned}
\]

where $A_{e}(x) = A\left(\frac{x}{\varepsilon}\right)$, $\mu_{e}(x) = \mu\left(\frac{x}{\varepsilon}\right)$, $\alpha_{e}(x) = \alpha\left(\frac{x}{\varepsilon}\right)$, $g_{e}(x) = g\left(x, \frac{x}{\varepsilon}\right)$ with $g \in L^{2}(\Omega \times \Sigma)$, and $v_{e}$ is the external normal to $\Sigma_{e}$.

We make the following assumptions:

1. $f_{e} \in L^{2}\left(\Omega_{e} \times (0, T)\right)$, $g_{e} \in L^{2}\left(\Sigma_{e} \times (0, T)\right)$. $\Sigma_{e} = \partial S_{e}$ represents the border of the holes from the domain $\Omega_{e}$. $\partial S_{e} = \Sigma$. The estimation $\|f_{e}\|_{L^{2}(\Omega_{e})} + \sqrt{e}\|g_{e}\|_{L^{2}(\Sigma_{e})} \leq c$ is true, where $c$ is a positive constant, independent of $\varepsilon$.

2. $A \in L^{\infty}_{\text{per}}(Y)$, $m|\xi|^{2} \leq A_{ij}(y)\xi_{i}\xi_{j} \leq \beta|\xi|^{2}$, $\forall \xi \in \mathbb{R}^{n}$ a.e. $y \in Y$.

3. $\mu \in L^{\infty}_{\text{per}}(Y)$, $\int_{Y}\mu(y)\,dy \geq \mu_{0} > 0$; $\alpha \in L^{2}_{\text{per}}(\Sigma)$ so that $\int_{\Sigma}\alpha(y)\,d\sigma(y) = 0$, $u_{0}^{e} \in L^{2}(\Omega_{e})$.

### 4. THE HOMOGENIZATION

The limit problem for the equation (1) is

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \text{div}\left(A^{\text{eff}}\nabla u(x, t)\right) + B\nabla u(x, t) + \lambda u(x, t) &= F(x, t) \\
u &= 0 \quad \text{on} \quad \partial \Omega \times (0, T) \\
u(0) &= u_{0} \quad \text{on} \quad \Omega
\end{aligned}
\]

where $\chi_{\Omega_{e}}u_{e}^{0} \overset{\text{weak}}{\rightharpoonup} u_{0}$, when $\varepsilon \to 0$ in $L^{2}(\Omega)$ and the entries of matrix $A^{\text{eff}}$ are given by:

\[
a^{\text{eff}}_{ik} = \frac{1}{|Y|} \int_{Y} a_{ij}(y) \frac{\partial (\chi_{k}(y) + y_{k})}{\partial y_{j}}\,dy
\]
and the correctors $\chi_k$ satisfy the local problem

$$
\begin{cases}
-\text{div}_y \left[ A(y) \nabla_y (y_j - \chi_j(y)) \right] = 0 & \text{in } Y \\
\chi_j \text{ is } Y - \text{periodically}, & j = 1, n.
\end{cases}
$$

(4)

where $B = (b_i)_{1 \leq i \leq n}$ is the convection vector.

$$
\lambda = \bar{\mu} + \int_{\Sigma} \alpha(y) \gamma(y) \, d\sigma(y),
$$

(6)

where $\bar{\mu} = \frac{1}{|Y|} \int_{Y} \mu(y) \, dy$, and $\gamma$ satisfies the local problem:

$$
\begin{cases}
-\text{div}_y \left[ A(y) \nabla_y (\gamma) \right] = 0 & \text{in } Y^*, \\
[A(y) \nabla_y (\gamma)] \nu = -\alpha(y) & \text{on } \Sigma, \\
\gamma \text{ is } Y - \text{periodically},
\end{cases}
$$

(7)

We consider the bidimensional case and we choose $Y = [0, 1]^2$, $S = \left(\frac{3}{8}, \frac{5}{8}\right)^2$.

5. THE SPATIAL DISCRETIZATION

The spatial discretization of the local problems is made with the finite element method. Because the local problems (4) and (7) are considered on two different domains $Y$, respectively $Y^*$, and these two problems are independent of each other, we will consider two different triangulations: $T_{h/2}$ on $Y$ and respectively $T_{h/4}$ on $Y^*$, with $h > 0$. In the following figures we present these two triangulations.

About the discretized coefficients $a^{eff}_{i\ell, k}, b_{i, h}$ and $\lambda_h$ we apply a quadrature scheme.

We consider the bidimensional case and we choose $Y = [0, 1]^2$, $S = \left(\frac{3}{8}, \frac{5}{8}\right)^2$.

Fig. 1.: The triangulation $T_{h/2}$. 
Fig. 2.: The triangulation $\mathcal{T}_{h/4}$.

We consider the finite dimensional spaces

$$V_{h/2} = \left\{ v_h \mid v_h \in C^0(\bar{Y}), v_{hK} \in P_{11}, \forall K \in \mathcal{T}_{h/2} \right\},$$

$$P_{11} = \left\{ p(x_1, x_2) = \sum_{0 \leq i, j \leq 1} c_{ij} x_1^i x_2^j, c_{ij} \in \mathbb{R} \right\},$$

$$W_{h/4} = \left\{ w_h \mid w_h \in C^0(Y^*), w_{hK} \in P_{11}, \forall K \in \mathcal{T}_{h/4} \right\},$$

and the subspaces

$$V_{\text{per}, h/2} = \left\{ v_h \in V_h \mid v_h(0, y_2) = v_h(1, y_2), v_h(y_1, 0) = v_h(y_1, 1), \forall y_1, y_2 \in [0, 1] \right\},$$

$$W_{\text{per}, h/4} = \left\{ w_h \in W_h \mid w_h \text{ is periodic} \right\}.$$

In this case, the spatial discretization of the local problems (4) and (7) is:

Find $\chi_{j,h} \in V_{\text{per}, h/2}$ so that

$$\int_Y A_{h/2}(y) \nabla y \chi_{j,h}(y) \cdot \nabla y v_h(y) \, dy = \int_Y A_h(y) \cdot e_j \cdot \nabla y v_h(y) \, dy, \quad \forall v_h \in V_{\text{per}, h/2},$$

respectively,

Find $\gamma_{h} \in W_{\text{per}, h/4}$ so that

$$\int_Y A_{h/4}(y) \nabla y \gamma_{h}(y) \cdot \nabla y w_h(y) \, dy = \int_{\Sigma} a_h(y) w_h(y) \, d\sigma(y), \quad \forall w_h \in W_{\text{per}, h/4},$$

where $A_{h/2}$, $A_{h/4}$ represents the approximations of the matrix $A(y)$ relative to $\mathcal{T}_{h/2}$, respectively $\mathcal{T}_{h/4}$.

The relations (9) and (10) are linear algebraic systems.
The discretized coefficients are obtained from the equations (3), (5) and (6):

\[
\begin{align*}
a_{ik}^{\text{eff}} &= \sum_{K \in T_h/2} \int_K a_{i;k} (y) \frac{\partial \chi_{i;k} (y)}{\partial y_j} dy, \\
b_{i;k} &= - \sum_{K \in T_h/4} \int_K a_{i;k} (y) \frac{\partial \gamma_{i;k}}{\partial y_j} dy + \\
&\quad \sum_{K \in T_h/4} \int_{\partial K \cap \Sigma} a_{i;k} (y) \chi_{i;k} (y) d\sigma (y), \\
\lambda_h &= \sum_{K \in T_h/2} \int_K \mu_{i;k} (y) dy + \\
&\quad \sum_{K \in T_h/4} \int_{\partial K \cap \Sigma} \alpha_{i;k} (y) \gamma_{i;k} (y) d\sigma (y),
\end{align*}
\]

where $\alpha_{i;k}$ and $\mu_{i;k}$ are the approximations of the functions $\alpha$ (to $\mathcal{T}_h/4$) and $\mu$ (to $\mathcal{T}_h/2$).

We apply the quadrature schemes to integrals.

Regarding the free term $F(x,t)$, the two integrals of the relation (8) are calculated using the quadrature scheme.

6. THE DISCRETIZATION

The discretization of the problem (2) using the operator splitting method and the finite element method.

We discretize the global problem (2) by combining the operator splitting method - the Glowinski's scheme of the fractional step - with the finite element method. We consider $\Omega$ a bounded polygonal domain from $\mathbb{R}^2$. Let $\mathcal{T}_h$ be a triangulation of $\Omega$. We introduce the spaces:

\[ W_h = \{ v_h \in C^0 (\bar{\Omega}) | v_{h,T} \in P_{11}, \forall T \in \mathcal{T}_h \}, \]

where $P_{11}$ is the space of polynomials in two variables with degree at most one;

\[ W_{0h} = \{ v_h \in W_h | v_h = 0 \text{ on } \partial \Omega \}. \]

We consider the partition of the interval $[0, T]$:

\[ 0 = t^0 < t^1 = \Delta t < \cdots < t^n = n \Delta t < t^{n+1} = (n + 1) \Delta t < \cdots < t^N = N \Delta t = T, \]

$\Delta t = \frac{T}{N}$ and $t^{n+\theta} = (n + \theta) \Delta t, \theta \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$.

We have the following scheme: we denote by $u^0$ = $u_0$ and let $u_h^0 = (u_0)_h$ an approximation of $u_0$. We assume that $u_h^n$ is known and we consider the following discrete variational problems:
Let find \( u_h^{n+\frac{1}{2}}, u_h^{n+\frac{3}{2}} \in W_h \) and \( u_h^{n+1} \in W_{0h} \) so that

\[
3 \int_{\Omega} \frac{u_h^{n+\frac{3}{2}}-u_h^{n+\frac{1}{2}}}{\Delta t} v_h dx + \int_{\Omega} A_{h}^{\text{eff}} \nabla u_h^{n+\frac{3}{2}} \nabla v_h dx =
\int_{\Omega} \left( F^n_h - B_h \nabla u^n_h - \lambda_h u^n_h \right) v_h dx,
\forall v_h \in W_h,
\]

\[
3 \int_{\Omega} \frac{u_h^{n+\frac{3}{2}}-u_h^{n+\frac{1}{2}}}{\Delta t} v_h dx + \int_{\Omega} B_h \nabla u_h^{n+\frac{3}{2}} v_h dx + \int_{\Omega} \lambda_h u_h^{n+\frac{3}{2}} v_h dx =
\int_{\Omega} \left( F^{n+\frac{1}{2}}_h v_h - A_{h}^{\text{eff}} \nabla u^{n+\frac{3}{2}} \nabla v_h \right) dx,
\forall v_h \in W_h,
\]

\[
3 \int_{\Omega} \frac{u_h^{n+1}-u_h^{n+\frac{3}{2}}}{\Delta t} v_h dx + \int_{\Omega} A_{h}^{\text{eff}} \nabla u_h^{n+1} \nabla v_h dx =
\int_{\Omega} \left( F^{n+\frac{1}{2}}_h v_h - A_{h}^{\text{eff}} \nabla u^{n+\frac{3}{2}} \nabla v_h \right) dx,
\forall v_h \in W_h,
\]

where \( F^n_h = F_h(t^n) \) in \( \Omega, F_h^{n+\frac{1}{2}} = F_h \left( t^{n+\frac{1}{2}} \right), F_h^{n+\frac{3}{2}} = F_h \left( t^{n+\frac{3}{2}} \right), \) and \( F_h \) is an approximation of \( F \) relative to \( T_h \) and \( t^{n+\theta} = (n + \theta) \Delta t, \theta \in \left\{ 0, \frac{1}{2}, \frac{2}{3}, 1 \right\} \).

Therefore, the switching from \( u^n_h \) to \( u_h^{n+1} \) is made passing through \( u_h^{n+\frac{1}{2}} \) and \( u_h^{n+\frac{3}{2}} \), practically breaking the interval \( (t^n, t^{n+1}) \) with the intermediary points \( t^{n+\frac{1}{2}} \) and \( t^{n+\frac{3}{2}} \), where \( t^{n+\theta} = (n + \theta) \Delta t, \theta \in \left\{ 0, \frac{1}{3}, \frac{2}{3}, 1 \right\}, n \in \{0,1,\ldots,N-1\} \), where we have the partition of the interval \([0,T]\):

\[
0 = t^0 < t^1 < \cdots < t^n < t^{n+1} < \cdots < t^N = T, \quad t^n = n \Delta t,
\]

and also by splitting the convection-diffusion operator, we denote by \( \mathcal{L}_1, \mathcal{L}_2 \) the operators:

\[
\begin{align*}
\mathcal{L}_1 &= -\text{div}(A_{h}^{\text{eff}} \nabla) \\
\mathcal{L}_2 &= B \nabla + \lambda.
\end{align*}
\]

This is the decomposition of the operator

\[
\mathcal{L} = -\text{div}(A_{h}^{\text{eff}} \nabla) + B \nabla + \lambda.
\]

7. CONCLUSION

In the section 6 we combined the finite element method with the operator splitting for the convection-diffusion operator and for the breaking of the interval \( (t^n, t^{n+1}) \), but before we partitioned the time interval \((0, T)\) such that

\[
0 = t^0 < t^1 = \Delta t < \cdots < t^n = n \Delta t < t^{n+1} =
\]
The splitting operator method for convection-diffusion

Fig. 3.: The interval \((t^n, t^{n+1})\).

\[ = (n + 1) \Delta t < \cdots < t^N = N \Delta t = T, \Delta t = \frac{T}{N}. \]

By the other hand, from the ellipticity of the coefficients and the Schwars’s inequality, we find the estimation

\[
\sum_{n=0}^{N-1} \| u_h^{n+\theta} \|_h^2 \leq \text{constant},
\]

for \(\theta \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}\), and \(\| \cdot \|_h\) is the norm on \(W_{0h}\) - the space which is the approximation of \(H^1_0(\Omega)\).

Also, we obtain the convergence

\[
u_h^{n+\theta} \xrightarrow{h \to 0} u(\cdot, t^{n+\theta}) \text{ strong in } L^2(\Omega),
\]

\(\theta \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}\).

References


ON THE ACCURACY OF PSEUDOSPECTRAL DIFFERENTIATION
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Abstract
For various grids on a finite interval we measure the accuracy of pseudospectral (collocation) differentiation matrices using two parameters. The first one is the the rank deficiency of the differentiation matrices. The second one quantifies the extent at which such matrices transform a constant vector into the null vector.

Keywords: pseudospectral differentiation; Chebyshev-Gauss-Lobatto grid; Legendre grid; equidistant grid; accuracy; floating-point arithmetic.

2010 MSC: 65D25 (Primary), 65M70; 65N35 (Secondary).

1. INTRODUCTION

Fundamental results from approximation theory refers to the best uniform approximation of a smooth function by polynomials and the uniform approximation of a smooth $2\pi$ periodic function by trigonometric polynomials. These results are reviewed for instance in the monograph [1], Ch. 3. Thus, the Chebyshev equioscillation theorem states that a best approximation is unique for important classes of approximating functions but the set of nodes on which this approximation is realized is not necessarily unique. On the other hand, the spectral collocation methods essentially depend on the set of nodes on which the differential equation is collocated. In order to implement these methods one needs very accurate differentiation matrices of various orders.

Consequently, the accuracy at which the pseudospectral (collocation) differentiation matrices operate is of utmost importance in numerical analysis. In this note we will address the issue of the dependence of the accuracy of differentiation process on the node distribution in a grid covering a finite interval. We will consider an equispaced grid, an arbitrary one and then Legendre and Chebyshev grids. For non algebraic polynomials we will consider the Fourier interpolate.
2. ARBITRARY AND EQUIDISTANT GRIDS

It is well established that spectral collocation methods for solving differential equations is based on weighted interpolants of the form

\[ u(x) \approx p_{N-1}(x) := \sum_{j=1}^{N} \frac{\alpha(x)}{\alpha(x_j)} \phi_j(x) u_j, \]  

(1)

where \( u_j := u(x_j), \ u: [a, b] \to \mathbb{R} \), the set of interpolating functions \( \phi_j(x) \), \( j = 1, \ldots, N \) satisfy \( \phi_j(x_k) := \delta_{jk} \) (the Kronecker delta), the set of nodes \( x_j, \ j = 1, \ldots, N \) in \([a, b] \subset \mathbb{R}\) is distinct but otherwise arbitrary and the weight \( \alpha(x) \) is an arbitrary continuously differentiable positive function (see for instance our contribution [3]). Typically, \( u(x) \) is the solution of an initial/boundary value problem.

The baricentric form of interpolation polynomial writes (see for instance the seminal paper [7])

\[ p_{N-1}(x) = \frac{\alpha(x)}{\sum_{j=1}^{N} \frac{w_j}{x-x_j} \alpha(x_j) \phi_j(x)} \]

where \( w_j^{-1} := \prod_{m=1, m \neq j}^{N} \left( a - x_m \right) \). This means that \( p_{N-1}(x) \) defined above is an interpolant of the function \( u(x) \) in the sense that

\[ u(x_k) = p_{N-1}(x_k), \ k = 1, \ldots, N. \]

The collocation derivative operators are generated by taking various order derivatives of (1) and evaluating them at nodes \( x_k, \ k = 1, \ldots, N \), i.e.,

\[ u^{(l)}(x_k) \approx \sum_{j=1}^{N} \frac{d^l}{dx^l} \left[ \frac{\alpha(x)}{\alpha(x_j)} \phi_j(x) \right]_{x=x_k} u_k, \ k = 1, \ldots, N. \]

Consequently, the \( l-th \) order differentiation matrices associated to this operator are computed by

\[ D^{(l)}_{k,j} = \frac{d^l}{dx^l} \left[ \frac{\alpha(x)}{\alpha(x_j)} \phi_j(x) \right]_{x=x_k}, \ k, j = 1, \ldots, N, \ l \in \mathbb{N}, \]

(2)

where \( \phi_j(x) \) are given by Lagrange’s formula

\[ \phi_j(x) := \prod_{m=1, m \neq j}^{N} \left( \frac{x-x_m}{x_j-x_m} \right), \ j = 1, \ldots, N. \]
The approximation theory dictates that the set of nodes \( x_j, j = 1, \ldots, N \) cannot be just any set of nodes. The main aim of this note is to make this statement more clear. Thus we use the MATLAB code `poldif.m` from [7] in order to perform the differentiation in (2).

In order to quantify the performances of every set of nodes we compute two specific parameters, namely:

- the norm of the error in approximating the zero vector, i.e., \( \|D^{(1)} \cdot 1_N\| \) where \( 1_N := \text{ones}(N, 1) \);
- the rank \( \text{rank}(D^{(1)}) \) for various values of approximation parameter \( N \).

Instead of the first parameter we could use another one reflecting the fact that the matrix \( D^{(1)} \) has to satisfy

\[
\sum_{j=1}^{N} D^{(1)}_{ij} = 0, \ 1 \leq i \leq N,
\]

i.e., the derivative of a constant vanishes.

Let’s consider a set of equidistant nodes

\[
x_j := \frac{j - 1}{N - 1}, \ j = 1, \ldots, N,
\]

where \( N \) takes in turn the value from the first row of Table 3.1. Thus the interval \([0, 1]\) is successively divided in \( N - 1 \) subintervals each of length \( 1/(N - 1) \).
Along with the equidistant grid (3) we consider first an arbitrary one, namely

$$\tilde{x}_j := \left( \frac{j-1}{N-1} \right)^2, \ j = 1, \ldots, N.$$  \hspace{1cm} (4)

It is fairly clear that this new grid is a non uniform one with nodes clustering to the left margin 0. The extent at which the differentiation matrices perform on both grids is depicted in Fig. 1. The error in approximating the zero vector takes huge values. The collocation derivatives of the hat function and of a highly oscillatory but fairly smooth function, i.e., $\exp(\sin(nx))$, with $n = 10$ on 20 equispaced nodes are depicted in Fig. 2. They show large oscillations which cluster to the ends of the interval $[-1, 1]$. They are the direct consequence of the numerical instability of the differentiation process.

It is well known that given $N$ nodes each of the differentiating matrices should be rank $N - 1$, a differentiating has the constant vector as its null space. Thus, it is important to point out that this equispaced approach works accurately for a very small number of nodes. As differentiating matrix $D^{(1)}$ should be rank one deficient, this simple test shows that even for a very rough approximation $D^{(1)}$ has additional null-spaces. Moreover, in case of quadratically spaced grid $\tilde{x}_j$, the differentiation matrices
On the accuracy of pseudospectral differentiation

Table 1: The evolution of rank deficiency of differentiation matrices for equidistant grid (3) and squared grid. (4)

<table>
<thead>
<tr>
<th>N</th>
<th>11</th>
<th>21</th>
<th>31</th>
<th>41</th>
<th>51</th>
<th>61</th>
<th>71</th>
<th>81</th>
<th>91</th>
<th>101</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{rank}(D^{(1)})-(3)$</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>37</td>
<td>42</td>
<td>9</td>
<td>8</td>
<td>8</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>$\text{rank}(D^{(1)})-(4)$</td>
<td>10</td>
<td>19</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

become much more degenerated. Table 3.1 shows that the situation dramatically deteriorates as $N$ increases.

3. CONSECRATED GRIDS

In this section we will analyze some well known grids such as Legendre, Chebyshev and Fourier applied to a finite interval. First, let’s reconsider two illustrative examples, the Chebyshev and Legendre derivatives of the hat function $h(x) := \max(0, 1 - \text{abs}(x))$ and of the smooth but fairly oscillatory function $\exp(\sin(nx))$, $n > 2$.

![Fig. 3: Chebyshev and Legendre derivatives of hat function and $\exp(\sin(nx))$, $n = 10$. The order of approximations equals 128.](image)

The differentiation on the roots of Legendre polynomials presents an intermediate situation. As it is apparent from Fig. 4 the differentiation matrices for $N \leq 700$
perform satisfactorily well. Our numerical experiments have showed that all differentiation matrices keep a rank of order \(N - 1\). Most notably, for \(N\) larger than 700 the Legendre differentiation process rapidly becomes unstable.

The best situation with polynomial differentiation is encountered when Chebyshev nodes of second kind

\[ x_k := \cos \left( \frac{(k - 1)\pi}{N - 1} \right), \; k = 1, \ldots, N, \tag{5} \]

or equivalently Chebyshev-Gauss-Lobatto quadrature nodes (see for instance [5] or our contribution [4] p. 11) are used. The corresponding differentiation matrix has the entries (see for instance [2], p. 69)

\[
D^{(1)}_{kj} = \begin{cases} 
\frac{c_k(-1)^{j-k}}{c_j(x_k-x_j)}, & j \neq k, \; j, k = 1, 2, \ldots, N, \\
\frac{2}{N(1-x_k^2)}, & j = k \neq 1, N, \\
\frac{2(N-1)^2+1}{6}, & j = k = 1, \\
\frac{2(N-1)^2+1}{6}, & j = k = N.
\end{cases}
\]

The above differences \((x_k - x_j)\) may be subject to floating-point cancellation errors for large \(N\). Various tricks based on simple trigonometric identities have been used in [7] in order to avoid such errors in floating-point arithmetic. Thus, the MATLAB code chebdif.m has been fairly stable algorithm in computing these matrices.

The upper curve in Fig 5 correspond to this situation.
Beyond this polynomial differentiation we will pay a particular attention to the Fourier differentiation matrices (see [7]). The Fourier interpolate reads

\[ t_N(x) := \sum_{j=1}^{N} \phi_j(x) u_j, \]

where

\[ \phi_j(x) := \frac{1}{N} \sin \frac{N}{2}(x-x_j) \cot \frac{1}{2}(x-x_j), \ N \text{ even,} \]
\[ \phi_j(x) := \frac{1}{N} \sin \frac{N}{2}(x-x_j) \csc \frac{1}{2}(x-x_j), \ N \text{ odd,} \ j = 1, 2, \ldots, N. \]

Using the baricentric form of the interpolate (see [6] Sect. 13.6) the MATLAB code fourdif.m from [7] provides the Fourier differentiation matrices on the nodes

\[ x_k := (k-1)h, \ h = \frac{2\pi}{N}, \ k = 1, \ldots, N. \]

Their performances are the best as it is apparent from Fig. 5 (see the lower curve). For \( N = O\left(2^6\right) \) it is of utmost importance to underline that Fourier and even Chebyshev differentiation matrices work fairly close to the machine precision. Excellent approximations are also attained when the cut off parameter ranges up to \( 2^{11} \). It is also important to notice that all differentiation matrices conserve a correct rank. It is a practical illustration of the spectral accuracy.

Fig. 5.: Semilogy plots of the accuracy of Fourier differentiation (starred line Euclidean norm, circled line inf norm) vs \( N \) and of the accuracy of Chebyshev differentiation (diamonded line Euclidean norm, squared line inf norm) vs \( N \).
4. CONCLUDING REMARKS

In ([7]) p.478 the authors state that no general error analysis applicable to differentiation process on arbitrary grids has been undertaken. We hope that the present note fills this gap at least partially. Fourier and Chebyshev differentiation matrices have proved to be fairly reliable. This facts explain to some extent the success of the collocation spectral methods based on them.

References

A GOODNESS-OF-FIT TEST OF THE ERRORS IN NONLINEAR AUTOREGRESSIVE TIME SERIES MODELS WITH STATIONARY \( \alpha \)-MIXING ERROR TERMS

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Abstract
In this work we deal with the problem of fitting an error density to the goodness-of-fit test of the errors in nonlinear autoregressive time series models with stationary \( \alpha \)-mixing error terms. The test statistic is based on the integrated squared error of the nonparametric error density estimate and the null error density. By deriving the asymptotic normality of test statistics in these models, we extend the result of Cheng and Sun (Statist. Probab. Lett. 78, 1(2008), 50-59) in the model with i.i.d error terms to the more general case.

Keywords: autoregressive process, goodness-of-fit test, error density estimation.


1. INTRODUCTION

The purpose of the goodness-of-fit tests is to test hypotheses on the empirical distributions fitting some theoretical law. Much recent work has been devoted to the goodness-of-fit tests of the errors in variant models; see for example [1, 5, 6]. In autoregressive time series models, the goodness-of-fit tests based on the residual empirical process have been extensively studied; for more details concerning them, we refer to [4, 7]. Bachmann and Dette [2] studied the Bickel-Rosenblatt test by considering the asymptotic behaviour of the test statistic under a fixed alternative. They proved that, under such conditions, a standardized version of the Bickel-Rosenblatt test statistic based on i.i.d. observations is asymptotically normal distributed, but with a different rate of convergence. Cheng and Sun [5] derived asymptotic normality of the Bickel-Rosenblatt test statistic in nonlinear autoregressive time series models with i.i.d. errors. We extend this result to nonlinear autoregressive time series models with stationary \( \alpha \)-mixing error terms.

Let \( \{X_i, i = 0, \pm1, \pm2, \cdots \} \) be a strictly stationary process of real random variables obeying the model

\[
X_i = r_\theta(X_{i-1}, \cdots , X_{i-p}) + \varepsilon_i
\]

for some \( \theta = (\theta_1, \cdots , \theta_d)^T \in \Theta \subset R^q \), where \((r_\theta; \theta \in \Theta)\) is a family of known measurable functions from \(R^p\) to \(R\). Unlike in [5], we assume that the errors \(\varepsilon_i\) are \(\alpha\)-

mixing random variables with common density $f$ and $X_{i-1}, \cdots, X_{i-p}$ are independent of $\{e_i, i = 1, 2, \cdots\}$. We focus on the problem of testing the hypothesis

$$H_0 : f = f_0 \ vs. \ H_1 : f \neq f_0,$$

where $f_0$ is a prescribed density based on the data $\{X_{1-p}, \cdots, X_0, X_1, \cdots, X_n\}$. We perform a test using the integrated square deviation of a kernel type density estimator based on the residuals from the expectation of the kernel error density based on the true errors. Let $\hat{\theta} = (\hat{\theta}_1, \cdots, \hat{\theta}_q)^T$ be an estimator of $\theta$, and define the residuals for $i = 1, 2, \cdots$,

$$\hat{e}_i = X_i - r_\theta(X_{i-1}, \cdots, X_{i-p}).$$

For a kernel density function $K$, the kernel type estimator of the error density $f(t)$ is defined as

$$\hat{f}_n(t) = \frac{1}{n} \sum_{i=1}^{n} K_{h_n}(t - \hat{e}_i), \quad t \in \mathbb{R},$$

where $K_h(\cdot) = (1/h)K(\cdot)$ is a scaled kernel and $h = h_n$ denotes a bandwidth tending to zero. We also define the kernel error density based on the true errors $e_1, \cdots, e_n$, which we cannot observe, as follows:

$$f_n(t) = \frac{1}{n} \sum_{i=1}^{n} K_{h_n}(t - e_i), \quad t \in \mathbb{R}.$$

For the problem of testing the hypothesis (1.2) we use the integrated squared deviation of $\hat{f}_n$ from

$$E f_n(t) = \int K(x)f(t - h_nx)dx = K_h * f(t),$$

where $K_h * f$ denotes the convolution of the functions $K_h$ and $f$, i.e., we reject the null-hypothesis $H_0 : f = f_0$ for large values of the statistic

$$\hat{T}_n = \int [\hat{f}_n(t) - K_h * f_0(t)]^2 dt.$$

This $\hat{T}_n$ is an analogue of the Bickel-Rosenblatt statistic proposed in the case of the observable $e_i$’s

$$T_n = \int [f_n(t) - E(f_n(t))]^2 dt,$$

see [3].

2. BASIC ASSUMPTIONS AND PRELIMINARIES

In this section we introduce some basic assumptions on the nonlinear autoregressive model (1) and the estimator and give some preliminaries which can be used to
prove our main results. The same assumptions on the autoregression function $r_\theta$ and the estimator $\hat{\theta}$ for $\theta$ as in [5] are adopted here. Throughout the paper we assume that limits are taken as $n \to \infty$ unless otherwise specified.

(A_1). Let $U \subset \Theta \subset \mathbb{R}^q$ be an open neighborhood of $\theta$. We assume that, for all $y \in \mathbb{R}^p, \theta = (\theta_1, \cdots, \theta_q) \in U$ and $j, k = 1, \cdots, q$,

$$\left| \frac{\partial}{\partial \theta_j} r_\theta(y) \right| \leq M_1(y)$$

$$\left| \frac{\partial^2}{\partial \theta_j \partial \theta_k} r_\theta(y) \right| \leq M_2(y),$$

where $EM_1^1(X_{i-1}, \cdots, X_{i-p}) < +\infty$ and $EM_2^1(X_{i-1}, \cdots, X_{i-p}) < +\infty$ for $i \geq 1$.

For all $1 \leq i \leq n$ and $1 \leq j \leq q$, let

$$Y_{ij} = \frac{\partial}{\partial \theta_j} r_\theta(X_{i-1}, \cdots, X_{i-p}).$$

(A_2). We assume that there exists $\alpha < 1$ such that $Y_{ij}$’s satisfy

$$\sum_{i=1}^n Y_{ij} = O_p(n^\alpha), \: j = 1, 2, \cdots, q.$$

(A_3). We assume that the estimator $\hat{\theta} = (\hat{\theta}_1, \cdots, \hat{\theta}_q)^T$ for $\theta$ (based on $X_0, X_1, \cdots, X_n$) satisfies the law of iterated logarithm, i.e., there exists a constant $C_1(0 < C_1 < \infty)$ such that

$$\lim \sup_{n \to \infty} \sqrt{\frac{n}{\log(\log n)}} |\hat{\theta} - \theta| \leq C_1,$$

where $|\hat{\theta} - \theta| = \sqrt{\sum_{j=1}^q (\hat{\theta}_j - \theta_j)^2}$.

In this work we derive the asymptotic distribution of $\hat{T}_n$ under $H_0$. In order to calculate the probability of type II error when $\hat{T}_n$ is used to test hypothesis (2), we consider the asymptotic distribution of $\hat{T}_n$ under one fixed alternative in $H_1$ of (2) in the sense of

$$d(f, f_0) = \int (f - f_0)^2(x)dx > 0.$$ 

Next we describe some basic assumptions on the error density $f$, the kernel density $K$ and the bandwidth $h_n$.

(D). $f$ is two time continuously differentiable with bounded first and second derivatives, and $f^2$ is integrable.

(K). (K_1) $K$ is a continuous bounded symmetric kernel with compact support.
(K2) \( K''' \) exists and is bounded. \( K', (K')^2, K'' \) and \( (K'')^2 \) are integrable.

(H) \( nh_n^2 \to \infty \) and \( h_n \to 0 \).

Note that assumption (K1) implies that
\[
\int x^2 K(x) dx < \infty \quad \text{and} \quad \int K^2(x) dx < \infty.
\]

Under the above assumptions, Cheng and Sun [5] established the following results.

**Lemma 2.1.** Under assumptions (A1) and (A3), we have
\[
\sum_{i=1}^{n} (\hat{\varepsilon}_i - \varepsilon_i)^2 = O_p(\log(\log n)).
\]

**Lemma 2.2.** Under assumptions (D) and (K), we have
\[
\begin{align*}
(i) & \int \left[ E(K'\left(\frac{t - \varepsilon_i}{h_n}\right)) \right]^2 dt = O(h_n^2), \quad \int E(K'\left(\frac{t - \varepsilon_1}{h_n}\right))^2 dt = O(h_n) \\
(ii) & \int \left[ E(K''\left(\frac{t - \varepsilon_i}{h_n}\right)) \right]^2 dt = O(h_n^2), \quad \int E(K''\left(\frac{t - \varepsilon_1}{h_n}\right))^2 dt = O(h_n).
\end{align*}
\]

**Lemma 2.3.** Suppose that assumptions (A1) – (A3), (D), (K) and (H) hold and the bandwidth \( h_n \) satisfies the following condition
\[
n^{-1}h_n^4(\log(\log n))^2 \to 0
\]
and, moreover,
\[
n^{2(\alpha-1)}h_n^{-3/2}\log(\log n) \to 0.
\]

Then we have
\[
\int \left[ \hat{f}_n(t) - f_n(t) \right]^2 dt = O_p \left( \frac{\log(\log n)^2}{n^2h_n^4} + \frac{\log(\log n)}{n^3-2\alpha h_n^2} \right) = o_P \left( \frac{1}{n \sqrt{h_n}} \right) \quad (3)
\]

**3. MAIN RESULTS**

In this section we derive the asymptotic normality of the Bickel-Rosenblatt test statistic in nonlinear autoregressive time series models with stationary \( \alpha \)-mixing error terms.

We start with the following property of stationary \( \alpha \)-mixing random variables.

**Lemma 3.1.** Suppose that the stationary sequence \( \{X_t\} \) satisfies \( \alpha \)-mixing condition. If the random variables \( \xi \) and \( \eta \) are measurable for \( \mathcal{F}_{-\infty}^t \) and \( \mathcal{F}_{t+\tau}^\infty \) and \( |\xi| < C_1, |\eta| < C_2 \) then we obtain
\[
|E\xi \eta - E\xi \cdot E\eta| \leq 4C_1C_2\alpha(\tau).
\]
A goodness-of-fit test of the errors in nonlinear autoregressive time series models... 

The proof of this lemma is simple, so is omitted.

We are now in position to formulate our main results in this exposition.

**Theorem 3.1.** Suppose that assumptions (D), (K) and (H) are satisfied. Then Bickel-Rosenblatt test statistics

\[ T_n = \int [f_n(t) - K_h * f_0(t)]^2 dt \]

has the following properties:

(i) Under the null hypothesis \( H_0 : f = f_0 \), as \( n \to \infty \)

\[ n \sqrt{h_n} \left[ T_n - \frac{1}{n h_n} \int K^2(x) dx \right] \to N(0, 2 \int f_0^2(x) dx \int (K * K)^2(x) dx). \]  

(ii) Under the alternative \( H_1 : f \neq f_0 \), as \( n \to \infty \)

\[ \sqrt{n} \left[ T_n - \int (K_h * (f - f_0))^2(x) dx \right] \to N(0, 4 \text{Var}[(f - f_0)(\varepsilon_1)]). \]

**Proof.** Recall that we are establishing the asymptotic normality under the null hypothesis \( f = f_0 \) and under fixed alternatives \( f \neq f_0 \) with different rates of convergence in both cases. Let \( f \) denote the “true” density of the random variables \( \varepsilon_i \). By the definition of the statistic \( T_n \) and the density estimate \( f_n \), we obtain the following decomposition:

\[ T_n = \int [f_n - K_h * f_0]^2(x) dx \]

\[ = \int [f_n - K_h * f]^2(x) dx + 2 \int [f_n - K_h * f](x) g_h(x) dx + \int g_h^2(x) dx \]

\[ = \frac{2}{n^2} \sum_{i \neq j} \int [K_h(x - \varepsilon_i) - e_h(x)][K_h(x - \varepsilon_i) - e_h(x)] dx \]

\[ + \frac{2}{n} \sum_{i=1}^{n} [(K_h * g_h)(\varepsilon_i) - E[(K_h * g_h)(\varepsilon_i)]] \]

\[ + \frac{1}{n^2} \sum_{i=1}^{n} [K_h(x - \varepsilon_i) - e_h(x)]^2 dx + g_h^2(x) dx. \]

where the functions \( e_h, g_h \) are defined by \( e_h = K_h * f \) and \( g_h = K_h * (f - f_0) \), respectively. Simple calculation implies

\[ \frac{1}{n^2} \sum_{i=1}^{n} \int [K_h(x - \varepsilon_i) - e_h(x)]^2 dx = \frac{1}{nh} \int K^2(x) dx + O_p(\frac{1}{n}). \]
and therefore we have the stochastic expansion

\[ T_n = \frac{1}{nh} \int K^2(x)dx - \int [K_h * (f - f_0)]^2(x)dx = \frac{2}{n^2} \sum_{i<j} H_n(e_i, e_j) + \frac{2}{n} \sum_{i=1}^{n} Y_i + O_p\left(\frac{1}{n}\right), \]

where

\[ H_n(e_i, e_j) = \int [K_h(x - e_i) - e_h(x)][K_h(x - e_j) - e_h(x)]dx, \]

\[ Y_i = (K_h * g_h)(e_i) - E[K_h * g_h](e_i). \]

Denote the first term in this decomposition as

\[ U_n = \frac{2}{n^2} \sum_{i<j} H_n(e_i, e_j). \]

Note that \( U_n \) does not depend on the density \( f_0 \) specified by the null hypothesis. It is clear that \( H_n \) is symmetric and

\[ \lim_{n \to \infty} E[H_n(\epsilon_1, \epsilon_2)|\epsilon_1] = 0, \quad \lim_{n \to \infty} E[H_n^2(\epsilon_1, \epsilon_2)] < \infty \]

for each \( n \in \mathbb{N} \). In fact,

\[ E[H_n(\epsilon_i, \epsilon_j)|\epsilon_i] = \int E[(K_h(x - \epsilon_i) - e_h(x))(K_h(x - \epsilon_j) - e_h(x))|\epsilon_i]dx. \]

Denote the random variable in the integrate symbol by \( \eta \), then we have

\[ E\left[ E[\eta|\mathcal{Y}^{0}_{-\infty}] - E\eta\right] = E[\xi_1(E[\eta|\mathcal{Y}^{0}_{-\infty}] - E\eta)], \]

where \( \xi_1 = \text{sgn}(E[\eta|\mathcal{Y}^{0}_{-\infty}] - E\eta) \) and it is measurable for \( \mathcal{Y}^{0}_{-\infty} \). It follows (via \( |\eta| \leq 4 \)) that

\[ |E\xi_1 \eta - E\xi_1 E\eta| \leq 4|E\xi_1 \eta_1 - E\xi_1 E\eta_1|, \]

where \( \eta_1 = \text{sgn}(E[\xi_1|\mathcal{Y}^{0}_{-\infty}] - E\xi_1). \)

By Lemma 3.1, we have

\[ |E\xi_1 \eta_1 - E\xi_1 E\eta_1| \leq 4\alpha(\tau), \]

and therefore

\[ E\left[ E[\eta|\mathcal{Y}^{0}_{-\infty}] - E\eta\right] \leq 16\alpha(\tau), \]

where \( \tau = |i - j| \). Since \( \{\epsilon_i\} \) is a strictly mixing sequence with coefficient \( \alpha(\tau) \), the left-hand side of above equation converges zero as \( \tau \to \infty \). And we have

\[ EK_h(x - \epsilon_i) = K_h * f(x) = e_h. \]
Let 
\[ \xi = K_h(x - \varepsilon_i), \eta = K_h(x - \varepsilon_j), \]
then by Lemma 3.1 we obtain \( |E \eta| \leq 4 \alpha(\tau) \), which implies that 
\[ \lim_{n \to \infty} E[H_n(\varepsilon_i, \varepsilon_j) | \varepsilon_j] = 0. \]
The other moment limit results can be proved in the same way. Applying the central limit theorem for degenerate U-statistics completes the proof. \[ \blacksquare \]

**Theorem 3.2.** Suppose that assumptions (A1) – (A3), (D), (K) and (H) are satisfied and that the bandwidth \( h_n \) satisfies the following:
\[ n^{2(a-1)} h_n^{-2} \log(\log n) \to 0, \] 
\[ n^{-1} h_n^{-4}(\log(\log n))^2 \to 0. \]
Then the test statistics \( \hat{T}_n \) has the following properties:

(i) Under the null hypothesis \( H_0 : f = f_0 \), as \( n \to \infty \),
\[ n \sqrt{h_n} \left( \hat{T}_n - \frac{1}{nh_n} \int K^2(x)dx \right) \to N\left(0, 2 \int f_0^2(x)dx \int (K * K)^2(x)dx \right). \]

(ii) Under the alternative \( H_1 : f \neq f_0 \), as \( n \to \infty \),
\[ \sqrt{n} \left( \hat{T}_n - \int (K_h * (f - f_0))^2(x)dx \right) \to N\left(0, 4 \text{Var}[(f - f_0)(\varepsilon_1)]\right). \]

**Proof.** First we prove (i). By (4), it suffices to show that 
\[ n \sqrt{h_n}(\hat{T}_n - T_n) = o_p(1) \]
From the definition of \( \hat{T}_n \) and \( T_n \), we obtain 
\[ |\hat{T}_n - T_n| \leq \int (f_n(t) - \dot{f}_n(t))^2 dt + 2 \int (\dot{f}_n(t) - f_n(t))^2 dt \left\{ 1/2 \sqrt{T_n} \right. \]
Using Lemma 2.2 and the fact 
\[ T_n = O_p\left( \frac{1}{nh_n} \right) \]
obtained from (4), we have
\[ |\hat{T}_n - T_n| = o_p\left( \frac{1}{n \sqrt{h_n}} \right) + O_p\left( \frac{1}{n \sqrt{h_n}} \sqrt{ \frac{(\log(\log n))^2}{nh_n^4} + \frac{\log(\log n)}{n^{2-2\alpha} h_n^2} } \right) \]
\[ = o_p\left( \frac{1}{n \sqrt{h_n}} \right). \]
Here (6) and (7) were also used. This completes the proof of (i).

Next we prove (ii). By (5), it suffices to show that
\[ \sqrt{n}(\hat{T}_n - T_n) = o_p(1). \]

Again by (5) we obtain
\[ T_n = O_p(1), \tag{9} \]
and from (8), (9) and (3), it follows that
\[
|\hat{T}_n - T_n| = o_p\left(\frac{1}{\sqrt{n}} \sqrt{nh_n}\right) + o_p\left(\frac{1}{\sqrt{n}} \sqrt{\frac{\left(\log(\log n)^2\right)}{nh_n^4} + \frac{\log(\log n)}{n^{2-2\alpha}h_n^2}}\right)
\]
\[ = o_p\left(\frac{1}{\sqrt{n}} \sqrt{nh_n}\right). \]

Here we also used (6), (7) and the fact \( nh_n \to \infty \), which is guaranteed by (7) and assumption \( H \). Now the proof of (ii) is straightforward. \( \blacksquare \)

References


MULTIPLE POSITIVE SOLUTIONS FOR A SECOND-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM
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Abstract
By using the fixed point index theory, we prove the existence and multiplicity of positive solutions of a coupled system of second-order nonlinear ordinary differential equations with multi-point boundary conditions.

Keywords: second-order differential system, multi-point boundary conditions, positive solutions.
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1. INTRODUCTION
We consider the coupled system of nonlinear second-order ordinary differential equations

\[
\begin{align*}
(a(t)u'(t))' - b(t)u(t) + f(t, v(t)) &= 0, \quad 0 < t < T, \\
(c(t)v'(t))' - d(t)v(t) + g(t, u(t)) &= 0, \quad 0 < t < T,
\end{align*}
\]

with the multi-point boundary conditions

\[
\begin{align*}
a u(0) - \beta a(0) u'(0) &= m \sum_{i=1}^{m} a_i u(\xi_i), \quad \gamma u(T) + \delta a(T) u'(T) = \sum_{i=1}^{n} b_i u(\eta_i), \\
\tilde{a} v(0) - \tilde{\beta} c(0) v'(0) &= \sum_{i=1}^{r} c_i v(\zeta_i), \quad \tilde{\gamma} v(T) + \tilde{\delta} c(T) v'(T) = \sum_{i=1}^{l} d_i v(\rho_i),
\end{align*}
\]

where \(m, n, r, l \in \mathbb{N} = \{1, 2, \ldots\}\).

By using some fixed point index theorems, we prove that under sufficient conditions on \(f\) and \(g\), the above problem has positive solutions. By a positive solution of problem \((S) - (BC)\) we mean a pair of functions \((u, v) \in (C^2([0, T]))^2\) satisfying \((S)\) and \((BC)\) with \(u(t) \geq 0, v(t) \geq 0\) for all \(t \in [0, T]\) and \(\sup_{t \in [0, T]} u(t) > 0\), \(\sup_{t \in [0, T]} v(t) > 0\). This problem is a generalization of the one studied in [2], where \(a(t) = c(t) = 1, b(t) = d(t) = 0\) for all \(t \in [0, T]\) in the system \((S)\) (denoted by \((\tilde{S})\))

1
and \( \alpha = \bar{\alpha} = 1, \beta = \bar{\beta} = 0, a_i = 0 \) for all \( i = 1, \ldots, p \), \( c_i = 0 \) for all \( i = 1, \ldots, r \), \( \gamma = \bar{\gamma} = 1 \) and \( \delta = \bar{\delta} = 0 \) in \( (BC) \). Some particular cases of the problem from [2] have been investigated in [7] (the system \( (S) \) with the boundary conditions \( (BC) \)): \( u(0) = 0, u(1) = \alpha u(\eta), v(0) = 0, v(1) = \alpha v(\eta), \eta \in (0, 1), 0 < \alpha \eta < 1, T = 1 \), and in [5] (where the authors studied the problem \( (S) - (BC) \) with \( f, g \) singular functions at \( t = 0 \) and/or \( t = 1 \) \( (T = 1) \) by using the Guo-Krasnosel’skii fixed point theorem of cone expansion and compression). For some systems of higher-order nonlinear ordinary differential equations subject to multi-point boundary conditions, we mention the papers [3] and [6].

In Section 2, we present some auxiliary results from [4] which investigate a boundary value problem for second-order equations. In Section 3, we prove some existence and multiplicity results for positive solutions with respect to a cone for our problem \( (S) - (BC) \) which are based on three fixed point index theorems of Amann ([1], see also Lemma 1 and Lemma 2 from [7]) and Zhou et al. (Lemma 3 from [7]). An example is presented in Section 4 which illustrates our main results.

## 2. AUXILIARY RESULTS

In this section, we present some auxiliary results from [4] related to the second-order differential equation with multi-point boundary conditions

\[
\begin{align*}
(a(t)u'(t))^\prime - b(t)u(t) + \gamma(t) & = 0, \quad t \in (0, T), \\
au(0) - \beta au'(0) & = \sum_{i=1}^{m} a_i u(\xi_i), \\
\gamma u(T) + \delta a(T)u'(T) & = \sum_{i=1}^{n} b_i u(\eta_i),
\end{align*}
\]

where \( m, n \in \mathbb{N} \).

For \( a \in C([0, T], (0, \infty)), b \in C([0, T], [0, \infty]), \alpha, \beta, \gamma, \delta \in \mathbb{R}, |\alpha| + |\beta| \neq 0, |\gamma| + |\delta| \neq 0 \), we denote by \( \psi \) and \( \phi \) the solutions of the following linear problems

\[
\begin{align*}
(a(t)\psi'(t))^\prime - b(t)\psi(t) & = 0, \quad 0 < t < T, \\
\psi(0) & = \beta, \quad a(0)\phi'(0) = \alpha,
\end{align*}
\]

and

\[
\begin{align*}
(a(t)\phi'(t))^\prime - b(t)\phi(t) & = 0, \quad 0 < t < T, \\
\phi(T) & = \delta, \quad a(T)\phi'(T) = -\gamma,
\end{align*}
\]

respectively.

We denote by \( \theta_1 \) the function \( \theta_1(t) = a(t)(\phi(t)\psi'(t) - \phi'(t)\psi(t)) \) for \( t \in [0, T] \). Because \( \theta_1'(t) = 0 \) for all \( t \in (0, T) \), we deduce that \( \theta_1(t) = \text{const.} \) for all \( t \in [0, T] \). We denote this constant by \( \tau_1 \). Then \( \theta_1(t) = \tau_1 \) for all \( t \in [0, T] \), and so \( \tau_1 = \theta_1(0) = a(0)(\phi(0)\psi'(0) - \phi'(0)\psi(0)) = a(0)\phi(0) - \beta a(0)\phi'(0) \) and \( \tau_1 = \theta_1(T) = a(T)(\phi(T)\psi'(T) - \phi'(T)\psi(T)) = \delta a(T)\psi'(T) + \gamma \psi(T) \).

**Lemma 2.1.** ([4]) We assume that \( a \in C^1([0, T], (0, \infty)), b \in C([0, T], [0, \infty]), \alpha, \beta, \gamma, \delta \in \mathbb{R}, |\alpha| + |\beta| \neq 0, |\gamma| + |\delta| \neq 0, m, n \in \mathbb{N}, a_i \in \mathbb{R} \) for all \( i = 1, \ldots, m, b_i \in \mathbb{R} \).
for all $i = 1, \ldots, n$, and $0 < \xi_1 < \cdots < \xi_m < T$, $0 < \eta_1 < \cdots < \eta_n < T$. If $\tau_1 \neq 0$,

$$\Delta_1 = -\frac{1}{\tau_1} \left[ \phi(T) \left( \sum_{i=1}^{n} b_i \phi(\xi_i) \right) + \phi(0) \left( \tau_1 - \sum_{i=1}^{n} b_i \psi(\eta_i) \right) \right]$$

and $y \in C([0, T])$, then the unique solution of (1)-(2) is given by

$$u(t) = \int_{0}^{T} G_1(t, s)y(s) \, ds,$$

where the Green’s function $G_1$ is defined by

$$G_1(t, s) = g_1(t, s) + \frac{1}{\tau_1} \left[ \phi(T) \left( \sum_{i=1}^{n} b_i \phi(\xi_i) \right) + \phi(0) \left( \tau_1 - \sum_{i=1}^{n} b_i \psi(\eta_i) \right) \right] \prod_{i=1}^{n} a_i g_1(\xi_i, s)$$

for all $(t, s) \in [0, T] \times [0, T]$, and

$$g_1(t, s) = \frac{1}{\tau_1} \left( \phi(t) \phi(s), \quad 0 \leq s \leq t \leq T, \right.$$

$$\left. \phi(s) \phi(t), \quad 0 \leq t \leq s \leq T. \right)$$

Now, we introduce the assumptions

(A1) $\alpha \in C^1([0, T], (0, \infty))$, $b \in C([0, T], [0, \infty)).$

(A2) $\alpha, \beta, \gamma, \delta \in [0, \infty)$ with $\alpha + \beta > 0$ and $\gamma + \delta > 0$.

(A3) If $b(t) \equiv 0$, then $\alpha + \gamma > 0$.

(A4) $m, n \in \mathbb{N}, a_i \geq 0$ for all $i = 1, \ldots, m$, $b_i \geq 0$ for all $i = 1, \ldots, n$; $0 < \xi_1 < \cdots < \xi_m < T$, $0 < \eta_1 < \cdots < \eta_n < T$.

(A5) $\tau_1 - \sum_{i=1}^{m} a_i \phi(\xi_i) > 0$, $\tau_1 - \sum_{i=1}^{n} b_i \psi(\eta_i) > 0$ and $\Delta_1 > 0$.

**Lemma 2.2.** ([4]) Let (A1) – (A5) hold. Then the Green’s function $G_1$ of problem (1)-(2) (given by (3)) is continuous on $[0, T] \times [0, T]$ and satisfies $G_1(t, s) \geq 0$ for all $(t, s) \in [0, T] \times [0, T]$. Moreover, if $y \in C([0, T])$ satisfies $y(t) \geq 0$ for all $t \in [0, T]$, then the unique solution $u$ of problem (1)-(2) satisfies $u(t) \geq 0$ for all $t \in [0, T]$.

**Lemma 2.3.** ([4]) Assume that (A1) – (A5) hold. Then the Green’s function $G_1$ of problem (1)-(2) satisfies the inequalities

$$a) \quad G_1(t, s) \leq J_1(s), \quad \forall (t, s) \in [0, T] \times [0, T],$$

where

$$J_1(s) = g_1(s, s) + \frac{1}{\tau_1} \left[ \phi(T) \left( \sum_{i=1}^{n} b_i \phi(\xi_i) \right) + \phi(0) \left( \tau_1 - \sum_{i=1}^{n} b_i \psi(\eta_i) \right) \right] \prod_{i=1}^{n} a_i g_1(\xi_i, s)$$

for all $(t, s) \in [0, T] \times [0, T]$, and satisfies $G_1(t, s)$
b) For every $\sigma \in (0, T/2)$, we have

$$\min_{t \in [\sigma, T-\sigma]} G_1(t, s) \geq v_1 J_1(s) \geq v_1 G_1(t', s), \quad \forall t', s \in [0, T],$$

where $v_1 = \min \left\{ \phi(T-\sigma), \psi(T) \right\}$.

**Lemma 2.4.** ([44]) Assume that (A1)–(A5) hold and let $\sigma \in (0, T/2)$. If $y \in C([0, T])$, $y(t) \geq 0$ for all $t \in [0, T]$, then the solution $u(t)$, $t \in [0, T]$ of problem (1)-(2) satisfies the inequality $\min_{t \in [\sigma, T-\sigma]} u(t) \geq v_1 \max_{t \in [0, T]} u(t')$.

We can also formulate similar results as Lemmas 2.1-2.4 above for the boundary value problem

$$c(t)v'(t) - d(t)v(t) + h(t) = 0, \quad 0 < t < T, \quad (4)$$

$$\tilde{\alpha}v(0) - \tilde{\beta}c(0)v'(0) = \sum_{i=1}^{r} c_i v(\xi_i), \quad \tilde{\gamma}v(T) + \tilde{\delta}c(T)v'(T) = \sum_{i=1}^{l} d_i v(\rho_i), \quad (5)$$

under similar assumptions as (A1)–(A5) and $h \in C([0, T])$. We denote by $\tilde{\psi}, \tilde{\phi}, \tilde{\phi}, \tilde{\tau}, \Delta_2, g_2, G_2, v_2$ and $J_2$ the corresponding constants and functions for problem (4)-(5) defined in a similar manner as $\psi, \phi, \phi, \tau, \Delta, g, G, v_1$ and $J_1$, respectively.

### 3. MAIN RESULTS

In this section, we investigate the existence and multiplicity of positive solutions for our problem (5)–(BC), under various assumptions on $f$ and $g$.

We present now the basic assumptions for our main results.

**(H1)** The functions $a, c \in C^1([0, T], (0, \infty))$ and $b, d \in C([0, T], [0, \infty))$.

**(H2)** $\alpha, \beta, \gamma, \delta, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \in [0, \infty)$ with $\alpha + \beta > 0, \gamma + \delta > 0, \tilde{\alpha} + \tilde{\beta} > 0, \tilde{\gamma} + \tilde{\delta} > 0$; if $b \equiv 0$ then $\alpha + \gamma > 0$; if $d \equiv 0$ then $\tilde{\alpha} + \tilde{\gamma} > 0$; $m, n, r, l \in \mathbb{N}, a_i \geq 0$ for all $i = 1, \ldots, m, b_i \geq 0$ for all $i = 1, \ldots, n, c_i \geq 0$ for all $i = 1, \ldots, r$,

d_i \geq 0 for all $i = 1, \ldots, l$; $0 < \xi_1 < \cdots < \xi_m < T, 0 < \eta_1 < \cdots < \eta_n < T$,

$$0 < \xi_1 < \cdots < \xi_r < T, 0 < \rho_1 < \cdots < \rho_l < T; \quad \tau_1 - \sum_{i=1}^{m} a_i \phi(\xi_i) > 0,$$

$$\tau_1 - \sum_{i=1}^{n} b_i \psi(\eta_i) > 0, \Delta_1 > 0,$$

$$\tau_2 - \sum_{i=1}^{r} c_i \tilde{\phi}(\xi_i) > 0, \tau_2 - \sum_{i=1}^{l} \tilde{d}_i \tilde{\psi}(\rho_i) > 0, \Delta_2 > 0,$$

where $\psi, \phi, \tilde{\psi}, \tilde{\phi}, \tau_1, \tau_2, \Delta_1, \Delta_2$ are defined in Section 2.

**(H3)** The functions $f, g \in C([0, T] \times [0, \infty), [0, \infty))$ and $f(t, 0) = 0, g(t, 0) = 0$ for all $t \in [0, T]$. 

The pair of functions \((u, v) \in (C^2([0, T]))^2\) is a solution for our problem \((S) - (BC)\) if and only if \((u, v) \in (C([0, T]))^2\) is a solution for the nonlinear integral system
\[
\begin{align*}
  u(t) &= \int_0^T G_1(t, s) f \left( s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) \, d\tau \right) \, ds, \quad t \in [0, T], \\
  v(t) &= \int_0^T G_2(t, s) g(s, u(s)) \, ds, \quad t \in [0, T].
\end{align*}
\]

We consider the Banach space \(X = C([0, T])\) with supremum norm \(\| \cdot \|\) and define the cone \(P \subset X\) by \(P = \{ u \in X, \ u(t) \geq 0, \ \forall t \in [0, T] \}\).

We also define the operators \(A : P \to X\) by
\[
(Au)(t) = \int_0^T G_1(t, s) f \left( s, \int_0^T G_2(s, \tau) g(\tau, u(\tau)) \, d\tau \right) \, ds, \quad t \in [0, T],
\]
and \(B : P \to X, \ C : P \to X\) by
\[
(Bu)(t) = \int_0^T G_1(t, s) u(s) \, ds, \quad (Cu)(t) = \int_0^T G_2(t, s) u(s) \, ds, \quad t \in [0, T].
\]

Under the assumptions \((H1) - (H3)\), using also Lemma 2.2, it is easy to see that \(A\), \(B\) and \(C\) are completely continuous from \(P\) to \(P\). Thus the existence and multiplicity of positive solutions of the problem \((S) - (BC)\) are equivalent to the existence and multiplicity of fixed points of the operator \(A\).

**Theorem 3.1.** Assume that \((H1) - (H3)\) hold and \(\sigma \in (0, T/2)\). If the functions \(f\) and \(g\) also satisfy the conditions
\[
(H4) \text{ There exists a positive constant } p_1 \in (0, 1) \text{ such that}
\]
\[
i) f^i_\infty = \liminf_{u \to \infty} \inf_{t \in [\sigma, T-\sigma]} \frac{f(t, u)}{u^{p_1}} \in (0, \infty); \quad ii) g^i_\infty = \liminf_{u \to \infty} \inf_{t \in [\sigma, T-\sigma]} \frac{g(t, u)}{u^{1/p_1}} = \infty,
\]
and
\[
(H5) \text{ There exists a positive constant } q_1 \in (0, \infty) \text{ such that}
\]
\[
i) f^i_0 = \limsup_{u \to 0^+} \sup_{t \in [0, T]} \frac{f(t, u)}{u^{p_1}} \in [0, \infty); \quad ii) g^i_0 = \limsup_{u \to 0^+} \sup_{t \in [0, T]} \frac{g(t, u)}{u^{1/q_1}} = 0,
\]
then the problem \((S) - (BC)\) has at least one positive solution \((u(t), v(t))\), \(t \in [0, T]\).

**Proof.** By \((H4) \ i)\), we deduce that there exist \(C_1, C_2 > 0\) such that
\[
f(t, u) \geq C_1 u^{p_1} - C_2, \quad \forall (t, u) \in [0, T] \times [0, \infty). \tag{6}
\]
For any \(u \in P\), by using \((6)\) and Lemma 2.3, we conclude after some computations
\[
(Au)(t) \geq \tilde{C}_1 \int_0^T G_1(t, s) \left( \int_0^T (G_2(s, \tau))^{p_1} (g(\tau, u(\tau)))^{p_1} \, d\tau \right) \, ds - C_3, \tag{7}
\]
for all \( t \in [0, T] \), where \( \bar{C}_1 = C_1 T^{p_1/q_0} \) for \( p_1 \in (0, 1) \) and \( q_0 = p_1/(p_1 - 1) \), \( \bar{C}_1 = C_1 \) for \( p_1 = 1 \) and \( C_3 = C_2 \int_0^T J_1(s) \, ds \).

We define the cone \( P_0 = \{ u \in P ; \inf_{t \in [\tau, T - \sigma]} u(t) \geq v_1||u|| \} \). From our assumptions and Lemma 2.4, we obtain \( B(P) \subset P_0 \) and \( C(P) \subset P_0 \). Now we consider the function

\[
\begin{align*}
&u_0(t) = \int_0^T G_1(t, s) \, ds = (B_{Y_0})(t), \quad t \in [0, T], \text{ where } y_0(t) = 1 \text{ for all } t \in [0, T], \text{ and the set } \\
&M = \{ u \in P ; \text{ there exists } \lambda \geq 0 \text{ such that } u = Au + \lambda u_0 \}.
\end{align*}
\]

We will show that \( M \subset P_0 \) and \( M \) is a bounded subset of \( X \). If \( u \in M \), then there exists \( \lambda \geq 0 \) such that \( u(t) = (Au(t) + \lambda u_0(t)), \quad t \in [0, T] \). Hence, we have

\[
u(t) = (Au)(t) + \lambda(B_{Y_0})(t) = B(Fu(t)) + \lambda(B_{Y_0})(t) = B(Fu(t) + \lambda y_0(t)) \in P_0,
\]

where \( F : P \to P \) is defined by \( (Fu)(t) = f(t, \int_0^T G_2(t, s)g(s, u(s)) \, ds) \). Therefore, \( M \subset P_0 \), and

\[
||u|| \leq \frac{1}{v_1} \inf_{t \in [\tau, T - \sigma]} u(t), \quad \forall u \in M. \tag{8}
\]

From (H4) ii), we conclude that for \( \varepsilon_0 = (2/(\bar{C}_1 v_1 \tau^{p_1}) m_1 m_2) \) \( > 0 \) there exists \( C_4 > 0 \) such that

\[
(g(t, u))^{p_1} \geq \varepsilon_0^{p_1} u - C_4, \quad (t, u) \in [0, T] \times [0, \infty), \tag{9}
\]

where \( m_1 = \int_{\tau}^{T - \sigma} J_1(\tau) \, d\tau > 0, m_2 = \int_{\tau}^{T - \sigma} (J_2(\tau))^{p_1} \, d\tau > 0 \).

For \( u \in M \) and \( t \in [\tau, T - \sigma] \), by using Lemma 2.3 and the relations (7), (9), we obtain

\[
\begin{align*}
u(t) &= (Au)(t) + \lambda u_0(t) \geq (Au)(t) \\
&\geq \bar{C}_1 \int_{\tau}^{T - \sigma} G_1(t, s) \left[ \int_{\tau}^{T - \sigma} (G_2(s, \tau))^{p_1} (g(\tau, u(\tau)))^{p_1} \, d\tau \right] \, ds - C_3 \\
&\geq \bar{C}_1 v_1 \tau^{p_1} \left( \int_{\tau}^{T - \sigma} J_1(s) \, ds \right) \left( \int_{\tau}^{T - \sigma} (J_2(\tau))^{p_1} \left( \varepsilon_0^{p_1} u(\tau) - C_4 \right) \, d\tau \right) - C_3 \\
&\geq \bar{C}_1 v_1 \tau^{p_1} \varepsilon_0^{p_1} \left( \int_{\tau}^{T - \sigma} J_1(s) \, ds \right) \left( \int_{\tau}^{T - \sigma} (J_2(\tau))^{p_1} \, d\tau \right) \inf_{\tau \in [\tau, T - \sigma]} u(\tau) - C_5 \\
&= 2 \inf_{\tau \in [\tau, T - \sigma]} u(\tau) - C_5,
\end{align*}
\]

where \( C_5 = C_3 + C_4 \bar{C}_1 v_1 \tau^{p_1} m_1 m_2 > 0 \).

Hence, \( \inf_{\tau \in [\tau, T - \sigma]} u(t) \geq \inf_{\tau \in [\tau, T - \sigma]} u(t) - C_5 \), and so

\[
\inf_{\tau \in [\tau, T - \sigma]} u(t) \leq C_5, \quad \forall u \in M. \tag{10}
\]

From relations (8) and (10), we obtain \( ||u|| \leq C_5/\gamma_1 \), for all \( u \in M \), that is, \( M \) is a bounded subset of \( X \).
Moreover, there exists a sufficiently large $L > 0$ such that

$$u \neq Au + \lambda u_0, \quad \forall u \in \partial B_L \cap P, \quad \forall \lambda \geq 0,$$

where $B_L$ is the open ball of radius $L$ centered at 0, and $\partial B_L$ is its boundary. From [1] (or Lemma 2 from [7]), we deduce that the fixed point index of operator $A$ is

$$i(A, B_L \cap P, P) = 0. \quad (11)$$

Next, from (H5), we conclude that there exist $M_0 > 0$ and $\delta_1 \in (0, 1)$ such that

$$f(t, u) \leq M_0 u^{\eta_1}, \quad \forall (t, u) \in [0, T] \times [0, 1];$$

$$g(t, u) \leq \varepsilon_1 u^{1/q_1}, \quad \forall (t, u) \in [0, T] \times [0, \delta_1], \quad (12)$$

where $\varepsilon_1 = \min \left\{ 1/M_2, \left( 1/(2M_0 M_1 M_2^q) \right)^{1/q_1} \right\} > 0$, $M_1 = \int_0^T J_1(s) ds > 0$, $M_2 = \int_0^T J_2(s) ds > 0$. Hence, for any $u \in \overline{B}_{\delta_1} \cap P$ and $t \in [0, T]$, we obtain

$$\int_0^T G_2(t, s) g(s, u(s)) ds \leq \varepsilon_1 \int_0^T J_2(s) (u(s))^{1/q_1} ds \leq \varepsilon_1 M_2 ||u||^{1/q_1} \leq 1. \quad (13)$$

Therefore, by (12) and (13), we deduce that for any $u \in \overline{B}_{\delta_1} \cap P$ and $t \in [0, T]$

$$\left( Au \right)(t) \leq M_0 \int_0^T G_1(t, s) \left( \int_0^T G_2(s, \tau) g(\tau, u(\tau)) d\tau \right)^{q_1} ds \leq M_0 \varepsilon_1^{q_1} M_1 M_2^q ||u||.$$ 

This gives us $||Au|| \leq ||u||/2$, $\forall u \in \partial B_{\delta_1} \cap P$. From [1] (or Lemma 1 from [7]), we conclude that the fixed point index of $A$ is

$$i(A, B_{\delta_1} \cap P, P) = 1. \quad (14)$$

Combining (11) and (14), we obtain

$$i(A, (B_L \setminus \overline{B}_{\delta_1}) \cap P, P) = i(A, B_L \cap P, P) - i(A, B_{\delta_1} \cap P, P) = -1.$$ 

Hence, we deduce that $A$ has at least one fixed point $u_1 \in (B_L \setminus \overline{B}_{\delta_1}) \cap P$, that is $\delta_1 < ||u_1|| < L$. Let $v_1(t) = \int_0^T G_2(t, s) g(s, u_1(s)) ds$. Then $(u_1, v_1) \in P \times P$ is a solution of $(S) - (BC)$. By using (H3), we also have $||v_1|| > 0$. 

Using similar arguments as those used in the proofs of Theorem 3.2 and Theorem 3.3 from [2], we also obtain the following results for our problem $(S) - (BC)$.

**Theorem 3.2.** Assume that (H1) - (H3) hold and $\sigma \in (0, T/2)$. If the functions $f$ and $g$ also satisfy the conditions

(H6) There exists a positive constant $r_1 \in (0, \infty)$ such that

$$i) f^{\infty}_\sigma = \lim_{u \to \infty} \sup_{t \in [0, T]} \frac{f(t, u)}{u^\sigma} \in [0, \infty); \quad ii) g^{\infty}_\sigma = \lim_{u \to \infty} \sup_{t \in [0, T]} \frac{g(t, u)}{u^{1/r_1}} = 0,$$

then...
and

(H7) The following conditions are satisfied

\[ i) \liminf_{t \to 0^+} \inf_{u \in [0,T]} \frac{f(t,u)}{u} \in (0, \infty]; \quad ii) \liminf_{t \to 0^+} \inf_{u \in [0,T]} \frac{g(t,u)}{u} = \infty, \]

then the problem \((S) - (BC)\) has at least one positive solution \((u(t), v(t)), \ t \in [0, T]\).

Theorem 3.3. Assume that (H1) – (H3) hold and \(\sigma \in (0, T/2)\). If the functions \(f\) and \(g\) also satisfy the conditions (H4), (H7) and

(H8) For each \(t \in [0, T], f(t,u)\) and \(g(t,u)\) are nondecreasing with respect to \(u\), and there exists a constant \(N > 0\) such that

\[ f(t,m_0 \int_0^T g(s, N) \, ds) < \frac{N}{m_0}, \quad \forall \ t \in [0, T], \]

where \(m_0 = \max\{K_1T, K_2\}, K_1 = \max_{s \in [0,T]} J_1(s), K_2 = \max_{s \in [0,T]} J_2(s)\) and \(J_1, J_2\) are defined in Section 2, then the problem \((S) - (BC)\) has at least two positive solutions \((u_1(t), v_1(t)), (u_2(t), v_2(t)), t \in [0, T]\).

4. AN EXAMPLE

Let \(T = 1, a(t) = 1, b(t) = 1, c(t) = 1, d(t) = 0, p(t) = 1, q(t) = 1\) for all \(t \in (0, 1), m = 2, n = 1, r = 1, l = 2, \alpha = 1, \beta = 2, \gamma = 2, \delta = \tilde{\alpha} = 1, \beta = 1, \gamma = 2, \delta = 1, \xi_1 = 1/3, \xi_2 = 2/3, a_1 = 3/2, a_2 = 1, \eta_1 = 1/2, b_1 = 1, \zeta_1 = 1/4, c_1 = 4/3, \rho_1 = 1/5, \rho_2 = 4/5, d_1 = 1, d_2 = 1/3, f(t, x) = a(x^\tilde{\alpha} + x^\beta), g(t, x) = b(x^\gamma + x^\delta), a, b > 0, \tilde{\alpha} > 1, \beta < 1, \gamma > 2, \delta < 1.

We consider the second-order differential system

\[
(S) \begin{cases}
    u''(t) - u(t) + a(\bar{v}(t) + \bar{v}(t)) = 0, & 0 < t < 1, \\
    \bar{v}''(t) + b(u(t) + \bar{u}(t)) = 0, & 0 < t < 1,
\end{cases}
\]

with the boundary conditions

\[
(BC_0) \begin{cases}
    u(0) - 2u'(0) = \frac{1}{3}u\left(\frac{1}{2}\right) + u\left(\frac{1}{2}\right), & 2u(1) + 3u'(1) = u\left(\frac{1}{2}\right), \\
    v(0) - v'(0) = \frac{1}{3}v\left(\frac{1}{2}\right), & 2v(1) + v'(1) = v\left(\frac{1}{2}\right) + \frac{1}{2}v\left(\frac{1}{2}\right).
\end{cases}
\]
Multiple positive solutions for a second-order multi-point boundary value problem

The functions $\psi$ and $\phi$ from Section 2 are $\psi(t) = \frac{3e^{2t} + 1}{2e^t}$ and $\phi(t) = \frac{1 + 5e^{2-2t}}{2e^{1-t}}$ for all $t \in [0, 1]$ (see also [4]). Besides, we have $\tau_1 = \frac{15e^2 - 1}{2e}$.

\[ \Lambda_1 := \tau_1 - \sum_{i=1}^{n} b_i \psi(\eta_i) = (15e^2 - 3e^{3/2} - e^{1/2} - 1)/(2e) \approx 17.42682676 > 0, \]

\[ \Lambda_2 := \tau_1 - \sum_{i=1}^{m} a_i \phi(\xi_i) = (30e^2 - 15e^{5/3} - 10e^{4/3} - 2e^{2/3} - 3e^{1/3} - 2)/(4e) \approx 8.66681178 > 0, \]

\[ \Lambda_3 := \sum_{i=1}^{m} a_i \psi(\xi_i) = (6e^{5/3} + 9e^{4/3} + 3e^{2/3} + 2e^{1/3})/(4e) \approx 6.85583606, \]

\[ \Lambda_4 := \sum_{i=1}^{n} b_i \phi(\eta_i) = (5e^{3/2} + e^{1/2})/(2e) \approx 4.42506851, \]

\[ \Delta_1 = -\Lambda_3 \Lambda_4 + \Lambda_1 \Lambda_2 \approx 120.69748322 > 0. \]

The functions $g_1$ and $J_1$ are given by

\[ g_1(t, s) = \frac{2e}{15e^2 - 1} \left\{ \frac{(1 + 5e^{2-2t})(3e^{2s} + 1)}{4e^{1-s+t}} - \frac{(1 + 5e^{2-2s})(3e^{2t} + 1)}{4e^{1+s-t}} \right\}, \quad s \leq t, \]

\[ J_1(s) = g_1(s, s) + \frac{1}{\Delta_1} (\Lambda_4 \psi(1) + \Lambda_1 \phi(0)) \left( \frac{3}{2} g_1 \left( \frac{1}{3}, s \right) + g_1 \left( \frac{2}{3}, s \right) \right) + \frac{1}{\Delta_1} (\Lambda_2 \psi(1) + \Lambda_3 \phi(0)) g_1 \left( \frac{1}{2}, s \right). \]

The functions $\tilde{\psi}$ and $\tilde{\phi}$ from Section 2 are $\tilde{\psi}(t) = t + 1$ and $\tilde{\phi}(t) = -2t + 3$ for all $t \in [0, 1]$, $\tau_2 = 5, \tilde{\Lambda}_1 := \tau_2 - \sum_{i=1}^{l} d_i \tilde{\psi}(\rho_i) = 16/5 > 0, \tilde{\Lambda}_2 := \tau_2 - \sum_{i=1}^{r} c_i \tilde{\phi}(\xi_i) = 5/3 > 0, \tilde{\Lambda}_3 := \sum_{i=1}^{r} c_i \tilde{\psi}(\xi_i) = 5/3, \tilde{\Lambda}_4 := \sum_{i=1}^{l} d_i \tilde{\phi}(\rho_i) = 46/15, \Delta_2 = -\tilde{\Lambda}_3 \tilde{\Lambda}_4 + \tilde{\Lambda}_1 \tilde{\Lambda}_2 = 2/9 > 0.$

The functions $g_2$ and $J_2$ are given by

\[ g_2(t, s) = \frac{1}{5} \left\{ \frac{(3 - 2t)(s + 1)}{(3 - 2s)(t + 1)} \right\}, \quad s \leq t, \]

\[ J_2(s) = g_2(s, s) + \frac{1}{\Delta_2} (\tilde{\Lambda}_4 \tilde{\psi}(1) + \tilde{\Lambda}_1 \tilde{\phi}(0)) \frac{4}{3} g_2 \left( \frac{1}{3}, s \right) + \frac{1}{\Delta_2} (\tilde{\Lambda}_2 \tilde{\psi}(1) + \tilde{\Lambda}_3 \tilde{\phi}(0)) \left( g_2 \left( \frac{1}{5}, s \right) + \frac{1}{3} g_2 \left( \frac{4}{5}, s \right) \right). \]
We also have $K_1 = \max_{s \in [0,1]} J_1(s) \approx 2.62878282$, $K_2 = \max_{s \in [0,1]} J_2(s) = 86.5$. Then $m_0 = \max\{K_1, K_2\} = 86.5$. The functions $f(t,u)$ and $g(t,u)$ are nondecreasing with respect to $u$, for any $t \in [0,1]$, and for $p_1 = 1/2$ and $\sigma \in (0,1/2)$ fixed, the assumptions (H4) and (H7) are satisfied; indeed we obtain $f^{i}_{\infty} = \infty$, $g^{i}_{\infty} = \infty$, $f^{1}_0 = \infty$, $g^{1}_0 = \infty$.

We take $N = 1$ and then $\int_0^1 g(s,1) \, ds = 2b$ and $f(t,2bm_0) = a[(2bm_0)^{\overline{\alpha}} + (2bm_0)^{\overline{\beta}}]$. If $a[m_0^{\overline{\alpha}+1}(2b)^{\overline{\alpha}} + m_0^{\overline{\beta}+1}(2b)^{\overline{\beta}}] < 1$, then the assumption (H8) is satisfied. For example, if $\overline{\alpha} = 3/2$, $\overline{\beta} = 1/2$, $b = 1/2$ and $a < 1/(m_0^{5/2} + m_0^{3/2})$ (e.g. $a \leq 1.4 \cdot 10^{-5}$), then the above inequality is satisfied. By Theorem 3.3, we deduce that the problem $(S_0) - (BC_0)$ has at least two positive solutions.

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References
SOME CONSEQUENCES OF THE GENERALIZED HAMILTON-CAYLEY THEOREM FOR MATRICES POLYNOMIALLY DEPENDENT ON E. SCHMIDT SPECTRAL PARAMETER

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Abstract

At the beginning of previous century E. Schmidt has introduced for integral operators the eigenvalue systems \( \{ \lambda_k \} \), taking into account their multiplicities and the sets of eigenelements \( \{ \varphi_k \}_1^\infty \), \( \{ \psi_k \}_1^\infty \) for which \( A \varphi_k = \lambda_k \psi_k \), \( A^* \psi_k = \lambda_k \varphi_k \). In this article the generalized matrix spectral problems polynomially dependent on Schmidt’s spectral parameter are considered. I.S. Arzhanykh in 1951 has proved the generalized Hamilton-Cayley theorem for polynomial matrix pencils with identity matrix at the spectral parameter highest degree with the aim of applications in numerical methods of linear algebra. Below it is given the extension of Hamilton-Cayley theorem for matrix spectral problems on E. Schmidt, polynomially dependent on spectral parameter with identity matrix at the parameter highest degree (sec. 2) and also with identity (invertible) matrix at parameter zero degree (sec. 3) together with development of the relevant characteristic polynomial.

Keywords: E. Schmidt spectral parameter; polynomial dependence; Hamilton-Cayley theorem; development of the characteristic polynomial.

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1. INTRODUCTION

In the cycle of works on linear and nonlinear integral equations, in the beginning of XX-th century, E. Schmidt [1] had introduced in a Hilbert space \( H \) the systems of eigenvalues \( \lambda_k \) taking into account their multiplicities and the relevant eigenelements \( \{ \varphi_k \}_1^\infty \), \( \{ \psi_k \}_1^\infty \), satisfying the relations \( B \varphi_k = \lambda_k \psi_k \), \( B^* \psi_k = \lambda_k \varphi_k \) and allowing to extend Hilbert-Schmidt theory on nonselfadjoint completely continuous operators in separable Hilbert space \( H \) [2, 3]. Under the title of \( s \)-numbers this system has an application in numerical mathematics and ill-posed problems theory. As far as no one of mathematicians which has applied the \( s \)-numbers haven’t given references on Schmidt in our work for the justice recovery we say about spectral problems on E. Schmidt.

In the work of I.S. Arzhanykh [4, 5] and common his articles with V.I. Gugnina [6, 7] and also in the V.I. Gugnina’s dissertation [8] the variants of Hamilton-Cayley theorem were proved for matrices polynomially dependent on spectral parameter with
identity matrix at higher its degree with the aims of application in numerical methods of linear algebra (extension of Krylov, Leverrie and Faddeev methods for the eigenvalues computation) and also to stability theory of ODE solutions [9, 10].

In the article [11] the generalized Hamilton-Cayley theorem was proved for polynomial matrices with identity matrix of parameter zero degree.

In this article the extension of Hamilton-Cayley theorem is given for matrix generalized spectral problems on E. Schmidt of the form:

\[
(A_s + \lambda A_{s-1} + \lambda^2 A_{s-2} + ... + \lambda^{s-1} A_1)\varphi = \lambda^s \psi \\
(A_s^* + \lambda A_{s-1}^* + \lambda^2 A_{s-2}^* + ... + \lambda^{s-1} A_1^*)\psi = \lambda^s \varphi
\]  

(1)

and

\[
(\lambda^s A_s^* + \lambda^{s-1} A_{s-1}^* + ... + \lambda^2 A_2^* + \lambda A_1^*)\varphi = \psi \\
(\lambda^s A_s + \lambda^{s-1} A_{s-1} + ... + \lambda^2 A_2 + \lambda A_1)\psi = \varphi.
\]  

(2)

In general cases of invertible matrices the passage to spectral problems of the form (1) and (2) is realized with the aid of the relevant matrices inversion. Further it is convenient to use the matrix designations, i.e. to present the generalized E. Schmidt eigenvalue problems in the form of the equations

\[
\Phi(\lambda) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv \left[ \lambda^s \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} - \sum_{1 \leq k \leq s} \lambda^{s-k} \begin{pmatrix} 0 & A_k \\ A_k^* & 0 \end{pmatrix} \right] \times \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0
\]  

(3)

\[
\Phi(\lambda) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \equiv \left[ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} - \lambda \begin{pmatrix} A_s & 0 \\ 0 & A_1 \end{pmatrix} \right] - \sum_{1 \leq k \leq s} \lambda^{s-k} \begin{pmatrix} 0 & b_k \\ b_k^* & 0 \end{pmatrix} \times \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0
\]  

(4)

correspondingly with the matrix \( \mathcal{3} \) at the higher and lower degree of \( \lambda \).

In the dissertation [12] the other writing of the problem (1) is used

\[
\left[ \lambda^s \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \lambda \begin{pmatrix} A_s & A_{s-1} \\ A_{s-1}^* & 0 \end{pmatrix} \right] - \sum_{1 \leq k \leq s-1} \lambda^{s-1-k} \begin{pmatrix} 0 & A_k \\ A_k^* & 0 \end{pmatrix} \times \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0.
\]  

(5)

It should be noted that the system (2) at the passage to characteristic numbers is reduced to the system (1) with the aid of the division on \( \lambda^s \) and changing \( \varphi \) on \( \psi \) and vice versa.
2. THE GENERALIZED HAMILTON-CAYLEY THEOREM FOR EIGENVALUE PROBLEMS WITH THE MATRIX 3 AT THE \( \lambda \) HIGHER DEGREE

Follow to [6] introduce the symbolic degrees of matrix operators: 
\[ a^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \]
\[ a^{-k} = 0, \quad a^t = \sum_{0 < m \leq t} \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right) a_m [a^{t-m}] + \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right) a_t a^{0}, \quad t > 0, \] 
\[ a_t = \sum_{r \leq m \leq s} a_m a^{t-(m-r)}, \]
where obviously, that
\[ 0 < r \leq s, t > 0. \]
Set also \( \varphi_t(a^t) = a^t + \sum_{0 < m \leq t} a_t a^{t-m} \), where \( a_t \) are the coefficients of the corresponding to \( \Phi(\lambda) \) characteristic polynomial, i.e.
\[ \det \Phi(\lambda) = \lambda^{2n_3} + \sum_{0 < i \leq 2n_3} a_i \lambda^{2n_3-i}. \]

Theorem 2.1. (Generalized Hamilton-Cayley theorem). Matrices \( a_t, 0 < r \leq s \) comply with the equations:
\[ a_1 \varphi_{(2n-1)}(a^t) + a_2 \varphi_{(2n-1)s-1}(a^t) + \ldots + a_s \varphi_{(2n-2)s+1}(a^t) + A_{(2n-1)s+1} = 0 \]
\[ a_1 \varphi_{(2n-1)}(a^t) + a_2 \varphi_{(2n-1)s-1}(a^t) + \ldots + a_s \varphi_{(2n-2)s+2}(a^t) + A_{(2n-1)s+2} = 0 \]

\[ \vdots \]
\[ a_{s-1} \varphi_{(2n-1)s}(a^t) + a_s \varphi_{(2n-1)s-1}(a^t) + A_{2n_3-1} = 0 \]

\[ a_s \varphi_{(2n-1)s}(a^t) + A_{2n_3} = 0, \quad \overline{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \]

Proof. Let \( \Psi(\lambda) \) be the matrix adjoint to the matrix \( \Phi(\lambda) \)
\[ \Phi(\lambda) \cdot \Psi(\lambda) = \overline{I} \text{der} \Phi(\lambda), \]
where, obviously, that
\[ \Psi(\lambda) \left( \begin{array}{c} \varphi \\ \psi \end{array} \right) = \left( \begin{array}{c} \lambda^{(2n-1)s} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + \lambda^{(2n-1)s-1} \begin{pmatrix} g^{(1)}_{11} & g^{(1)}_{12} \\ g^{(1)}_{21} & g^{(1)}_{22} \end{pmatrix} + \ldots + \\ + A \begin{pmatrix} g^{(2n-1)s-1} & g^{(2n-1)s-1} \\ g^{(2n-1)s-1} & g^{(2n-1)s-1} \end{pmatrix} \end{array} \right) \left( \begin{array}{c} \varphi \\ \psi \end{array} \right). \]

The undetermined coefficients method leads to the following three groups of identities beginning with higher \( \lambda \) degree and finishing its first degree (at zero \( \lambda \) degree \( \overline{A}^2 = \overline{I} \)).

1-th group \((1 < t \leq s)\):
\[ \begin{pmatrix} g^{(1)}_{11} & g^{(1)}_{12} \\ g^{(1)}_{21} & g^{(1)}_{22} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_1^* \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + a_1 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \Rightarrow \]
\[ \begin{align*}
\Rightarrow \mathcal{B}^{(1)} &= 3a_1 \mathcal{I} + a_1 \mathcal{J} \\
\left( \begin{array}{ll}
\mathcal{B}^{(2)}_{11} & \mathcal{B}^{(2)}_{12} \\
\mathcal{B}^{(2)}_{21} & \mathcal{B}^{(2)}_{22}
\end{array} \right) &= \left( \begin{array}{ll}
0 & I \\
I & 0
\end{array} \right) \left( \begin{array}{ll}
A_1 & 0 \\
0 & A_1^*
\end{array} \right) \left( \begin{array}{ll}
\mathcal{B}^{(1)}_{11} & \mathcal{B}^{(1)}_{12} \\
\mathcal{B}^{(1)}_{21} & \mathcal{B}^{(1)}_{22}
\end{array} \right) + \left( \begin{array}{ll}
0 & I \\
I & 0
\end{array} \right)
\times \left( \begin{array}{ll}
A_2 & 0 \\
0 & A_2^*
\end{array} \right) \left( \begin{array}{ll}
0 & I \\
I & 0
\end{array} \right) + a_2 \left( \begin{array}{ll}
0 & I \\
I & 0
\end{array} \right) \Rightarrow \mathcal{B}^{(2)} = 3a_1 \mathcal{B}^{(1)} + 3a_2 \mathcal{I} + a_2 \mathcal{J}
\end{align*} \]

\[ \begin{align*}
\Rightarrow \mathcal{B}^{(k)} &= 3a_1 \mathcal{B}^{(k-1)} + 3a_y \mathcal{B}^{(k-2)} + \ldots + 3a_{k-1} \mathcal{B}^{1} + 3a_k \mathcal{I} + a_k \mathcal{J} \\
\Rightarrow \mathcal{B}^{(s)} &= 3a_1 \mathcal{B}^{(s-1)} + 3a_2 \mathcal{B}^{(s-2)} + \ldots + 3a_{s-1} \mathcal{B}^{1} + 3a_s \mathcal{I} + a_s \mathcal{J}
\end{align*} \]

II-th group (s < t \leq (2n - 1)s):

\[ \begin{align*}
\mathcal{B}^{(s+1)} &= 3a_y \mathcal{B}^{(s)} + 3a_y \mathcal{B}^{(s-1)} + \ldots + 3a_y \mathcal{B}^{(1)} + a_{s+1} \mathcal{J} \\
\mathcal{B}^{(s+2)} &= 3a_y \mathcal{B}^{(s+1)} + 3a_y \mathcal{B}^{(s)} + \ldots + 3a_y \mathcal{B}^{(2)} + a_{s+2} \mathcal{J}
\end{align*} \]

\[ \begin{align*}
\Rightarrow \mathcal{B}^{((2n-1)s-r)} &= 3a_y \mathcal{B}^{((2n-1)s-r-1)} + 3a_y \mathcal{B}^{((2n-1)s-r-2)} + \ldots + \\
&+ 3a_y \mathcal{B}^{((2n-2)s-r)} + a_{(2n-1)s-r} \mathcal{J}
\end{align*} \]

\[ \begin{align*}
\Rightarrow \mathcal{B}^{((2n-1)s)} &= 3a_y \mathcal{B}^{((2n-1)s-1)} + 3a_y \mathcal{B}^{((2n-1)s-2)} + \ldots + \\
&+ 3a_y \mathcal{B}^{((2n-2)s)} + a_{(2n-1)s}
\end{align*} \]
The first two groups of equalities determine the matrices $\mathcal{I}^{(1)}$, $1 < t \leq (2n - 1)s$:

$$\mathcal{I}^{(1)} = 3a_0 3 + a_1 3 = a^1 + a_1 a^0 = \varphi_1(a^0)$$

$$\mathcal{I}^{(2)} = 3a_1 3^{(1)} + 3a_2 3 + a_2 3 = (3a_0 3^2 + 3a_1 3 + 3a_2 3 + 3a_2 3^2 =
\begin{align*}
&= a^2 + a_1 a^1 + a_2 a^0 = \varphi_2(a^0) \\
\end{align*}$$

$$\mathcal{I}^{(k)} = 3a_0 3^{(k-1)} + 3a_2 3^{(k-2)} + \ldots + a_{s-1} 3 =
\begin{align*}
&= a^k + a_1 a^{k-1} + a_2 a^{k-2} + \ldots + a_{k-1} a^1 + a_k a^0 = \varphi_k(a^0) \\
\end{align*}$$

$$\mathcal{I}^{(s-1)} = 3a_0 3^{(s-2)} + 3a_2 3^{(s-3)} + \ldots + a_{s-1} 3 =
\begin{align*}
&= a^{s-1} + a_1 a^{s-2} + a_2 a^{s-3} + \ldots + a_{s-2} a^1 + a_{s-1} a^0 = \varphi_{s-1}(a^0) \\
\end{align*}$$

$$\mathcal{I}^{(s)} = 3a_0 3^{(s-1)} + 3a_2 3^{(s-2)} + \ldots + a_s 3 =
\begin{align*}
&= a^s + a_1 a^{s-1} + a_2 a^{s-2} + \ldots + a_{s-1} a^1 + a_s a^0 = \varphi_s(a^0) \\
\end{align*}$$

$$\mathcal{I}^{(s+1)} = 3a_1 3^{(s)} + 3a_2 3^{(s-1)} + \ldots + 3a_s 3^{(1)} + a_{s+1} 3 =
\begin{align*}
&= a^{s+1} + a_1 a^s + a_2 a^{s-1} + \ldots + a_s a^1 + a_{s+1} a^0 = \varphi_{s+1}(a^0) \\
\end{align*}$$

$$\mathcal{I}^{(s+2)} = 3a_1 3^{(s+1)} + 3a_2 3^{(s)} + \ldots + 3a_s 3^{(2)} + a_{s+2} 3 =
\begin{align*}
&= a^{s+2} + a_1 a^{s+1} + a_2 a^s + \ldots + a_{s+1} a^1 + a_{s+2} a^0 = \varphi_{s+2}(a^0) \\
\end{align*}$$

$$\mathcal{I}^{((2n-1)s)} = 3a_0 3^{((2n-1)s-1)} + 3a_2 3^{((2n-1)s-2)} + \ldots + 3a_s 3^{((2n-2)s)} +
\begin{align*}
&+ a_{(2n-1)s} 3 = a^{(2n-1)s} + a_1 a^{(2n-1)s-1} + a_2 a^{(2n-1)s-2} + \ldots +
\begin{align*}
&+ a_{(2n-1)s-1} a^1 + a_{(2n-1)s} a^0 = \varphi_{(2n-1)s}(a^0) \\
\end{align*}$$

---

**III-th group** $((2n - 1)s + r < t \leq 2ns, 0 < r \leq s)$:

$$-a_y \mathcal{I}^{((2n-1)s)} - a_y \mathcal{I}^{((2n-1)s-1)} - \ldots - a_{(2n-2)s+1} \mathcal{I}^{((2n-2)s+2)} =
\begin{align*}
&-a_x \mathcal{I}^{((2n-2)s+1)} = \mathcal{I}a^{(2n-1)s+1} \\
&-a_y \mathcal{I}^{((2n-1)s)} - a_y \mathcal{I}^{((2n-1)s-1)} - \ldots - a_{(2n-2)s+1} \mathcal{I}^{((2n-2)s+3)} =
\begin{align*}
&-a_x \mathcal{I}^{((2n-2)s+2)} = \mathcal{I}a^{(2n-1)s+2} \\
\end{align*}$$

$$\mathcal{I}^{((2n-1)s)} - a_x \mathcal{I}^{((2n-1)s-1)} = \mathcal{I}a^{2ns-1} \\
-a_x \mathcal{I}^{((2n-1)s)} = \mathcal{I}a^{2ns}$$

---

Some consequences of the generalized Hamilton-Cayley...
The substitution of

\[ g^{(1)} = \varphi_1(a'), \quad B^{(2)} = \varphi_2(a'), \quad B^{(3)} = \varphi_3(a'), \ldots, B^{(2n-1)s} = \varphi_{(2n-1)s}(a') \]

into the third group gives the equalities (6), i.e. the generalized Hamilton-Cayley theorem.

**Corollary 2.1.** (Explicit form of Hamilton-Cayley theorem). Matrices \( a_r, 0 < r \leq s \) comply with the equations:

\[
\begin{align*}
    a_1^{(2n-1)s} + a_1 a_1^{(2n-1)s-1} + a_2 a_1^{(2n-1)s-2} + \ldots + a_{(2n-1)s-1} a_1^1 + \lambda a_{(2n-1)s+1} &= 0 \\
    a_2^{(2n-1)s} + a_1 a_2^{(2n-1)s-1} + a_2 a_2^{(2n-1)s-2} + \ldots + a_{(2n-1)s-1} a_2^1 + \lambda a_{(2n-1)s+2} &= 0 \\
    &\vdots \\
    a_{s-1}^{(2n-1)s} + a_1 a_{s-1}^{(2n-1)s-1} + a_2 a_{s-1}^{(2n-1)s-2} + \ldots + a_{(2n-1)s-1} a_{s-1}^1 + \lambda a_{2s-1} &= 0 \\
    a_s^{(2n-1)s} + a_1 a_s^{(2n-1)s-1} + a_2 a_s^{(2n-1)s-2} + \ldots + a_{(2n-1)s-1} a_s^1 + \lambda a_{2s} &= 0. \quad (8)
\end{align*}
\]

**Proof.** For \( 0 < r \leq s \) have

\[
\begin{align*}
    0 &= a_1 B^{(2n-1)s} + a_2 B^{(2n-1)s-1} + \ldots + a_{s-1} B^{(2n-2)s+2} + a_s B^{(2n-2)s+1} + \\
    &+ \lambda a_{(2n-1)s+1} = a_1 \left[ a^{(2n-1)s} + a_1 a^{(2n-1)s-1} + a_2 a^{(2n-1)s-2} + \ldots + \right] \\
    &+ a_{(2n-1)s-1} a^0 + a_{(2n-1)s-2} a^0 + \ldots + a_{(2n-1)s} a^0 + a_{(2n-1)s-1} a^0 + \ldots + \\
    &+ a_{s-1} a^{(2n-2)s+1} + a_1 a_{s-1}^{(2n-1)s-1} + a_2 a_{s-1}^{(2n-1)s-2} + \ldots + a_{s-1} a^{(2n-2)s+1} + \\
    &+ a_s a^{(2n-2)s} \ldots + \lambda a_{(2n-1)s+1} \left[ a_1 a^0 + a_2 a^0 + \ldots + \right] \\
    &= a_1 a^{(2n-1)s} + a_2 a^{(2n-1)s-2} + \ldots + a_{s-1} a^{(2n-2)s+1} + \\
    &+ a_s a^{(2n-2)s} + \ldots + \lambda a_{(2n-1)s+1} = \\
    &= \lambda a_{(2n-1)s+1} \left[ a_1 a^0 + a_2 a^0 + \ldots + \right].
\end{align*}
\]
On this way the following symbolic degrees of the matrices \( s \)

**Remark 2.1.** Instead of the identity (7) it can be used the identity

\[
\Psi(\lambda)\Phi(\lambda) = \text{Ider}(\Phi(\lambda))
\]

On this way the following symbolic degrees of the matrices \( a^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \mathcal{J}, \ a(-i) = \sum_{0 < m \leq \frac{s}{2}} a^{(i-m)}a_m \mathcal{J} + a^0a_s \mathcal{J}, \ s > 0, \ a(r) = \sum_{r \leq m \leq s} a^{(-m-r)}a_m, \ 0 < r \leq s, \ s > 0 \) allowing formulate analogues of the theorem (2.1) and its corollary (2.1).
Theorem 2.2. Matrices \( a_r, 0 < r \leq s \) comply with the equations:

\[
\begin{align*}
    a_1 \varphi_{(2n-1)s}(a) + a_2 \varphi_{(2n-1)s-1}(a) + \ldots + a_s \varphi_{(2n-2)s+1}(a) + \lambda \alpha_{(2n-1)s+1} &= 0 \\
    a_s \varphi_{(2n-1)s}(a) + a_2 \varphi_{(2n-1)s-1}(a) + \ldots + a_s \varphi_{(2n-2)s+2}(a) + \lambda \alpha_{(2n-1)s+2} &= 0
\end{align*}
\]

\[
\begin{align*}
    a_{s-1} \varphi_{(2n-1)s}(a) + a_{s-1} \varphi_{(2n-1)s-1}(a) + \lambda \alpha_{2ns-1} &= 0 \\
    a_s \varphi_{(2n-1)s}(a) + \lambda \alpha_{2ns} &= 0.
\end{align*}
\]

Corollary 2.2 Matrices \( a_r, 0 < r \leq s \) comply with the equations:

\[
\begin{align*}
    a_1^{(2n-1)s} + a_1' a_1^{(2n-1)s-1} + a_2' a_1^{(2n-1)s-2} + \ldots + a_s' a_1^{(2n-1)s-1} a_1^{(2n-1)s-2} + \lambda \alpha_{(2n-1)s+1} &= 0 \\
    a_1^{(2n-1)s} + a_1' a_2^{(2n-1)s-1} + a_2' a_2^{(2n-1)s-2} + \ldots + a_s' a_2^{(2n-1)s-1} a_2^{(2n-1)s-2} + \lambda \alpha_{(2n-1)s+2} &= 0 \\
    a_{s-1}^{(2n-1)s} + a_1' a_{s-1}^{(2n-1)s-1} + a_2' a_{s-1}^{(2n-1)s-2} + \ldots + a_s' a_{s-1}^{(2n-1)s-1} a_{s}^{(2n-1)s-2} + \lambda \alpha_{2ns-1} &= 0 \\
    a_s^{(2n-1)s} + a_1' a_s^{(2n-1)s-1} + a_2' a_s^{(2n-1)s-2} + \ldots + a_s' a_s^{(2n-1)s-1} a_s^{(2n-1)s-2} + \lambda \alpha_{2ns} &= 0.
\end{align*}
\]

3. **THE GENERALIZED HAMILTON-CAYLEY THEOREM FOR EIGENVALUE PROBLEMS WITH THE MATRIX 3 AT THE \( \lambda \) LOWER DEGREE**

Similarly to sec.2 the introduction of the symbolic degree of matrix operators

\[
a^0 = b^0, \ b_t = \begin{pmatrix} A_t^r & 0 \\ 0 & A_t \end{pmatrix}, \ b^{(-k)} = 0, b^t = \sum_{0 \leq m \leq t} \lambda m b^{t-m} + \lambda \phi b^0, \ t > 0, \ b^t = \sum_{r \leq m \leq s} b^t (-m-r), 0 < r \leq s, \ t > 0\]

and the matrix \( \lambda (A) \) adjoint to the matrix \( \lambda (A) \) and the usage of the identity (4) allow to prove the follow result.

Theorem 3.1. Matrices \( b_r, 0 < r \leq s \) comply with the equations:

\[
\begin{align*}
    b_1 \varphi_{(2n-1)s}(b) + b_2 \varphi_{(2n-1)s-1}(b) + \ldots + b_s \varphi_{(2n-2)s+1}(b) + \lambda \alpha_{(2n-1)s+1} &= 0 \\
    b_s \varphi_{(2n-1)s}(b) + b_2 \varphi_{(2n-1)s-1}(b) + \ldots + b_s \varphi_{(2n-2)s+2}(b) + \lambda \alpha_{(2n-1)s+2} &= 0 \\
    b_{s-1} \varphi_{(2n-1)s}(b) + b_{s-1} \varphi_{(2n-1)s-1}(b) + \lambda \alpha_{2ns-1} &= 0 \\
    b_s \varphi_{(2n-1)s}(b) + \lambda \alpha_{2ns} &= 0.
\end{align*}
\]
Proof. In the identity (7) the matrix $Ψ(λ)$ and $det Φ(λ)$ have the forms

$$Ψ(λ) \begin{pmatrix} φ \\ ψ \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + λ \begin{pmatrix} q_{11}^{(1)} & q_{12}^{(1)} \\ q_{21}^{(1)} & q_{22}^{(1)} \end{pmatrix} + \lambda^2 \begin{pmatrix} q_{11}^{(2)} & q_{12}^{(2)} \\ q_{21}^{(2)} & q_{22}^{(2)} \end{pmatrix} + \ldots + \lambda^{(2n−1)s−1} \begin{pmatrix} q_{11}^{(2n−1)s−1} & q_{12}^{(2n−1)s−1} \\ q_{21}^{(2n−1)s−1} & q_{22}^{(2n−1)s−1} \end{pmatrix} + \lambda^{(2n−1)s} \begin{pmatrix} q_{11}^{(2n−1)s} & q_{12}^{(2n−1)s} \\ q_{21}^{(2n−1)s} & q_{22}^{(2n−1)s} \end{pmatrix} \begin{pmatrix} φ \\ ψ \end{pmatrix}$$

$$det Φ(λ) = 1 + \alpha_1λ + \alpha_2λ^2 + \ldots + \alpha_{2ns−1}λ^{2ns−1} + \alpha_{2ns}λ^{2ns}.$$

The undetermined coefficient method again gives three groups of equalities corresponding beginning with the first degree of $λ$ up to $λ^{2ns}$. Here the matrices $q_{11}^{(1)}, q_{12}^{(1)}, \ldots, q_{1s}^{(1)}, q_{1s}^{(s+1)}, \ldots, q_{1s}^{(2n−1)s}$ are represented by the same formulae as in section 2, where $α_2$ must be changed on $b_s$. Their substitution into the third group of formulae proves the theorem (3.1).

**Corollary 3.1 (Explicit form of Hamilton-Cayley theorem).** Matrices $b_r, 0 < r ≤ s$ comply with the equations:

$$b_1^{(2n−1)s} + \alpha_1 b_1^{(2n−1)s−1} + \alpha_2 b_1^{(2n−1)s−2} + \ldots + \alpha_{(2n−1)s−1} b_1^0 + 3α_{(2n−1)s−1} = 0$$

$$b_2^{(2n−1)s} + \alpha_1 b_2^{(2n−1)s−1} + \alpha_2 b_2^{(2n−1)s−2} + \ldots + \alpha_{(2n−1)s−1} b_2^0 + 3α_{(2n−1)s−1} = 0$$

$$b_s^{(2n−1)s} + \alpha_1 b_s^{(2n−1)s−1} + \alpha_2 b_s^{(2n−1)s−2} + \ldots + \alpha_{(2n−1)s−1} b_s^0 + 3α_{(2n−1)s−1} = 0.$$ (13)

**Remark 3.1** At the usage of the identity $Ψ(λ)Φ(λ) = \lambda det Φ(λ)$ the explicit form of Hamilton-Cayley theorem is obtained with symbolic degrees of the form $b_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = 3, b^{(−r)} = 0, b^{(t)} = \sum_{0≤m≤t} b^{(t−m)} b_m 3 + b^0 3, t > 0, b^r = \sum_{r≤m≤s} b^{(r−m)} b_m, 0 < r ≤ s, t > 0.$

4. COMPUTATION OF THE INVARIANTS AT THE LINEAR DEPENDENCE ON SPECTRAL PARAMETER

The computation of the coefficient of the characteristic equation $det(A − λI) = 0$, which are invariant with respect to the base changing in $n$-dimensional space $E^n$ is given here with the aim of the investigation the simplest problems on E. Schmidt spectrum of the form $(a − α3)(φ, φ)^T = 0$ and $(b − α3)(φ, ψ)^T = 0$. Here the result of the work [13] based on the following statement are essentially used.
Lemma 4.1 For any matrix polynomial \( F(\lambda) \) of \( n \)-th degree the following equality is true
\[
\frac{d}{d\lambda} \det F(\lambda) = \det F(\lambda) \text{tr} \left[ F^{-1}(\lambda) F'(\lambda) \right].
\] (14)

The application of lemma 4.1 to \( F(\lambda) = A - \lambda I, F'(\lambda) = -I, \det F(\lambda) = (-1)^n \lambda^n + \sum_{k=1}^{n} (-1)^{n-k} i_k \lambda^{n-k} \) allows to write
\[
\frac{d}{d\lambda} \det F(\lambda) = -(\det F(\lambda)) \text{tr} \left[ (A - \lambda I)^{-1} \right] = -(\det F(\lambda)) \times \text{tr} \left[ (-\lambda)(I - \lambda^{-1} A) \right]^{-1} = \frac{1}{\lambda} \det F(\lambda) \text{tr} \left[ (I - \lambda^{-1} A)^{-1} \right]
\]
\[
\frac{d}{d\lambda} \det F(\lambda) = -(\det F(\lambda)) \text{tr} \left[ (A - \lambda I)^{-1} \right] = -(\det F(\lambda)) \times \text{tr} \left[ (-\lambda)(I - \lambda^{-1} A) \right]^{-1} = \frac{1}{\lambda} \det F(\lambda) \text{tr} \left[ (I - \lambda^{-1} A)^{-1} \right]
\]
whence it follows
\[
\lambda \frac{d}{d\lambda} \det F(\lambda) = (-1)^n n \lambda^n + (-1)^{n-1} (n - 1) i_1 \lambda^{n-1} + ...
\]
\[
+(-1)^2 2 i_{n-2} \lambda^2 - i_{n-1} \lambda = \left[ (-1)^n \lambda^n + (-1)^{n-1} i_1 \lambda^{n-1} + ...ight.
\]
\[
\left. (-1)^{n-k} i_k \lambda^{n-k} + ... + (-1)^2 i_{n-2} \lambda^2 - i_{n-1} \lambda + i_n \right] \times \left[ n + \lambda^{-1} \text{tr} A + \lambda^{-2} \text{tr} A^2 + ... + \lambda^{-s} \text{tr} A^s + ... \right].
\] (15)
The undetermined coefficients method leads to the following group of formulae
\[
(-1)^n n = (-1)^n n
\]
\[
(-1)^{n-1} (n - 1) i_1 = (-1)^n \text{tr} A + (-1)^{n-1} i_1 n
\]
\[
(-1)^{n-2} (n - 2) i_2 = (-1)^n \text{tr} A^2 + (-1)^{n-1} i_1 \text{tr} A + (-1)^{n-2} i_2 n
\]
\[
\cdots
\]
\[
(-1)^{n-k} (n - k) i_k = (-1)^n \text{tr} A^k + (-1)^{n-1} i_1 \text{tr} A^{k-1} + ...
\]
\[
\left. +(-1)^{n-k+1} i_{k-1} \text{tr} A + (-1)^{n-k} i_k n \right]
\]
\[
\cdots
\]
\[
(-1)^2 2 i_{n-2} = (-1)^n \text{tr} A^{n-2} + (-1)^{n-1} i_1 \text{tr} A^{n-3} + ...
\]
\[
\left. +(-1)^3 i_{n-3} \text{tr} A + (-1)^2 i_{n-2} n \right \}
\]
\[-i_{n-1} = (-1)^n \text{tr} A^{n-1} + (-1)^{n-1} i_1 \text{tr} A^{n-2} + ... + (-1)^2 i_{n-2} \text{tr} A - i_{n-1} n
\] (16)
0 = (-1)^n trA^n + (-1)^{n-1} i_1 trA^{n-1} + ... - i_n trA + i_n which determines the invariants i_1, i_2, ..., i_8, ...

\[
i_1 = trA
\]
\[
i_2 = \frac{1}{2} i_1^2 - \frac{1}{2} trA^2
\]
\[
i_3 = \frac{1}{3} i_1^3 - \frac{1}{2} i_1 trA^2 + \frac{1}{3} trA^3
\]
\[
i_4 = \frac{1}{4} i_1^4 - \frac{1}{2} i_1^2 trA^2 + \frac{1}{3} i_1 trA^3 + \frac{1}{23} (trA^3)^2 - \frac{1}{4} trA^4
\]
\[
i_5 = \frac{1}{5} i_1^5 - \frac{1}{2 \cdot 3} i_1^3 trA^2 + \frac{1}{2} i_1 (trA^2)^2 + \frac{1}{2 \cdot 3} i_1 trA^3 - \frac{1}{2} i_1 trA^4 - \frac{1}{2 \cdot 3} trA^2 trA^3 + \frac{1}{5} trA^5
\]
\[
i_6 = \frac{1}{6} i_1^6 - \frac{1}{24 \cdot 3} i_1^4 trA^2 + \frac{1}{24} i_1^2 (trA^2)^2 + \frac{1}{24 \cdot 3} i_1 trA^3 - \frac{1}{23} i_1^2 trA^4 + \frac{1}{23} trA^2 trA^3 - \frac{1}{12} trA^6
\]
\[
i_7 = \frac{1}{7} i_1^7 - \frac{1}{24 \cdot 3 \cdot 5} i_1^5 trA^2 + \frac{1}{24 \cdot 3} i_1^3 (trA^2)^2 + \frac{1}{24 \cdot 3} i_1^2 trA^3 - \frac{1}{7} i_1^2 trA^4 + \frac{1}{24 \cdot 3 i_1} (trA^3)^2 - \frac{1}{2} i_1 trA^5 - \frac{1}{2 \cdot 5} trA^2 trA^3 - \frac{1}{24 \cdot 3} trA^3 trA^4 + \frac{1}{7} trA^7
\]
\[
i_8 = \frac{1}{8} i_1^8 - \frac{1}{25 \cdot 3 \cdot 32 \cdot 5} i_1^6 trA^2 + \frac{1}{26 \cdot 3} i_1^4 (trA^2)^2 + \frac{1}{25 \cdot 3 \cdot 32 \cdot 5} i_1^6 trA^3 - \frac{1}{25 \cdot 3} i_1^4 trA^4 + \frac{1}{23 \cdot 3 \cdot 5} i_1^3 trA^5 - \frac{1}{24 \cdot 3} i_1^3 trA^2 trA^3 - \frac{1}{24 \cdot 3 i_1^2} (trA^3)^2 + \frac{1}{23 \cdot 3} i_1 trA^4 + \frac{1}{25} (trA^4)^2 trA^3 - \frac{1}{25} (trA^4)^2 trA^5 + \frac{1}{25} trA^6.
\]

However, on this way to obtain the general presentation of the invariants is not succeeded. Nevertheless the formulae (15) can serve as the base of computer simulation for the computation of invariant coefficients of det \(F(\lambda)\).
References


A COMPARATIVE STUDY OF SOME ALGORITHMS FOR FACE RECOGNITION
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Abstract
Face recognition is one of the most prevalent problems of pattern recognition and a current issue in the context of nowadays technology progress. This real-life problem needs real-time answers, so a variety of algorithms have been developed to address this issue. In this paper we present a comparative study (in terms of recognition rate) for some numerical linear algebra standard algorithms and an own method. The algorithms are tested on different datasets considered as individual items and joined together.

Keywords: pattern recognition, dimension reduction, eigenvalue, eigenvector, singular value.

1. INTRODUCTION

The face recognition problem comes down to identifying a specific image of a person by analyzing and comparing patterns from a face dataset. Facial recognition systems are commonly used for security purposes but are becoming more and more used in a variety of other applications. There have been developed different types of algorithms for solving the face recognition problem (see [5], [11]).

The problem under consideration is the following. Given a dataset of images belonging to $P$ persons, all images are transformed into vectors $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_N\}$ of the same size $M \times 1$ and all vectors become columns of a matrix, $A = [\Gamma_1 \ \Gamma_2 \ \ldots \ \Gamma_N]$. The matrix $A$ will be our (face/image) dataset. We split the dataset into two nonoverlapping subsets: training subset and testing subset. Given an image $\Gamma$ (the image of a person) from the testing set, we want to find out if the algorithm classifies correctly that person.

In this paper we present a comparative study of some algorithms used to solve the face recognition problem. The algorithms in question are the following: the Eigenfaces (PCA) algorithm (see [15] and [16]), an own algorithm, named COD algorithm (see [9]), and Lanczos and Block Lanczos algorithms (see [1], [2], [6], [10], [17]). We compare the results of our COD algorithm with those obtained with an algorithm on blocks, namely the Block Lanczos algorithm. The conclusions are encouraging since the results closely resemble.

The paper is structured as follows. In Section 2 we present some algorithms for face recognition: the Eigenfaces (PCA) algorithm, Lanczos and Block Lanczos algorithms and an own algorithm, COD algorithm. In the next section, Section 3, we
present experiments and comparison between the results obtained with all the algorithms described in this paper. All algorithms are tested on three datasets: the well known ORL dataset (also known as AT&T dataset), an own face dataset CTOVF, and a bigger dataset obtained by putting together the previous two datasets. We use two performance indicators: recognition rate and average query time. In the last section we draw conclusions regarding the obtained results.

As a novelty, the experiments were performed from two different viewpoints: first, we take each dataset (ORL and CTOVF) as a different entity in comparing the recognition rate obtained with each of them, latter, we consider a larger dataset, composed of the two datasets mentioned above.

In what follows we present a comparative study (in terms of recognition rate and average query time) for the mentioned above numerical linear algebra standard algorithms and an own method, the COD algorithm.

2. SOME ALGORITHMS FOR FACE RECOGNITION

2.1. THE EIGENFACES (PCA) ALGORITHM

The eigenfaces are the principal components of a set of faces, namely eigenvectors of the covariance matrix of a dataset (see [15] and [16]). They are called eigenfaces because when represented, they resemble human faces. The principal components are given by the eigenvectors of the covariance matrix. The first principal component is the eigenvector corresponding to the largest eigenvalue, the second principal component is the eigenvector corresponding to the next largest eigenvalue and so on.

All $N$ images (with $M = n_1 \times n_2$ resolution) are transformed into vectors $\Gamma_i : M \times 1$. We compute the average face vector $\Psi = \frac{1}{N} \sum_{i=1}^{N} \Gamma_i$, and subtract $\Psi$ from all vectors $\varphi_i = \Gamma_i - \Psi$.

We seek a set of orthonormal vectors $u_1, u_2, \ldots, u_N$ which best describes the patterns that appear in the dataset. Vectors $u_k$ are eigenvectors of the covariance matrix $C = AA^T$ for $A = [\varphi_1 \varphi_2 \ldots \varphi_N]$.

The covariance matrix $C = AA^T$ is $C \in R^{M \times M}$, where $M = n_1 \times n_2$ is the resolution of an image. Because in practice, the number $M$ is very large, the computational effort to determine the $M$ eigenvalues and $M$ eigenvectors for matrix $C$ is huge. In this case, the idea is to reduce the size and therefore the amount of calculations. Then, we consider $L = A^T A$, $L \in R^{N \times N}$. Usually $N$, the number of images in the dataset is much smaller then the size of a vector, $M$, and in this case it is much easier to compute $N$ eigenvectors and $N$ eigenvalues for a matrix of size $N \times N$. From among the eigenvectors $v_i$ of matrix $L$ we obtain the eigenvectors $Av_i$ of matrix $C$. From all $N$ eigenvectors of $L$, are kept only the first $K$ vectors corresponding to the largest $K$ eigenvalues.
For a new image, \(\Gamma\), we project \(\Gamma - \Psi\) onto subspace \(\{u_1, u_2, \ldots, u_K\}\), hence we have \(\omega_i = u_i^T (\Gamma - \Psi)\), \(i = 1 : K\). The vector \(\Omega^T = [\omega_1, \omega_2, \ldots, \omega_K]\) describes the contribution of each eigenface in representing the image \(\Gamma\) and is used to classify the new image \(\Gamma\).

**Eigenfaces algorithm (see [15]):**

1. All images are transformed into vectors: \(\Gamma_1, \Gamma_2, \ldots, \Gamma_N\).
2. Compute the average face vector: \(\Psi = \frac{1}{N} \sum_{i=1}^{N} \Gamma_i\).
3. Subtract the average face vector from all vectors: \(\varphi_i = \Gamma_i - \Psi, i = 1, N\).
4. Compute the covariance matrix: \(C = \frac{1}{N} \sum_{i=1}^{N} \varphi_i \varphi_i^T = \frac{1}{N} AA^T\).
5. Compute the eigenvectors \(u_i, i = 1, N\) of matrix \(C\) and keep only \(K\) vectors corresponding to the \(K\) largest eigenvalues.
6. Obtain the vector: \(\Omega^T = [\omega_1, \omega_2, \ldots, \omega_K]\) where \(\varphi_i \approx \hat{\varphi}_i = \sum_{j=1}^{K} \omega_j u_j\).
7. Given a image \(\Gamma\), normalize it: \(\varphi = \Gamma - \Psi\).
8. Project \(\Gamma\) on the eigenvectors space: \(\hat{\varphi} = \sum_{j=1}^{K} \omega_j u_j\).
9. Represent \(\hat{\varphi}\) as \(\Omega^T = [\omega_1, \omega_2, \ldots, \omega_K]\).
10. Find \(i_0 \in \{1, \ldots, N\}\) satisfying \(\|\Omega - \Omega_{i_0}\| = \min_{1 \leq i \leq N} \|\Omega - \Omega_i\|\).

### 2.2. COD ALGORITHM

In order to obtain better results (as recognition rate) than with the PCA algorithm (a truncated SVD algorithm), we have to find a good low-rank approximation for matrix \(A\). Another option for computing a rank \(k\) truncated SVD is using orthogonalization via deflation algorithm proposed in [3] and [4]. For a matrix \(A \in \mathbb{R}^{M \times N}, M \geq N\), is generated a sequence of matrices \(A_1, A_2, \ldots, A_{k+1}\), for which

\[
A_{k+1} = A - \tilde{\sigma}_k \tilde{u}_k \tilde{v}_k^T = A - \sum_{j=1}^{k} \tilde{\sigma}_j \tilde{u}_j \tilde{v}_j^T = A - \tilde{U}_k \tilde{D}_k \tilde{V}_k^T = A - \tilde{B}_k
\]

(1)

where \(\tilde{u}_k, \tilde{v}_k\) and \(\tilde{\sigma}_k\) are given by formula (2), (3), and respectively (4), \(\tilde{U}_k = [\tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_k]\), \(\tilde{V}_k = [\tilde{v}_1 \tilde{v}_2 \ldots \tilde{v}_k]\), \(\tilde{D}_k = \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_k)\), and \(\tilde{B}_k = \tilde{U}_k \tilde{D}_k \tilde{V}_k^T\). Hence, matrix \(\tilde{B}_k\) serves as a low-rank approximation of matrix \(A\).

Let \(\tilde{u}_k \in \text{Range}(A_k)\) and \(\tilde{v}_k \in \text{Range}(A_k^T)\) be an arbitrary pair of unit vectors that satisfy \(\tilde{u}_k^T A_k \tilde{v}_k > 0\). We have

\[
\tilde{u}_k = \frac{A_k \tilde{v}_k}{\|A_k \tilde{v}_k\|_2}.
\]

(2)
\[ \tilde{v}_k = \frac{A_k^T \hat{u}_k}{\|A_k^T \hat{u}_k\|_2}, \]  
\text{(3)}

and

\[ \tilde{\sigma}_k = \frac{\left(\|A_k \tilde{v}_k\|_2 \|A_k^T \hat{u}_k\|_2\right)}{\left(\hat{u}_k^T A_k \tilde{v}_k\right)}. \]  
\text{(4)}

**Theorem 2.1.** (see [3] and [4]) Let the matrices \( \hat{U}_k \in \mathbb{R}^{M \times k} \), \( \hat{V}_k \in \mathbb{R}^{N \times k} \), and \( \hat{D}_k \in \mathbb{R}^{k \times k} \) be defined by the equalities

\[
\hat{U}_k = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_k \end{bmatrix}, \quad \hat{V}_k = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_k \end{bmatrix} \quad \text{and} \quad \hat{D}_k = (\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_k),
\]  
\text{(5)}

where

\[ \hat{\sigma}_j = \hat{u}_j^T A \hat{v}_j, \quad \text{for} \ j = 1, \ldots, r = \text{rank}(A). \]  
\text{(6)}

Then, in exact arithmetic, the following relations hold for \( k = 1, \ldots, r \):

\[
\text{Range}(A_{k+1}) \subseteq \text{Range}(A_k), \quad \text{Range}(A_k^T) \subseteq \text{Range}(A_k^T),
\]  
\text{(7)}

\[
\text{Range}(\hat{U}_k) \subseteq \text{Range}(A), \quad \text{Range}(\hat{V}_k) \subseteq \text{Range}(A^T),
\]  
\text{(8)}

\[
\hat{U}_k^T A_{k+1} = 0, \quad A_{k+1} \hat{V}_k = 0,
\]  
\text{(9)}

\[
\hat{U}_k^T \hat{U}_k = I, \quad \hat{V}_k^T \hat{V}_k = I.
\]  
\text{(10)}

Thus, for \( k = r \) the columns of \( \hat{U}_r \) and \( \hat{V}_r \) constitute orthonormal basis for \( \text{Range}(A) \) and \( \text{Range}(A^T) \), respectively. Consequently,

\[ A_{r+1} = 0 \]  
\text{(11)}

and

\[ A = \hat{U}_r \hat{D}_r \hat{V}_r^T. \]  
\text{(12)}

The customization (see [9]) consists in a proper choice at each iteration for \( \hat{u}_{i+1} \in \text{Range}(A_i) \) and \( \hat{v}_{i+1} \in \text{Range}(A_i^T) \). We have tested several variants of choices for these initializations (each iteration), but the obtained results were not satisfactory. These customizations are presented in a submitted paper.

The COD algorithm is the following.

**COD algorithm**

1. Initialize \( \hat{u}_1 \in \text{Range}(A) \), \( \hat{v}_1 = \hat{u}_1 / \|\hat{u}_1\| \) and \( \hat{v}_1 \).
2. for \( i = 1, 2, \ldots, k \) compute
   \[
   \hat{u}_i = A_i \hat{v}_i / \|A_i \hat{v}_i\|_2, \quad \hat{v}_i = A_i^T \hat{u}_i / \|A_i^T \hat{u}_i\|_2,
   \]
\[ \tilde{\sigma}_i = \left( \| A_i * \hat{v}_i \|_2 \| A_i^T \hat{u}_i \|_2 \right) / \left( \hat{u}_i^T A_i \hat{v}_i \right) \]

\[ A_{i+1} = A_i - \tilde{\sigma}_i \hat{u}_i \hat{v}_i^T \]

choose \( \hat{u}_{i+1} \in \text{Range}(A_{i+1}) \), \( \hat{u}_{i+1} = \hat{u}_{i+1} / \| \hat{u}_{i+1} \| \)

choose \( \hat{v}_{i+1} \in \text{Range}(A_{i+1}^T) \), \( \hat{v}_{i+1} = \hat{v}_{i+1} / \| \hat{v}_{i+1} \| \)

end

3. Let \( U = [\hat{u}_1 \hat{u}_2 \ldots \hat{u}_k] \) and \( B = A_{k+1} \).

4. Obtain \( \Omega_i^T = [\omega_1, \omega_2, \ldots, \omega_k] \) where \( \text{col}_i A = \sum_{j=1}^k \omega_j \hat{u}_j \).

5. Given a image \( \Gamma \), obtain \( \Gamma = \sum_{j=1}^k \omega_j \hat{u}_j \).

6. Represent \( \Gamma \) as \( \Omega_i^T = [\omega_1, \omega_2, \ldots, \omega_k] \).

7. Find \( i_0 \in \{1, \ldots, k\} \) satisfying \( \| \Omega - \Omega_{i_0} \| = \min_{1 \leq i \leq k} \| \Omega - \Omega_i \| \).

As mentioned before, after extensive tests, in which we have tried to give a proper initialization to \( \hat{u}_{i+1} \) and \( \hat{v}_{i+1} \), we obtained that the best choices we can make for this type of face recognition problem are the ones described in [9].

2.3. LANCZOS AND BLOCK LANCZOS ALGORITHMS

The Lanczos method tries to find the optimal eigen-subspace of a matrix \( A \) in a Krylov subspace (see [7]):

\[ K(A, q_1, k) = \text{span} \{ q_1, Aq_1, \ldots, A^{k-1} q_1 \}. \tag{13} \]

The orthonormal basis \( Q_k \) of \( K(A, q_1, k) \) can be efficiently computed via the Lanczos procedure. Accordingly, \( A \) can be approximated as \( A \approx Q_k T_k Q_k^T \), where \( T_k \) is a tridiagonal matrix:

\[ T_k = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \alpha_2 & \beta_2 \\
& \ddots & \ddots & \ddots \\
& & \beta_{k-2} & \alpha_{k-1} & \beta_{k-1} \\
& & & \beta_{k-1} & \alpha_k
\end{bmatrix}. \tag{14} \]

The Lanczos method generates a sequence of tridiagonal matrices \( T_k \) with the property that the extremal eigenvalues of \( T_k \in \mathbb{R}^{k \times k} \) are progressively better estimates of matrix \( A \) extremal eigenvalues (see [7]).
Lanczos algorithm:

Input: $m \times m$ symmetric matrix $A$, $k$.
1. Initialization: $r_0 = q_1$; $\beta_0 = 1$; $q_0 = 0$; $l = 0$.
2. for $l = 1 : k-1$ do
   \begin{align*}
   q_{l+1} &= \frac{r_l}{\beta_l}; l = l + 1; \\
   r_l &= Aq_l - \alpha_l q_l - \beta_{l-1} q_{l-1}; \\
   \beta_l &= ||r_l||_2 ;
   \end{align*}
end for

Output: $Q_k = (q_1, \ldots, q_k)$ and $T_k$ as (14).

The block Lanczos procedure tries to find an approximation of $A$:
$$A \approx Q_k T_k Q_k^T,$$
where $T_k$ is a block tridiagonal matrix (see [7]):

$$T_k = \begin{bmatrix}
    M_1 & B_1^T \\
    B_1 & M_2 & B_2^T \\
    & \ddots & \ddots & \ddots \\
    & & B_{k-2} & M_{k-1} & B_{k-1}^T \\
    & & & B_{k-1} & M_k
\end{bmatrix}$$ (15)

and columns of $Q_k = (X_1, \ldots, X_k)$ are orthonormal. By comparing the left and right hand sides of $Q_k A = Q_k T_k$, we have
$$AX_r = X_{r-1} B_{r-1}^T + X_r M_r + X_{r+1} B_r, \ r = 1, \ldots, k - 1,$$ (16)
where $B_0$ is defined to be 0. From the orthogonality of $Q$, we have that

$$M_r = X_r^T A X_r, \ r = 1, \ldots, k.$$ (17)

Another version of Block Lanczos algorithm is Block Lanczos with Warm Start (BLWS) (see [17]) where, at each iteration matrix $X_r$ is initialized with the subspace obtained in the previous iteration.

In our experiments we propose as a warm start for $X_1$ a fixed number of singular values of matrix $A$ computed apriori as in [10]. This implementation offers us a good starting point and ensure us of satisfactory results.

Block Lanczos Algorithm:

Input: $m \times m$ symmetric matrix $A$, $m \times d$ orthogonal matrix $X_1$, $k$.
1. Initialization: $M_1 = X_1^T A X_1$; $B_0 = 0$.
2. for $r = 1 : k - 1$ do
   \begin{align*}
   R_r &= AX_r - X_r M_r - X_{r-1} B_{r-1}^T; \\
   (X_{r+1}, B_r) &= qr(R_r); (The \ QR \ decomposition) \\
   M_{r+1} &= X_{r+1}^T A X_{r+1};
   \end{align*}
end for

Output: $Q_k = (X_1, \ldots, X_k)$ and $T_k$ as in (15).
3. EXPERIMENTS

In our experiments we use three datasets.

1. The ORL dataset (also known as AT&T dataset) consists of pictures expressing
different facial expressions.

2. The CTOVF face dataset (generated by ourselves) consists of pictures expressing
different facial expressions and the head position of the subjects is not always
straight.

3. A larger dataset composed of the two datasets mentioned above.

<table>
<thead>
<tr>
<th></th>
<th>ORL</th>
<th>CTOVF</th>
<th>ORL+CTOVF</th>
</tr>
</thead>
<tbody>
<tr>
<td># subjects</td>
<td>40</td>
<td>11</td>
<td>51</td>
</tr>
<tr>
<td># images</td>
<td>400</td>
<td>110</td>
<td>510</td>
</tr>
<tr>
<td>resolution</td>
<td>$92 \times 112$</td>
<td>$92 \times 112$</td>
<td>$92 \times 112$</td>
</tr>
</tbody>
</table>

The experiments were performed from two different viewpoints: first, we take
each dataset (ORL and CTOVF) as a different entity in comparing the recognition
rate obtained with each of them, latter, we consider a larger dataset, composed of the
other two mentioned above.

We have divided the dataset into two nonoverlapping subsets: training subset and
testing subset. All three datasets have 10 images for every person. We have conducted
two types of experiments: for the first type we put 8 (80%) out of all 10 images in the
training set and 2 (20%) images in the testing set and for the second type we put 9
(90%) images in the training set and one image (10%) in the testing set. For all these
types of splitting we used different levels of truncation $k$.

We use two performance indicators: recognition rate and average query time.
Recognition rate is the ratio between the number of correct results given by the algo-
rithm (true positive) and the total number of tests performed with that algorithm. The average query time is the average of all query times when searching to identify a person. The preprocessing time is the dataset preparation stage, when we construct the smaller dimension subspace and project the dataset onto it. This stage is performed only once, before starting the search for a person. Since it is performed only once, in what follows, we will not take into account this performance indicator.

<table>
<thead>
<tr>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR</td>
<td>AQT</td>
<td>RR</td>
<td>AQT</td>
</tr>
<tr>
<td>k=40</td>
<td>95%</td>
<td>0.0027</td>
<td>70%</td>
</tr>
<tr>
<td>k=60</td>
<td>95%</td>
<td>0.0030</td>
<td>71.25%</td>
</tr>
<tr>
<td>k=80</td>
<td>95%</td>
<td>0.0034</td>
<td>72.25%</td>
</tr>
</tbody>
</table>

Table 1: ORL 80%+20%

where RR=Recognition Rate, AQT=Average Query Time (in seconds), and k=level of truncation.

In Table 1 are presented the results for the ORL dataset, 80% of images for training and 20% of the images for testing (80%+20%). In this case the highest recognition rate is 96.25% obtained for Lanczos and Block Lanczos algorithms.

<table>
<thead>
<tr>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR</td>
<td>AQT</td>
<td>RR</td>
<td>AQT</td>
</tr>
<tr>
<td>k=40</td>
<td>95%</td>
<td>0.0026</td>
<td>72.5%</td>
</tr>
<tr>
<td>k=60</td>
<td>92.5%</td>
<td>0.0035</td>
<td>72.5%</td>
</tr>
<tr>
<td>k=80</td>
<td>92.5%</td>
<td>0.0037</td>
<td>72.5%</td>
</tr>
</tbody>
</table>

Table 2: ORL 90%+10%

In Table 2 are presented the results for the ORL dataset, 90% of images for training and 10% of the images for testing (90%+10%). In this case the highest recognition rate is 95% obtained for COD, Lanczos, and Block Lanczos algorithms.

In Table 3 are presented the results for the CTOVF dataset, 80% of images for training and 20% of the images for testing (80%+20%). In this case the highest recognition rate is 90.91% obtained for COD, Lanczos, and Block Lanczos algorithms.

In Table 4 are presented the results for the CTOVF dataset, 90% of images for training and 10% of the images for testing (90%+10%). In this case the highest recognition rate is 90.91% obtained for Lanczos algorithm for $k = 40$, in the rest


### Table 3: CTOVF 80%+20%

<table>
<thead>
<tr>
<th>k=40</th>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RR</td>
<td>AQT</td>
<td>RR</td>
<td>AQT</td>
</tr>
<tr>
<td>86.36%</td>
<td>0.0016</td>
<td>86.36%</td>
<td>0.0012</td>
<td>86.36%</td>
</tr>
<tr>
<td>90.91%</td>
<td>0.0019</td>
<td>86.36%</td>
<td>0.0012</td>
<td>90.91%</td>
</tr>
<tr>
<td>90.91%</td>
<td>0.0023</td>
<td>86.36%</td>
<td>0.0015</td>
<td>90.91%</td>
</tr>
</tbody>
</table>

### Table 4: CTOVF 90%+10%

<table>
<thead>
<tr>
<th>k=40</th>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RR</td>
<td>AQT</td>
<td>RR</td>
<td>AQT</td>
</tr>
<tr>
<td>81.82%</td>
<td>0.0015</td>
<td>72.73%</td>
<td>0.0015</td>
<td>90.91%</td>
</tr>
<tr>
<td>81.82%</td>
<td>0.0020</td>
<td>72.73%</td>
<td>0.0015</td>
<td>81.82%</td>
</tr>
<tr>
<td>81.82%</td>
<td>0.0024</td>
<td>72.73%</td>
<td>0.0017</td>
<td>81.82%</td>
</tr>
</tbody>
</table>

### Table 5: ORL+CTOVF 80%+20%

<table>
<thead>
<tr>
<th>k=40</th>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RR</td>
<td>AQT</td>
<td>RR</td>
<td>AQT</td>
</tr>
<tr>
<td>93.14%</td>
<td>0.003</td>
<td>71.57%</td>
<td>0.0051</td>
<td>93.14%</td>
</tr>
<tr>
<td>92.16%</td>
<td>0.0035</td>
<td>71.57%</td>
<td>0.0050</td>
<td>93.14%</td>
</tr>
<tr>
<td>91.18%</td>
<td>0.0042</td>
<td>71.57%</td>
<td>0.0054</td>
<td>94.12%</td>
</tr>
</tbody>
</table>

### Table 6: ORL+CTOVF 90%+10%

<table>
<thead>
<tr>
<th>k=40</th>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RR</td>
<td>AQT</td>
<td>RR</td>
<td>AQT</td>
</tr>
<tr>
<td>92.16%</td>
<td>0.0032</td>
<td>78.43%</td>
<td>0.0055</td>
<td>94.12%</td>
</tr>
<tr>
<td>94.12%</td>
<td>0.0038</td>
<td>78.43%</td>
<td>0.0057</td>
<td>94.12%</td>
</tr>
<tr>
<td>90.2%</td>
<td>0.0044</td>
<td>78.43%</td>
<td>0.0061</td>
<td>94.12%</td>
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</tbody>
</table>

In Table 5 are presented the results for the ORL+CTOVF dataset, 80% of images for training and 20% of the images for testing (80%+20%). In this case the highest recognition rate is 95.1% obtained for the Block Lanczos algorithm.
In Table 6 are presented the results for the ORL+CTOVF dataset, 90% of images for training and 10% of the images for testing (90%+10%). In this case the highest recognition rate is 94.12% obtained for for COD, Lanczos, and Block Lanczos algorithms.

The results for the recognition rate from Table 7 are computed as a weighted average \((80 \times RR_{ORL} + 22 \times RR_{CTOVF})/102\), where \(RR_{ORL}\) is the recognition rate for ORL dataset and \(RR_{CTOVF}\) is the recognition rate for CTOVF dataset, for the splitting 80% of images for training and 20% of the images for testing (80%+20%). We wanted to check if the fact that we considered the two datasets as a single one, influenced in any way the results in terms of recognition rate. From Tables 5 and 7 we can conclude that the recognition rate obtained for the larger dataset is almost the same as the weighted average for the recognition rates for the ORL and CTOVF datasets.

<table>
<thead>
<tr>
<th>ORL+CTOVF 80%+20%</th>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td>k=40</td>
<td>93.13%</td>
<td>73.52%</td>
<td>93.13%</td>
<td>94.11%</td>
</tr>
<tr>
<td>k=60</td>
<td>94.11%</td>
<td>74.50%</td>
<td>94.11%</td>
<td>95.09%</td>
</tr>
<tr>
<td>k=80</td>
<td>94.11%</td>
<td>75.29%</td>
<td>95.09%</td>
<td>95.09%</td>
</tr>
</tbody>
</table>

Table 7: ORL+CTOVF 80%+20%

The results for the recognition rate from Table 8 are computed as a weighted average \((40 \times RR_{ORL} + 11 \times RR_{CTOVF})/51\), where \(RR_{ORL}\) is the recognition rate for ORL dataset and \(RR_{CTOVF}\) is the recognition rate for CTOVF dataset, for the splitting 90% of images for training and 10% of the images for testing (90%+10%). From Tables 6 and 8 we can conclude that the recognition rate obtained for the larger dataset is almost the same as the weighted average for the recognition rates for the ORL and CTOVF datasets.

<table>
<thead>
<tr>
<th>ORL+CTOVF 80%+20%</th>
<th>COD</th>
<th>PCA</th>
<th>Lanczos</th>
<th>Block Lanczos</th>
</tr>
</thead>
<tbody>
<tr>
<td>k=40</td>
<td>92.15%</td>
<td>72.54%</td>
<td>94.11%</td>
<td>92.15%</td>
</tr>
<tr>
<td>k=60</td>
<td>90.19%</td>
<td>72.54%</td>
<td>92.15%</td>
<td>92.15%</td>
</tr>
<tr>
<td>k=80</td>
<td>90.19%</td>
<td>72.54%</td>
<td>92.15%</td>
<td>92.15%</td>
</tr>
</tbody>
</table>

Table 8: ORL+CTOVF 90%+10%
4. CONCLUSIONS

In this paper we compare four algorithms: Eigenfaces, COD, Lanczos and Block Lanczos using two performance indicators: recognition rate and average query time. The results for the three datasets are different because they are also based on the specificity of the dataset, not only on the algorithm. All considered algorithms in this paper are real-time recognition methods (i.e. they give the answer in few milliseconds).

Our proposed algorithm COD (see [9]) has a higher recognition rate than the PCA algorithm and in most cases has about the same recognition rate as the Lanczos and Block Lanczos algorithms. It also has a smaller average query time than the Lanczos method in most cases.

Our initialization for Block Lanczos Warm Start algorithm gives the same recognition rate as the Lanczos method but smaller average query time. Other different initializations can be considered (see [1], [17], [10]).

The recognition rate obtained for the larger dataset (ORL and CTOVF as a single dataset) can be obtained as a weighted average of the recognition rate obtained for the ORL dataset and the one obtained for the CTOVF dataset.

References


ESSENTIAL SPECTRUM OF PERTURBED SINGULAR INTEGRAL OPERATORS

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Abstract
In the work there are studied perturbed singular integral operators by noncompact operators. It is proved that the property of singular operators to be Noetherian in the weighted space $L^p$ is stable with respect to these perturbations, moreover indexes and essential spectrums are also unchanged.

Keywords: singular integral operator, essential spectrum, spaces with weights, Noetherian operators.
2010 MSC: 45E05, 45E10.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM

Let $\Gamma$ be a closed Lyapunov contour on the complex plane, bounding some domain $G^+$ and let $L_{\infty}(\Gamma)$ be the set of all measurable essentially bounded on $\Gamma$ functions. By $L^p(\Gamma, \rho)$ we denote the space $L^p (1 < p < \infty)$ on $\Gamma$ with the weight

$$\rho(t) = \prod_{k=1}^s |t-t_k|^{\beta_k},$$

where $t_1, t_2, ..., t_s$ are some distinct points of contour $\Gamma$, and $\beta_1, \beta_2, ..., \beta_s$ are arbitrary complex numbers satisfying relations $-1 < \beta_k < p - 1$.

In monographs of N. Muskhelishvili [1] and F. Gahov [2] a complete singular integral operator is called an operator of the following form

$$(A \phi)(t) = a(t) \phi(t) + \int_{\Gamma} k(\tau, t) \frac{\phi(\tau)}{\tau - t} d\tau,$$  \hspace{1cm} (1)

where $a(t)$ and $k(\tau, t)$ are functions, satisfying Holder condition on $\Gamma$ and $\Gamma \times \Gamma$, respectively and the integral is understand in the sence of principal value. The operator $A$, defined by (1) is possible to be represented in the form of

$$A = aI + bS + K,$$

where $b(t) = \pi i k(t, t)$, $S$ is the operator of singular integration along $\Gamma$, 

$$(S \phi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\phi(t)}{\tau - t} d\tau,$$
and $K$ is an integral operator with kernel

$$
k_0(\tau, t) = \frac{k(\tau, t) - k(t, t)}{\tau - t}.
$$

(2)

In the case, when $k(\tau, t)$ satisfies H"older condition on $\Gamma \times \Gamma$, the kernel (2) has a weak singularity, therefore the operator $K$ is totally continuous in the space $L_p(\Gamma, \rho)$. By virtue of this operator $A$ is Noetherian in $L_p(\Gamma, \rho)$, if and only if the operator

$$
A_0 = aI + bS,
$$

(3)

which is called the characteristic part of $A$, has this property.

In connection with this the Noether theory for singular operators was developed in the main for characteristic operators (see [3, 4, 5, 6, 7] and the references they contain).

But, in many problems of mechanics, physics and other domains, leading to singular equations, appear complete operators, not characteristic ones. That is why it arises the necessity to study complete singular operators (1) with discontinuous functions $f(t)$ and $k(\tau, t)$. In this case the main difficulty is that operator $K$ with kernel (2) turned out to be not compact or (that is more important) stooped to be $\Phi$ - admissible perturbation for characteristic singular operators. As a result it turned out that the essential spectrums of operators $A$ and $A_0$ do not coincide: $\hat{\sigma}(A) \neq \hat{\sigma}(A_0)$. By $\hat{\sigma}(A)$ we have denoted the set of all complex numbers $\lambda$ ($\in \mathbb{C}$) for which operator $A - \lambda I$ is not Noetherian.

Let us show this on an example. Let $\Gamma$ be the unit circle, $h(t) = t^2$, $t = e^{i\theta}$ ($0 \leq \theta < 2\pi$) , $k(\tau, t) = h(t) - h(\tau)$, $\delta \in \mathbb{C}$ and

$$(A\phi)(t) = \delta \phi(t) + \int_{\Gamma} \frac{k(\tau, t) \phi(\tau)}{\tau - t} d\tau.
$$

Here $k(t, t) = 0$, hence, characteristic par of operator $A$ is a scalar operator $(A_0\phi)(t) = \delta \phi(t)$. It follows, that $\hat{\sigma}(A_0) = \{ \delta \}$. The operator $A$ may be presented in the form $A = \delta I + hS - shI$, whence it follows [4], that is belongs to algebra $A_{p,\rho}$, generated by singular integral operators with piece-wise continuous coefficients. In the work [5] (see also [8, 9, 10]) it is shown that on algebra $A_{p,\rho}$ it is defined the symbol $\gamma(A)((t, \xi) \in \Gamma \times R)$ which on the generators $aI$ and $S$ has the following form

$$
\gamma_{t, \mu}(aI) = \begin{bmatrix} a(t+0) & 0 \\ 0 & a(t-0) \end{bmatrix},
$$

(4)

$$
\gamma_{t, \nu}(S) = \begin{bmatrix} \csc \pi(\xi + iy(t)) & 1 \\ \frac{1}{\sin(\xi + iy(t))} & -ctg \pi(\xi + iy(t)) \end{bmatrix},
$$

(5)

$$\gamma(t) = \frac{1+\beta(t)}{p}, \beta(t) = \beta_k, \beta(t) = 0 \text{ for } t \in \Gamma \setminus \{t_1, t_2, ..., t_k\} \text{ and } \beta_k, \ t_k \text{ numbers and points participating in definition of weight } \rho(t). \text{ Particularly, at } p = 2 \text{ and } \rho(t) \equiv 1
for operator $A = \delta I + hS - ShI$ we have $\det \nu_{t, \xi}(A) = \delta^2$ if $t \neq 1$, and

$$\det \nu_{t, \xi}(A) = \delta^2 + 16 \frac{e^{2n\xi}}{(e^{2n\xi} + 1)^2} \quad (\xi \in \mathbb{R})$$

if $t = 1$. From results of [4, 9, 10] it follows that operator $A$ is Noetherian in $L^2(\Gamma)$ if and only if $\delta^2 + 16 \frac{e^{2n\xi}}{(e^{2n\xi} + 1)^2} \neq 0$ for all $\xi \in \mathbb{R}$. It is equivalent to the fact that $\delta \neq \mu i$, where $\mu \in [-2, 2]$. This, if $\delta = \mu i$, where $\mu \in [-2, 2] \setminus \{0\}$, operator $A$ is not Noetherian, but its characteristic part $A_0$ is Noetherian. It follows from this that operator $T = A - A_0$ is not $\Phi$-admissible perturbation for operator $A_0$. Moreover, $\hat{\sigma}(A_0) = \{\delta\}$ while $\hat{\sigma}(A) = \{\delta\} \cup \{\mu i\}$. 

For operator $A = \delta I + hS - ShI$ we succeeded to get criterium of noetherianness due to the fact that we embedded it in algebra $A_{p,p}$. Similarly one can deal with some other complete operators.

In this work it will be described a class of (noncompact) operators having the property that under perturbation of characteristic singular operators by operators from this class essential spectrum is not changed. Operator $K$ of this class has the following form

$$(K\phi)(t) = \sum_{k=1}^{m} c_k(t)(M_k\phi)(t) + T\phi \quad (c_k \in L_{\infty}(\Gamma)), \quad (6)$$

where

$$(M_k\phi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\phi(\tau)}{\tau - t - \alpha_k} \, d\tau (t \in \Gamma), \quad (7)$$

$T$ is a compact operator and $\alpha_k(\neq 0)$ are some complex numbers. It turns out that under some sufficient conditions on number $\alpha_k$ operators $A_0$ and $A_0 + K$ are or are not simultaneously Noetherian and their indices coincide. As a consequence of this assertion it follows that $\hat{\sigma}(A_0) = \hat{\sigma}(A_0 + K)$. These results are basic in the given work and for their receipt as a preliminary some properties of operators $M_k$, $M_jM_k$, $SM_k$, $M_kS$ are established. It is proved that $\{t + \alpha_k\} \in \Gamma \neq \emptyset$, then operator $M_k$, defined by (7), is not compact in the space $L^p_{\rho}(\Gamma, \rho)$. At the end the obtained results are carried over more large classes of operators

$$A = \sum_{j=1}^{r} A_{jl} \cdot A_{j2} \cdots A_{jm},$$

where $A_{jl}$ are operators of the form

$$A_{jl} = a_{jl}I + b_{jl}S + \sum_{k=1}^{n} c_{jl}^{(k)}M_k + T_{jl}.$$
2. SOME PROPERTIES OF OPERATORS $M_k$ AND THEIR COMPOSITION WITH OPERATORS $S, P$ AND $Q$

Let us agree to denote by $L(B, B_1)$ ($L(B)$) algebra of all linear bounded operators, acting from Banach space $B$ in Banach space $B_1(B)$, and by $T(B)$ the bilateral ideal of $L(B)$, consisting of all completely continuous operators.

1. Let $\Gamma_k = \{ \zeta : \zeta = t - \alpha_k, \ t \in \Gamma \}$ and $\bar{\Gamma}_k = \{ \zeta : \zeta = t + \alpha_k, \ t \in \Gamma \}$. It contour $\Gamma_k$ has no common points with $\Gamma$, then, obviously, operator $M_k$, defined by (7), is compact in the space $L_p(\Gamma, \rho)$ and, hence, it may be embedded in operator $T (T \in T(L_p(\Gamma, \rho))$. This, in this case operator $c_k M_k (c_k \in L_\infty(\Gamma))$ does not influence essential spectrum of operator $A_0 = al + bS$. In connection with this we shall suppose further that the numbers $\alpha_k (k = 1, ..., m)$ are such that $\Gamma \cap \Gamma_k \neq \emptyset$. For simplicity we shall consider that $\Gamma_0$ is the unit circle: $\Gamma = \{ t : |t| = 1 \}$. We note also here the results of the work are valid for every closed Lyapunov contour $\Gamma$, having the property that $\Gamma$ and $\Gamma_k (k = 1, ..., m)$ intersect in a finite number of points.

Let $t_k^{(1)}, t_k^{(2)}$ be point of intersection of contours $\Gamma$ and $\Gamma_k (k = 1, ..., m)$ and $t_k^{(3)}, t_k^{(4)}$ points of intersection of contours $\Gamma$ and $\bar{\Gamma}_k$: $t_k^{(3)} = t_k^{(1)} + \alpha_k$ and $t_k^{(4)} = t_k^{(2)} + \alpha_k$ (Fig. 1).

![Fig.1](image)

Denote by $N_k$ the set of all functions from $L_\infty(\Gamma)$, continuous in some neighbourhoods $u(t_k^{(j)})$ of points $t_k^{(j)} (j = 1, 2, 3, 4)$.

Let $a \in N_k$ and $u_k^{(j)} = u(t_k^{(j)}) (j = 1, 2, 3, 4)$ be some neighbourhoods of points $t_k^{(j)}$ in which function $a(t)$ is continuous. Part $\gamma_k = \bigcup_{j=1}^{4} u_k^{(j)}$.

**Theorem 2.1.** Let $a \in N_k$, then there exists function $a_k \in N_k$ such that $a_k(t) = 1$ at $t \in \Gamma\backslash \gamma_k$ and operator $N = M_k a_I - a_k M_k$ is compact in the space $L_p(\Gamma, \rho)$.

**Proof.** From Theorem 4 of [11] it may be deduced that there exists functions $\tilde{a}_k \in N_k$ such that its domain of values coincides with the set of values of function $a(t) (t \in \Gamma)$ and operator $\bar{N} = M_k a_I - \tilde{a}_k M_k$ is compact in the space $L_p(\Gamma, \rho)$. Define function $a_k(t)$ by equality

$$a_k(t) = \begin{cases} \tilde{a}_k, & \text{for } t \in \gamma_k, \\ 1, & \text{for } t \in \Gamma \backslash \gamma_k. \end{cases}$$
Denote by $R_1$ operator defined by the equality $(R_1 \phi)(t) = \chi_1(t) \phi(t)$, where $\chi_1$ is the characteristic function of the set $\gamma_k$. Obviously, $R_1\tilde{N}R_1 = R_1NR_1$, where $N = M_k a_I - a_k M_k$. Write operator $H = N - \tilde{N}$ in the form

$$H = R_1 HR_1 + R_1 HR_2 + R_2 HR_1 + R_2 HR_2,$$

where $R_2 = I - R_1$. It is easy to be convinced that operators $R_1 HR_1$, $R_2 HR_1$, and $R_2 HR_2$ are integral operators with continuous kernels, therefore they are compact. Since $R_1 HR_1 = R_1 NR_1 - R_1 \tilde{N}R_1 = 0$, then from (8) it follows that $H$ is also compact. This operator, $N = H + \tilde{N}$ is compact. The theorem is proved. 

**Theorem 2.2.** Let $\Gamma \cap \Gamma_k \neq \emptyset$, then operator $M_k$ is not compact in the space $L_p(\Gamma, \rho)$. 

**Proof.** Assume that operator $M_k$ is not compact in $L_p(\Gamma, \rho)$. Let $\gamma = \Gamma \cup \Gamma_k$ and $t_0$ be one of intersection point of contours $\Gamma$ from $\Gamma_k$. Consider in the space $L_p(\Gamma, \rho)$ a singular operator defined by the with equality

$$A = a_I + b S_\gamma,$$

where $a(t)$ and $b(t)$ are continuous in every point $t \in \gamma \setminus \{t_0\}$ and satisfy conditions:

$$(a(t \pm 0) + b(t \pm 0) \neq 0, (a(t_0 - 0) + b(t_0 - 0)) / (a(t_0 - 0) - b(t_0 - 0)) = i$$

and

$$(a(t_0 + 0) + b(t_0 + 0)) / (a(t_0 + 0) - b(t_0 + 0)) = 1.$$

Under these conditions operator $A$ is not Noetherian [3] in the space $L_2(\gamma)$. Operator $R$, which act by the rule

$$(R \phi)(t) = (\phi(t), \phi(t - a_k))(t \in \Gamma),$$

is reversible operator from $L\left(L_2(\gamma), L_2(\Gamma)\right)$. It is verified directly that

$$RAR^{-1} = \begin{bmatrix} a_1 I + b_1 S_\Gamma & b_1 M_k \\ b_2 N_k & a_2 I + b_2 S_\Gamma \end{bmatrix},$$

(9)

where $f_1(t) = f(t)$, $f_2(t) = f(t - a_k)$ ($t \in \Gamma$),

$$(S_\Gamma \phi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi \phi(\tau)}{\tau - t} d\tau, (N_k \phi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi \phi(\tau)}{\tau - t + a_k} d\tau.$$

Since

$$\mu a_j(t - 0) + b_j(t - 0) + (1 - \mu) a_j(t + 0) + b_j(t + 0) 
eq 0$$

($t \in \Gamma$, $0 \leq \mu \leq 1$, $j = 1, 2$), then operators $a_I + b_I S_\Gamma$ ($j = 1, 2$) are Noetherian in $L_2(\Gamma)$. Then from (9) (taking into account compactness of operators $M_k$ and $N_k$) it follows that operator $A$ is Noetherian in $L_2(\gamma)$. The obtained contradiction proved that $M_k$ is not compact in the space $L_2(\Gamma)$. Moreover, since operator $M_k$ is bounded
The following equalities are valid:

\[ \int \text{operators } \rho_{\Phi} \text{ of function } D \text{ment Plemeny-Sohotski } [1, 3], \]

\[ \phi \text{ and it means projectors. Denote by } D \Phi \Gamma \text{ which is analytic in domain } D \]\n
\[ \text{Proof.} \]

\[ \text{Theorem 2.3. The following equalities are valid:} \]

\[ M_k S = h_k M_k, \quad S M_k = M_k \tilde{h}_k I, \quad (10) \]

where \( h_k(t) = \begin{cases} 1, & \text{if } t \in \tilde{I}_k^{(1)} \\ -1, & \text{if } t \in \tilde{I}_k^{(2)} \end{cases} , \quad \tilde{h}_k(t) = \begin{cases} 1, & \text{if } t \in \tilde{I}_k^{(1)} \\ -1, & \text{if } t \in \tilde{I}_k^{(2)} \end{cases} . \]

\[ \text{Proof.} \]

Introduce operators \( P = (I + S)/2, \quad Q = (I - S)/2. \) Since \( P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \) and \( P + Q = I, \) the operators \( P \) and \( Q \) are mutually supplementary projectors. Denote by \( D_\ast \) the domain bounded by contour \( \Gamma \) and by \( D_\ast \) the supplement \( D_\ast \cup \Gamma \) to the extended complex plane. Let \( \phi \in L_p(\Gamma) \). Then from definition of function \( \rho(t) \) it follows that \( \rho^{-1/p} \in L^q(\Gamma) \left( p^{-1} + q^{-1} = 1 \right) \) hence \( \phi \rho^{1/p} \in L_p(\Gamma) \), and it means \( \phi \in L_1(\Gamma) \). With the help of this function compose function

\[ \Phi_\phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\tau)}{\tau - z} d\tau \quad (z \in D_\ast) , \]

which is analytic in domain \( D_\ast \) and for almost all points \( t \in \Gamma \) there exist limit values \( \Phi_\phi^\pm(t) \) as \( z \to t \in \Gamma \) \((z \in D_\ast)\). These limit values are defined by formula (formulae Plemeny-Sohotski [1, 3])

\[ \Phi_\phi^+(t) = \frac{1}{2} \phi(t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(\tau)}{\tau - t} d\tau = (P\phi)(t). \]

Thus, function \((P\phi)(t)\) admits analytic extension in domain \( D_\ast \) so that for all \( z \in D_\ast \) equality takes place

\[ \int_{\Gamma} \frac{(P\phi)(\tau)}{\tau - z} d\tau = 0. \]

Particularly, this is valid for points \( z = t + \alpha_k \), where \( t \in \tilde{I}_k^{(2)} \). Hence

\[ \int_{\Gamma} \frac{(P\phi)(\tau)}{\tau - z + \alpha_k} d\tau = 0 \quad (t \in \tilde{I}_k^{(2)}). \]

From this we obtain that \( (M_k S \phi)(t) = -(M_k \phi)(t) \) at \( t \in \tilde{I}_k^{(2)} \). Similarly it is established that \( (M_k S \phi)(t) = (M_k \phi)(t) \) at \( t \in \tilde{I}_k^{(1)} \). This \( M_k S = h_k M_k \) and the first equality of theorem is proved. Let us prove the second equality. Denote by \( M_k \) the set of all
functions from \( L_p(\Gamma, \rho) \), vanishing in some neighborhoods of points \( t^{(3)}_k \) and \( t^{(4)}_k \). Let \( \phi \in M_k \), then (see [1,2])

\[
\frac{1}{\pi i} \int_{\Gamma} \frac{d\tau}{\tau - t} \frac{1}{\pi i} \int_{\Gamma} \frac{d\xi}{\xi - \tau - \alpha_k} = \frac{1}{\pi i} \int_{\Gamma} \frac{d\xi}{\xi - \tau - \alpha_k} = \frac{1}{\pi i} \int_{\Gamma} \frac{d\tau}{\tau - t} \frac{1}{\pi i} \int_{\Gamma} \frac{d\xi}{\xi - \tau - \alpha_k},
\]

i.e. permutation of order of integration is admissible. The last equality may be written in the following way:

\[
\frac{1}{\pi i} \int_{\Gamma} \frac{d\tau}{\tau - t} \frac{1}{\pi i} \int_{\Gamma} \frac{d\xi}{\xi - \tau - \alpha_k} = \frac{1}{\pi i} \int_{\Gamma} \frac{d\xi}{\xi - t - \alpha_k} \left[ \frac{1}{\pi i} \int_{\Gamma} \frac{d\tau}{\tau - t} + \frac{1}{\pi i} \int_{\Gamma} \frac{d\tau}{\tau - \xi - \alpha_k} \right].
\]

By direct calculation we get

\[
\frac{1}{\pi i} \int_{\Gamma} \frac{d\tau}{\tau - t} + \frac{1}{\pi i} \int_{\Gamma} \frac{d\tau}{\tau - \xi - \alpha_k} = \tilde{h}_k(\xi).
\]

Hence, \((SM_k\phi)(t) = (M_k\tilde{h}_k\phi)(t)\) for all \( \phi \in M_k \). Since the set \( M_k \) is dense in the space \( L_p(\Gamma, \rho) \) then \( SM_k = M_k\tilde{h}_k \). Theorem is proved.

From the proved theorem it follows.

**Corollary 2.1.** The following equalities are valid:

\[
M_kP = \delta_kM_k, \quad M_kQ = (1 - \delta_k)M_k, \quad PM_k = M_k\tilde{\delta}_kI, \quad QM_k = M_k\left(1 - \tilde{\delta}_k\right)I,
\]

where \( \delta_k = \frac{1+\tilde{h}_k}{\tau} \) and \( \tilde{\delta}_k = \frac{1+\tilde{h}_k}{\tau} \).

Throughout what follows we shall consider that numbers \( \alpha_k(k = 1, \ldots, m) \) are such that \( \Gamma_j \cap \Gamma \cap \Gamma_k = \emptyset \) \((j, k = 1, \ldots, n)\). A remark relative to the case when \( \Gamma_j \cap \Gamma \cap \Gamma_k \neq \emptyset \) will be given at the end of work.

**Theorem 2.4.** Let \( a \in L_{\infty}(\Gamma) \), then operators \( M_jaM_k \) \((j, k = 1, \ldots, m)\) are compact in the space \( L_p(\Gamma, \rho) \).

**Proof.** Since \( \Gamma_j \cap \Gamma \cap \Gamma_k = \emptyset \), then contour \( \Gamma \) may be split on parts \( L_1 \) and \( L_2 \) \((L_1 \cup L_2 = \Gamma)\) such that points \( t^{(3)}_j, t^{(2)}_j \) in \( L_1 \), but \( t^{(3)}_k, t^{(2)}_k \in L_2 \). Note that the possibility for some \( r \) points \( t^{(3)}_j \) and \( t^{(2)}_k \) to coincide is not excluded. Let \( X_1(X_2) \) be the characteristic function on arch \( L_1(L_2) \). Operator of multiplication by function \( X_1(t) \) \((X_2(t))\) is also denoted by \( X_1(X_2) \). Then operator \( M_jaM_k \) may be presented in the form

\[
M_jaM_k = X_1M_jaM_k + X_2M_jaM_k.
\]
It is easy to observe that operators $X_1 M_j a M_k$ and $X_2 M_j a M_k$ are compact in the space $L_p(\Gamma, \rho)$. Theorem is proved.

Further we need the following.

**Theorem 2.5.** Let $a, b \in N_k$, then operators $P a Q b M_k, P a Q b P, M_k a Q b P, M_k a Q b$ are compact in $L_p(\Gamma, \rho)$.

**Proof.** of compactness of these operators are completely similar, they are based on theorem and on equalities (11) and (12). Let us prove, for example, that operators $P a Q b M_k$ and $M_k a Q b$ are compact. By Theorem 2.1 there exist functions $a_k$ and $b_k$ such that $b M_k - M_k b_k I$ and $a M_k - M_k a_k I$ are compact. From this and from equality (11) and (12) it follows

$$P a Q b M_k = P a M_k (1 - \delta_k) b_k + T_1 = P M_k a_k b_k (1 - \delta_k) + T_2 =$$

$$= M_k a_k b_k (1 - \delta_k) + T_2 = T_2,$$

$$M_k a Q b P = a_k (1 - \delta_k) M_k b P + \tilde{T}_1 = a_k b_k (1 - \delta_k) \delta_k M_k + \tilde{T}_2 = \tilde{T}_2,$$

where $T_j, \tilde{T}_j \in T(L_p(\Gamma, \rho))$ ($j = 1, 2$). Theorem is proved.

3. **THEOREM ABOUT ESSENTIAL SPECTRUM OF OPERATORS $A$ AND $A_0$**

In this section it is proved that if condition $\Gamma_j \cap \Gamma \cap \Gamma_k = \emptyset$ ($j, k = 1, \ldots, m$) is fulfilled, then essential spectra of operators $A_0 = a P + b Q$ and $A = a P + b Q + \sum_{k=1}^{m} c_k M_k$ ($a, b \in \cap_{k=1}^{m} N_k, c_k \in L_{\infty}(\Gamma)$) coincide. Moreover, of point $\lambda \in \mathbb{C}$ does not belong to essential spectrum of operators $A$ and $A_0$, then $\text{Ind}(A - \lambda) = \text{Ind}(A_0 - \lambda)$.

In other words, property of operator $A_0$ to be Noetherian $K = \sum_{k=1}^{m} c_k M_k$.

**Theorem 3.1.** Let $a, b \in \cap_{1 \leq k \leq m} N_k$ and $c_k \in L_{\infty}(\Gamma)$. For the operator

$$A = a P + b Q + \sum_{k=1}^{m} c_k M_k$$

(13)

to be Noetherian in $L_p(\Gamma)$ it is necessary and sufficient that operator $A_0 = a P + b Q$ should be Noetherian. It operator $A_0$ is Noetherian, then

$$\text{Ind} A = \text{Ind} A_0.$$ 

The proof of this theorem uses the following.

**Lemma 3.1.** Operator $H = I + \sum_{k=1}^{m} c_k M_k$ is Noetherian in $L_p(\Gamma)$ and its index is equal to zero.
Proof. of the first part of lemma is reduced to determinations of right and left regularizers for operator \( H \). As such operator will be the operator \( R = I - \sum_{k=1}^{n} c_k M_k \). Really, using Theorem 2.4, we see for ourselves that \( RH = I + T_1 \) and \( HR = I + T_2 \), where \( T_j (j = 1, 2) \) are compact operators. Thus, operator \( H \) is Noetherian. Operator \( H_\lambda = I + \lambda \sum_{k=1}^{n} c_k M_k \) at \( 0 \leq \lambda \leq 1 \) realizes homotopy of operator \( H \) in operator \( H_0 = I \). Hence, \( \text{Ind} \; H = 0 \). Lemma is proved. 

Proof of Theorem 3.1. With the help of results of [3] it is proved that conditions \( \text{essinf} \; \vert a(t) \vert > 0 \) and \( \text{essinf} \; \vert b(t) \vert > 0 \) are necessary for operators \( A \) and \( A_0 \) to admit \( \text{essinf} \) regularization. In connection with this we shall consider that these conditions are fulfilled. Let operator \( A_0 = aP + bQ \) be Noetherian and \( R_1 \) be one of its regularizing. By Theorem 2.1 there exists functions \( a_k \) and \( b_k (\in N_k) \), such that \( \text{essinf} \; \vert a_k (t) \vert > 0 \), \( \text{essinf} \; \vert b_k (t) \vert > 0 \), operators \( M_k aI - a_k M_k \) and \( M_k bI - b_k M_k \) are compact. Consider in \( L_p (\Gamma) \) bounded operator \( B \) defined by equality

\[
B = I + \sum_{k=1}^{m} \frac{c_k}{(1 - \delta_k)b_k + \delta_k a_k} M_k ,
\]

where \( \delta_k(t) \) are functions participating in (11). Using equalities (11), we see for ourselves that operator \( A \) is possible to be presented in form \( A = BA_0 + T \), where \( T \) is a compact. From this equality and lemma 1 it follow that \( A \) admits regularization, and operator

\[
R_2 = R_1(I - \sum_{k=1}^{m} \frac{c_k}{(1 - \delta_k)b_k + \delta_k a_k} M_k)
\]

is one of its regularizing. From equality \( \text{Ind} A = -\text{Ind} R_2 = -\text{Ind} R_1 \) and \( \text{Ind} A_0 = -\text{Ind} R_1 \) it follows that \( \text{Ind} A = \text{Ind} A_0 \). Sufficiency of conditions of theorem is proved. Let us prove their necessity.

Let operator \( A \) be Noetherian. If \( \text{essinf} \; \vert a(t) \vert > 0 \) and \( \text{essinf} \; \vert b(t) \vert > 0 \), then operator \( A_0 \) may be presented in form \( A_0 = H_1 A + T_1 \), where

\[
H_1 = I - \sum_{k=1}^{m} \frac{c_k}{\delta_k a_k + (1 - \delta_k)b_k} M_k , \quad (14)
\]

and \( T_1 \in T(L_p (\Gamma, \rho)) \). From here by virtue of Lemma 3.1 it follows that \( A_0 \) is Noetherian and \( \text{Ind} A_0 = \text{Ind} A \). This, if conditions \( \text{essinf} \; \vert a(t) \vert > 0 \) and \( \text{essinf} \; \vert b(t) \vert > 0 \), are fulfilled, then theorem is proved. 

Let us prove that if operator \( A \) is Noetherian, then

\[
\text{essinf} \; \vert a(t) \vert > 0, \text{essinf} \; \vert b(t) \vert > 0 . \quad (15)
\]
Suppose that one condition (15) is broken. For definiteness admit that \( \text{ess} \inf_{t} |a(t)| = 0 \). Denote by \( l \) the union of all neighborhood of points \( t_k \) \( (k = 1, \ldots, m, j = 1, 2, 3, 4) \) at which functions \( a \) and \( b \) are continuous. Show firstly that function \( a \) does not vanish on \( l \). Suppose that \( a(t_0) = 0 \) and \( t_0 \in l \). Let real function, continuous at every point \( t \in \Gamma \setminus \{ t_0 \} \), and one-sided limits at point \( t_0 \) satisfy conditions
\[
\psi(t_0 - 0) = \begin{cases} \frac{1 + \beta(t_0)}{p}, & \text{at } \frac{p}{1 + \beta(t_0)} \leq 2, \\ \frac{1 + \beta(t_0)}{p}, & \text{at } \frac{p}{1 + \beta(t_0)} > 2, \end{cases}
\]
\[
\psi(t_0 + 0) = \begin{cases} \frac{-1 + \beta(t_0)}{p}, & \text{at } \frac{p}{-1 + \beta(t_0)} \leq 2, \\ \frac{-1 + \beta(t_0)}{p}, & \text{at } \frac{p}{-1 + \beta(t_0)} > 2, \end{cases}
\]
where \( \beta(t) = \beta_k \), at \( t = t_k \) and \( \beta(t) = 0 \) at the other points of contour \( \Gamma \). \( \beta_k \) are respectively points and numbers, participating in the definition of weight \( \rho(t) = \prod_{k=1}^{n} |r - t_k|^p \). Define functions \( \tilde{a}(t) \) and \( \tilde{b}(t) \) by equalities
\[
\tilde{a}(t) = \begin{cases} a(t), & \text{at } |a(t)| \geq \epsilon, \\ \epsilon \in \rho(t), & \text{at } |a(t)| < \epsilon, \end{cases}
\]
\[
\tilde{b}(t) = \begin{cases} b(t), & \text{at } |b(t)| \geq \epsilon, \\ \epsilon, & \text{at } |b(t)| < \epsilon, \end{cases}
\]
where \( \epsilon \) is sufficiently small numbers. Obviously
\[
|\tilde{a}(t) - a(t)| < 2\epsilon, \quad |\tilde{b}(t) - b(t)| < 2\epsilon
\]
for all \( t \in \Gamma \) and \( \tilde{b}(t) \) is continuous at \( t_0 \). Since for the operator
\[
\tilde{A} = \tilde{a}P + \tilde{b}Q + \sum_{k=1}^{m} c_k M_k
\]
estimate \( \|\tilde{A} - A\| < 2\epsilon (\|P\| + \|Q\|) \) takes place and \( A \) is Noetherian, then for sufficiently small \( \epsilon \) operator \( \tilde{A} \) is Noetherian too. The same property has also operator
\[
\tilde{F} = \tilde{f}P + Q + \sum_{k=1}^{m} \tilde{c}_k M_k,
\]
where \( \tilde{f}(t) = \tilde{a}(t)/\tilde{b}(t) \) \( (\in \cap \set{N}_k \set{N}_k) \) and \( \tilde{c}_k(t) = c_k(t)/\tilde{b}_k(t) \) \( (\in L_{\infty}(\Gamma)) \). Since \( \text{ess} \inf_{t} |\tilde{f}(t)| > 0 \), then by what was proved above operator \( \tilde{F}_0 = \tilde{f}P + Q \) is also Noetherian in \( L_{p}(\Gamma, \rho) \). Consider operator \( V = gP + Q \), where \( g(t) = \epsilon/\tilde{b}(t_0) \cdot \exp(i\tilde{a}(t)) \). By the choice of \( \psi(t) \) operator \( V \) is not Noetherian (see [3, 13]) in \( L_{p}(\Gamma, \rho) \). From the other hand, at every point \( t \in \Gamma \setminus \{ t_0 \} \) operator \( V \) is locally Noetherian (see definition in [7]), and at point \( t = t_0 \) it is locally equivalent (see definition in [7]) to Noetherian operator \( \tilde{F}_0 = \tilde{f}P + Q \). This, operator \( V \) is locally Noetherian at every points \( t \in \Gamma \). Hence (see [7, 14]), operator \( V \) is Noetherian in \( L_{p}(\Gamma, \rho) \). The obtained contradiction proves that \( a(t) \neq 0 \) and \( b(t) \neq 0 \) at \( t \in l \). By Theorem 2.1 there are functions \( a_k, b_k \in \mathbb{N}_k \), such that operators \( M_{\lambda}aI - a_k M_k \) and \( M_{\lambda}bI - b_k M_k \) are compact.
Since \( a(t) \neq 0 \) and \( b(t) \neq 0 \) at \( t \in \mathcal{I} \), then \( \inf_{t \in \mathcal{I}} |a_k(t)| > 0 \), \( \inf_{t \in \mathcal{I}} |b_k(t)| > 0 \) \( (k = 1, \ldots, m) \). Therefore operator \( A_0 \) may be presented in the form
\[
A_0 = H_1 A + T_1,
\]
where \( H_1 \) is an operator defined by equality (14), and \( T_1 \in T(L_p(\Gamma, \rho)) \). From Lemma 3.1 it follows that \( H_1 \) is Noetherian and from this it follows that operator \( A_0 = H_1 A + T_1 \) is also Noetherian in \( L_p(\Gamma, \rho) \). The last it impossible, since by supposition \( \inf_{t \in \mathcal{I}} |a(t)| = 0 \). The obtained contradiction proved that \( \inf_{t \in \mathcal{I}} |a(t)| \neq 0 \). The theorem is proved.

**Corollary 3.1.** If operator
\[
A = aP + bQ + \sum_{k=1}^m c_k M_k
\]
is Noetherian in the space \( L_p(\Gamma, \rho) \), then
\[
\inf_{t \in \mathcal{I}} |a(t)| > 0, \inf_{t \in \mathcal{I}} |b(t)| > 0.
\]

**Corollary 3.2.** The essential spectrums of operators
\[
A = aP + bQ + \sum_{k=1}^m c_k M_k \quad \text{and} \quad A_0 = aP + bQ
\]
coincide \( \hat{\sigma}(A) = \hat{\sigma}(A_0) \). In addition, if \( \lambda \notin \hat{\sigma}(A) \), then
\[
\text{Ind}(A - \lambda I) = \text{Ind}(A_0 - \lambda I).
\]

Theorem 3.1 may be reformulated in the form which underlines the basic property of operators \( K = \sum_{k=1}^m c_k M_k \):

**Theorem 3.2.** Perturbation of operators \( A_0 = aP + bQ \) by operators \( K = \sum_{k=1}^m c_k M_k \) does not violate Noetherness of \( A_0 \) and does not change its index.

Using the result of I. Gohberg and N. Krupnik [5] and Theorem 3.1, we bring necessary and sufficient conditions for operator \( A = aP + bQ + \sum_{k=1}^m c_k M_k \) to be Noetherian under the conditions that functions \( a \) and \( b \) are piecewise continuous on \( \Gamma \).

**Theorem 3.3.** Operator \( A \) is Noetherian in \( L_p(\Gamma, \rho) \) if and only \( L_p(\Gamma, \rho) \) the following two conditions are fulfilled:
1) \( a(t \pm 0) \neq 0, b(t \pm 0) \neq 0, \forall t \in \Gamma \);
2) Function \( a(t)/b(t) \) is \( \omega \) nonsingular on \( \Gamma \).
Index of operator \( A \) are calculated by formula \( \text{Ind}A = -\text{ind}V_\omega(a \cdot b^{-1}) \).
4. THEOREM ABOUT SPECTRUM OF OPERATORS AND $A_0$ WITH MATRIX COEFFICIENTS

Denote by $L^l_p(\Gamma, \rho)$ the space of vector functions $\varphi = \{\phi_j\}_{j=1}^l$ with components from $L^p(\Gamma, \rho)$. Operators $\|\delta_{ij}P\|_{l_{ij,j=1}}$, $\|\delta_{ij}Q\|_{l_{ij,j=1}}$, $\|\delta_{ij}M_k\|_{l_{ij,j=1}}$ $(k = 1, \ldots, m)$, acting in the space $L^l_p(\Gamma, \rho)$, will be also denoted by letters $P$, $Q$ respectively $M_k$.

**Theorem 4.1.** Let $a$, $b$ be matrix-functions of order $l$ with elements from $\bigcap_{1 \leq k \leq m} N_k$ and $c_k$ be matrix-functions of order $l$ with elements from $L^\infty(\Gamma)$. For the operator

$$A = aP + bQ + \sum_{k=1}^m c_k M_k$$

to be Noetherian in the space $L^l_p(\Gamma, \rho)$ it is necessary and sufficient that operator

$$A_0 = aP + bQ$$

has the same property. If operator $A_0$ is Noetherian, then

$$\text{Ind}A = \text{Ind}A_0.$$  

Proof of this theorem is similar to the proof of Theorem 3.1, therefore we do not bring it. We note only, that if $A$ is Noetherian, then

$$\text{essinf}_{t \in \Gamma} |\det a(t)| > 0, \text{essinf}_{t \in \Gamma} |\det b(t)| > 0.$$  

Theorems 3.1 and 4.1 allowed us to study more complicated singular operators, namely operators, composed from sums of products of operator of form (13).

Let

$$A = \sum_{j=1}^v \prod_{i=1}^s (a_{ji}P + b_{ji}Q + \sum_{k=1}^m c_{ji}^{(k)} M_k)$$  \hspace{1cm} (16)

where $a_{ji}, b_{ji}$ from $\bigcap_{k=1}^m N_k$ and $c_{ji}^{(k)} \in L^\infty(\Gamma)$.

**Theorem 4.2.** Operator $A$ is Noetherian in the space $L^l_p(\Gamma, \rho)$ if and only if the same property has operator

$$A_0 = \sum_{j=1}^v \prod_{i=1}^s (a_{ji}P + b_{ji}Q).$$

If operator $A_0$ is Noetherian than $\text{Ind}A = \text{Ind}A_0$.

Proof. To operator $A$ ($A_0$), acting in $L^l_p(\Gamma, \rho)$, we associate its linear extension (see [5]), operator $\tilde{A}$ ($\tilde{A}_0$), acting in $L^l_p(\Gamma, \rho)$, where $l = vs + v + 1$. Operator $\tilde{A}$ has the form

$$\tilde{A} = \tilde{a}P + \tilde{b}Q + \sum_{k=1}^m \tilde{c}_k M_k,$$  \hspace{1cm} (17)
and operator $\tilde{A}_0$ has the form
\[
\tilde{A}_0 = \tilde{a}P + \tilde{b}Q,
\]
where $\tilde{a}, \tilde{b}$ (and $\tilde{c}_k$) are matrix-functions of order $l$ with elements from $\bigcap_{k=1}^n N_k(L_\infty(\Gamma))$. Operator $A(A_0)$ and $\tilde{A}(\tilde{A}_0)$ are or are not simultaneously Noetherian in respective spaces and $\text{Ind}A = \text{Ind}\tilde{A}$ ($\text{Ind}A_0 = \text{Ind}\tilde{A}_0$). It remains to apply Theorem 3.3 to finish the proof. Theorem is proved. ■

**Corollary 4.1.** Essential spectrums of operators

\[
A = \sum_{j=1}^\nu \prod_{i=1}^s (a_{ji}P + b_{ji}Q + \sum_{k=1}^m c_{ji}^{(k)} M_k)
\]

and

\[
A_0 = \sum_{j=1}^\nu \prod_{i=1}^s (a_{ji}P + b_{ji}Q) + \sum_{k=1}^m \lambda_{ji}^{(k)} M_k
\]

coincide: $\hat{\sigma}(A) = \hat{\sigma}(A_0)$. Moreover, if $\lambda \notin \hat{\sigma}(A)$, then

\[
\text{Ind}(A - \lambda I) = \text{Ind}(A_0 - \lambda I).
\]

Theorems 4.1 and 4.2 permit to transfer on operators of form (16) different propositions from the theory of singular equations with coefficients from $L_\infty(\Gamma)$, giving condition for singular operators

\[
A_0 = \sum_{j=1}^\nu \prod_{i=1}^s (a_{ji}P + b_{ji}Q)
\]
to be Noetherian (see [3, 15] and other). For example, if functions $a_{ji}$ and $b_{ji}$ are continuous on $\Gamma$, then operator $A_0$ differs from operator $B = aP + bQ$, where $a(t) = \sum_{j=1}^\nu \prod_{i=1}^s a_{ji}(t)$ and $b(t) = \sum_{j=1}^\nu \prod_{i=1}^s b_{ji}(t)$, by a compact summand. From here and Theorem 4.2 it follows that operator $A$, defined by equality (16), is Noetherian in $L_p(\Gamma, \rho)$ if and only if there are fulfilled conditions

\[
\sum_{j=1}^\nu \prod_{i=1}^s a_{ji}(t) \neq 0, \sum_{j=1}^\nu \prod_{i=1}^s b_{ji}(t) \neq 0, (t \in \Gamma).
\]

In this case $\text{Ind}A = \text{ind} \left\{ \sum_{j=1}^\nu \prod_{i=1}^s a_{ji}(t) \right\}_{t \in \Gamma} \cup \left\{ \sum_{j=1}^\nu \prod_{i=1}^s b_{ji}(t) \right\}_{t \in \Gamma}$.

5. **EXAMPLES**

We note that in the proofs of Theorems 3.1, 4.1 and 4.2 conditions $\Gamma_j \cap \Gamma \cap \tilde{\Gamma}_k = \emptyset$ ($j, k = 1, ..., m$) were used. Naturally, the question arises: how much essential are these conditions. In this section we construct concrete examples, which show that these conditions are "almost exact": an example, showing that if conditions $\Gamma_j \cap \Gamma \cap \tilde{\Gamma}_k = \emptyset$ are broken, then the proved theorems, generally speaking, are not valid, and an example in which $\Gamma_j \cap \Gamma \cap \tilde{\Gamma}_k = \emptyset$ but nevertheless Theorem 3.1 is
valid. That is, conditions $\Gamma_j \cap \Gamma \cap \Gamma_k = \emptyset$ (j, k = 1, ..., m) in Theorems 3.1, 3.3 and 4.1 are essential, but not necessary.

**Example 5.1.** Let $\alpha_1 = 1$ and $\alpha_2 = -1$. Obviously, (remind that $\Gamma$ is the unit circle) $\Gamma_1 \cap \Gamma \cap \Gamma_2 \neq \emptyset$. Consider in $L_\rho(\Gamma, \rho)$ operator $A = \delta I + M_1 + M_2$, where $\delta \in \mathcal{C}$ and

$$(M_1 \phi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\phi(t)}{\tau - t - 1} d\tau, (M_2 \phi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\phi(t)}{\tau - t + 1} d\tau.$$

Operator $A_0 = \delta I$ is Noetherian at all $\delta \neq 0$. Suppose that operator $A = \delta I + M_1 + M_2$ is also Noetherian for all $\delta \neq 0$. Then operator $B = \delta I - M_1 - M_2$ is Noetherian too for all $\delta \neq 0$. The same property has also operator

$$F = AB = \delta^2 - M_1^2 - M_2^2 = M_1 M_2 - M_2 M_1.$$ 

By Theorem 2.4 operators $M_1^2$ and $M_2^2$ are compact in $L_\rho(\Gamma, \rho)$. Hence, operator

$$\tilde{F} = \delta I - (M_1 M_2 + M_2 M_1)$$

is Noetherian at all $\delta \neq 0$. It is directly verified that

$$(M_1 M_2 \phi \phi + M_2 M_1 \phi \phi)(t) = \frac{g(t)}{\pi i} \int_{\Gamma} \frac{\phi(t)}{\tau - t} d\tau - \frac{1}{\pi i} \int_{\Gamma} \frac{g(t) \phi(t)}{\tau - t} d\tau,$$

where

$$g(t) = \begin{cases} 2, & \text{at } t \in \Gamma \setminus f_1^{(1)} \cup f_2^{(1)} ; \\ 0, & \text{at } t \in \Gamma \cap f_1^{(1)} \cup f_2^{(1)} . \end{cases}$$

Using last equalities, operator $\tilde{F}$, defined by equality (19), can be written in the form

$$\tilde{F} = \delta I - (gS - S gI),$$

where $S$ is a singular integral with Cauchy kernel. Operator $\tilde{F}$ belongs (see [4]) to algebra generated by singular integral operators with piecewise continuous coefficients. Let $F(t, \xi)$ ($t \in \Gamma, \xi \in R$) be its symbol (see [4]). It is easy to see (we do not dwell on details) that for $p = 2$, $\rho(t) \equiv 1$, $\xi_0 = 0$, $t_0 = s^{(1)}$ and $\delta_0 = 2i$ equality $\det F(t_0, \xi_0) = 0$ takes place. This contradicts the fact that $\tilde{F}$ is Noetherian in $L_2(\Gamma)$. The respective number $\delta (\neq 0)$ may be chosen in any space $L_\rho(\Gamma, \rho)$.

**Example 5.2.** Let $\alpha_1 = 2$ and $\alpha_2 = -2$. In this case $\tilde{\Gamma}_1 = \Gamma_2$, $\tilde{\Gamma}_2 = \Gamma_1$, $\Gamma_1 \cap \Gamma \cap \Gamma_2 = \{-1\}$ and

$$(M_1 \phi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\phi(t)}{\tau - t - 2} d\tau, (M_2 \phi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\phi(t)}{\tau - t + 2} d\tau.$$

Denote by $K$ operator $K = M_1 + M_2$ and by $N$–the set of piecewise continuous function on $\Gamma$ and continuous in points $\tau = \pm 1$. 
Theorem 5.1. Let $a, b \in \mathbb{N}$. For the operator $A = aI + bS + K$ to be Noetherian in the space $L_p(\Gamma, \rho)$, it is necessary and sufficient that the same property has operator $A_0 = aI + bS$. If operator $A_0$ is Noetherian, then $\text{Ind} A = \text{Ind} A_0$.

Proof of this theorem is based on series of property of operator $K$, which will be established in following lemmas.

Lemma 5.1. For every function $h \in \mathbb{N}$ there exists functions $\tilde{h} \in \mathbb{N}$ such that operator $K h - \tilde{h} K$ is compact in $L_p(\Gamma, \rho)$. Moreover, if $h(t \pm 0) \neq 0$ ($t \in \Gamma$), then $\tilde{h}(t \pm 0) \neq 0$ too.

This statement it is easy deduced from [11].

Lemma 5.2. The following relations are valid:

\[ SK = K, \quad KS = -K, \quad K^2 = 0. \quad (20) \]

Proof. Let $\phi(t) = \sum_{k=-n}^n a_k t^k$ be trigonometric polynomial, $\phi_+(t) = \sum_{k=0}^n a_k t^k$ and $\phi_-(t) = \sum_{k=-n}^{-1} a_k t^k$. Then $(S\phi)(t) = \phi_+(t) - \phi_-(t)$ and for every point $t \in \Gamma \setminus \{1, -1\}$ equality takes place

\[ (K\phi)(t) = -2 \sum_{k=-n}^{-1} a_k \left( (t + 2)^k + (t - 2)^k \right). \]

It is easy show that $S K \phi = K \phi$ and since the set of trigonometric polynomials is dens in $L_p(\Gamma, \rho)$, then $S K = K$. Further we have

\[ KS \phi = K(\phi_+ - \phi_-) = 2 \sum_{k=-n}^{-1} a_k \left[ (t + 2)^k + (t - 2)^k \right] = -K\phi. \]

This, $KS = -K$. The last relation from (20) easily follows from the first two. Lemma is proved.

We note yet that for operators of form $F = I + f K$ ($f \in \mathbb{N}$) it is valid assertion of Lemma 3.1:

Lemma 5.3. Operator $F = I + f K$ is Noetherian and $\text{Ind} F = 0$.

Proof of theorem 5.1. If operator $A_0 = aI + bS$ is Noetherian in $L_p(\Gamma, \rho)$, then (see [5]) conditions

\[ a(t \pm 0) + b(t \pm 0) \neq 0 \text{ and } a(t \pm 0) - b(t \pm 0) \neq 0 \quad (t \in \Gamma). \]

Denote by $f$ function $f = 1/(a + b)$ ($\in \mathbb{N}$). By Lemmas 5.1 and 5.2 we convince ourselves that operator $A$ may be presented in the form

\[ A = A_0 (I + f K) + T, \]
where $T$ is a compact operator. By Lemma 5.3 operator $F = I + fK$ is Noetherian and $\text{Ind} F = 0$. Hence, operator $A$ is Noetherian too and $\text{Ind} A = \text{Ind} A_0$. Sufficiency is proved.

Prove necessity of conditions of theorem. Suppose that operator $A = aI + bS + K$ is Noetherian but operator $A_0 = aI + bS$ is not Noetherian. Let $\varepsilon$ be a positive number such that operators $\|A - A'\| < \varepsilon$, are Noetherian and $\text{Ind} A' = \text{Ind} A$. As in [3], two Noetherian operators $B_j = a_j I + b_j S$ ($j = 1, 2$) may be constructed such that $\|A_0 - B_j\| < \varepsilon$ and $\text{Ind} B_1 \neq \text{Ind} B_2$. By what was proved above, operators $A_j = a_j I + b_j S + K$ ($j = 1, 2$) are Noetherian and $\text{Ind} A_j = \text{Ind} B_j$. Hence, $\text{Ind} A_1 \neq \text{Ind} A_2$. And since $\|A - A_j\| < \varepsilon$ ($j = 1, 2$), then $\text{Ind} A_1 = \text{Ind} A_2$. The obtained contradiction proves that $A_0$ is Noetherian. Theorem is proved.

References


Abstract

We expose an algorithmic method for determining the eigenoperators of the renormalization map for a given post critical finite self-similar fractal and apply it on the fractal called “generalized Sierpinski gasket”; the computation of the “highly-symmetric” irreducible eigenoperator for this fractal can be done by $\Delta - Y$ transform, because its boundary has three points. For fractals with boundary having more than three points there is no chance of success without computer assistance. A Java program was developed to help in such cases and is used on Lindstrom’s snowflake.

Keywords: post critical finite self-similar structure, “generalized Sierpinski gasket”, Lindstrom’s snowflake, renormalization map, eigenoperators, Dirichlet form, Laplacian, diffusion process.

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1. THE RENORMALIZATION PROBLEM FOR POST CRITICAL FINITE SELF SIMILAR STRUCTURES

In this section we review the main facts on renormalization problem for post critical finite self similar structures. Some particular forms and operators are used as in [14] . Some results of Lindstrom, Metz and Sabot are remembered ([9], [10], [11], [12], [16]).

1.1. INTRODUCTION

A family of $N$ contractions on a complete metric space form an iterated function system. The attractor of the iterated function system is the unique compact set which can be written as the union of its images throw the contractions (see [2]).

A self similar structure is a compact metric space $F$ together with $N$ injective contractions with the property that the entire space can be written as the union of its images throw the contractions, the so called level 1 copies of the structure (see [1], [6]). By considering an iterated function system on $\mathbb{R}^d$ formed with similitudes and restricting them to its attractor we get a self-similar structure.

Associated to a self-similar structure are its natural map, a continuous, surjective map that encodes the points of the structure; every point has at least one address in
Lemma 1.5.2 for details). The ramifying set of the structure $B$ is the set of the intersection points of the level 1 copies; the critical set $\Gamma$ is collecting the addresses of the points in the ramifying set and the post critical set $P$ is the union of all addresses obtained by removing words in the beginning of the addresses in $\Gamma$; the initial boundary of the structure is the set of points corresponding to the addresses in $P$ (see again [1], [6] for details).

A self-similar structure is called finitely ramified if $B$ is finite and post critical finite if $P$ is finite. A self-similar structure is called affine nested fractal if it satisfies three axioms: connectivity, symmetry and a so called nesting axiom. A classification of these types of structures can be found in [13].

1.2. FORMS, OPERATORS, CONDUCTANCES

If $V$ is a finite set, we consider the subspace of linear symmetric operators $\mathcal{L}_s(V)$ on $\mathbb{R}^V$ and the associated subspace of bilinear symmetric forms $\mathbb{F}_s(V)$ throw the natural isomorphism $\Pi, \Pi(H) := \mathcal{F}_H, \mathcal{F}_H(u,v) := -(u,Hv), u,v \in \mathbb{R}^V$.

Consider the cone $\mathbb{F}_{12}'(V)$ of positive semi-definite forms satisfying $\mathcal{F}(1,1) = 0$ and the cone $\mathbb{F}_{12}(V)$ of positive semi-definite forms satisfying $\mathcal{F}(u,u) = 0 \Leftrightarrow u \text{ constant}$. Consider also the cones $\mathbb{F}_{12}'(V)$ and $\mathbb{F}_{123}(V)$ of forms in $\mathbb{F}_{12}(V)$ (and $\mathbb{F}_{12}(V)$ respectively) which satisfy the so called Markov property, that is $\mathcal{F}(u,u) \leq \mathcal{F}(u,v)$ for all $u \in \mathbb{R}^V$, where $u := 0 \vee (1 \wedge u)$, for $u \in \mathbb{R}^V$; see [14] for details. An element from $\mathbb{F}_{12}'(V)$ (or $\mathbb{F}_{123}(V)$) is called Dirichlet form (or irreducible Dirichlet form) on $V$.

Consider the cones of symmetric operators $\mathcal{L}_{12}'(V), \mathcal{L}_{12}(V), \mathcal{L}_{123}(V)$ and $\mathcal{L}_{123}(V)$ in one-to-one correspondence with the cones of forms $\mathbb{F}_{12}'(V), \mathbb{F}_{12}(V), \mathbb{F}_{123}(V)$ and $\mathbb{F}_{123}(V)$ throw $\Pi$ by Proposition 2.1.3 from [6] and Proposition 2.2 from [14]. An element in $\mathcal{L}_{123}(V)$ is called Laplacian on $V$.

Finally we consider a conductance on $V$, that is a symmetric matrix $c := (c_{pq} := c(p,q))_{p,q \in V}$ with positive entries and "0 on the diagonal". We associate with $c$ the graph $\Gamma := (V,G), G := \{(p,q) \in V|c_{pq} > 0\}$. There is an obvious one-to-one correspondence between $\mathcal{L}_{123}(V)$ and conductance matrices.

For $U \subseteq V$ and $H \in \mathcal{L}_s(V)$, consider $T : \mathbb{R}^U \rightarrow \mathbb{R}^U, J : \mathbb{R}^U \rightarrow \mathbb{R}^{V\setminus U}, X : \mathbb{R}^{V\setminus U} \rightarrow \mathbb{R}^{V\setminus U}$, so that $H \cong \begin{pmatrix} T & J' \\ J & X \end{pmatrix}$. If $H \in \mathcal{L}_{12}(V)$ then $X$ is negative definite, and $[H]_U := T - J'X^{-1}J \in \mathcal{L}_{12}(U)$ (see Lemma 2.1.5 and Theorem 2.1.6 from [6]). $[H]_U$ is called the restriction of $H$ to $U$. $(V,H)$ and $(U,[H]_U)$ are called equivalent networks. The Ohm’s law and $\Delta - Y$ transform are examples of equivalence. For $\Delta - Y$ transform $V$ is a four points set, $U$ a three points subset of $V$, and the names $\Delta$ and $Y$ come from the shape of the graphs associated to $[H]_U$ and $H$ (see also [17], Lemma 1.5.2 for details).
1.3. THE RENORMALIZATION MAP

Consider the following objects
- \( r := (r_1, \ldots, r_n), r_i > 0; \)
- \( S := \{ F_i \phi_i \}_{i=1}^{\infty} \) a connected post critical finite self-similar structure;
- \( V_0 \) the initial frontier of \( S; \)
- the cones \( F_{12}(V_0) =: F_{12}, F_{12}'(V_0) =: F_{12}, F_{123}(V_0) =: F_{123} \) on \( V_0 \) defined as above and their associated cones of operators.
- \( \mathbb{B}(V_0) := F_{12}'(V_0) - F_{12}(V_0) \) the real normed space with respect to the norm \( \| E \|_2 := \sup \{ E(u, u) \}, u \in \mathbb{V}_0, \| u \|_{V_0} = 1 \).

I. Define the renormalization map associated to \( S \) and \( r \) by:
\[
\Lambda := \Lambda_r : F_{12}(V_0) \to F_{12}(V_0), \Lambda := \Phi \circ \Psi,
\]

- \( \Psi \) is the replication "operation": \( \Psi : F_{12}(V_0) \to F_{12}(V_1), \)
\[
\Psi(A_0) := A_1, A_1(u, u) := \sum_{i=1}^{n} r_i^{-1} A_0(u \circ \phi_i, u \circ \phi_i), u \in \mathbb{V}_1, A_0 \in F_{12}(V_0).
\]
- \( \Phi \) is the trace "operation": \( \Phi : F_{12}(V_1) \to F_{12}(V_0), \)
\[
\Phi(A_1) := A_0, A_0(u, u) := \inf \{ A_1(v, v) \}, v \in \mathbb{V}_1, v_{V_0} = u, u \in \mathbb{V}_0, A_1 \in F_{12}(V_1).
\]

The one-to-one correspondence between forms and operators allow \( \Lambda \) to be defined on operators instead of forms.

II. Let \( \mathbb{K} \in \{ F_{12}, F_{12}', F_{123} \} \) and \( A, B \in \mathbb{K} \setminus \{ 0 \} \). Define
- \( A \leq_{\mathbb{K}} B \iff B - A \in \mathbb{K} \); "\( \leq_{\mathbb{K}} \)" is denoted simply by "\( \leq \";"
- \( A \sim_{\mathbb{K}} B \iff \exists \alpha, \beta > 0, \alpha A \leq_{\mathbb{K}} B \leq_{\mathbb{K}} \beta A. \)

Then
- "\( \sim_{\mathbb{K}} \)" is an equivalence relation on \( \mathbb{K} \setminus \{ 0 \}; \)
- the equivalence classes are called parts (subcones of \( \mathbb{K} \)); \( \mathbb{K}^0 \) is the most important one and obviously every \( F_{123} \)-part is contained in a \( F_{12} \)-part.

III. For \( A \in F_{123}, \) define the graph
\[
\Gamma(A) := (V_0, G(A)), G(A) := \left\{ [p, q] \subset V_0 \mid c_A(p, q) > 0 \right\}.
\]
If \( \Lambda \) there exists an eigenform in every closed convex

\((\ref{16})\) For

if \( p \) and \( q \) are

\( \Psi \) variant under specific symmetry groups

Propozit \( \text{a} {} \) 1.1. Nuică Antonio-Mihail

VI. The properties of \( \Lambda \) with respect to \( h \) and \( \leq \) (V. Metz, 1995, 2005):

1 \( \Lambda (\mathbb{F}_{12}) \subset \mathbb{F}_{12}, \Lambda \left( \mathbb{F}_{12}' \right) \subset \mathbb{F}_{12}' \left( \mathbb{F}_{12} \right) \subset \mathbb{F}_{12}'; \)

2 \( \Lambda \) is positive homogeneous, \( \leq \)-monotone and superadditive;

3 \( \Lambda \) is non-expansive with respect to \( \Lambda \)-invariant \( \mathbb{F}_{12} \)-parts;

4 \( \Lambda \) "acts" on \( \mathbb{F}_{12} \)-parts and also on \( \mathbb{F}_{12} \)-parts;

5 there exists an eigenform in every closed convex \( \Lambda \)-invariant subcone (for example \( \mathbb{F}_{12} \)-part) of \( \mathbb{F}_{12} \);

6 if \( A, B \in \mathbb{F}_{12}, A \sim_{\mathbb{F}_{12}} B, \Lambda (A) = \alpha A, \Lambda (B) = \beta B \), then \( \alpha = \beta \).

VI. The renormalization map is studied with respect to subcones which are invariant under specific symmetry groups associated to the structure \( S \):

- \( \Theta \) is called symmetry group for \( S \) \( \iff \) \( \Theta \) is formed with continuous bijections

\[ g : F \longrightarrow F \] such that \( g(V_0) \subset V_0, \forall g \in \Theta \) and if

\[ \forall i \in \{1, \ldots, N\}, g \in \Theta, \exists j \in \{1, \ldots, N\}, g' \in \Theta \text{ with } g \circ \phi_i = \chi_j \circ g'. \]

- \( A \in \mathbb{F}_i(V_0) \) is called \( \Theta \)-invariant \( \iff \)

\[ \forall \theta \in \Theta, \forall u, v \in \mathbb{R}^{V_0}, A(u \circ \theta, v \circ \theta) = A(u, v); \]
Renormalization of a generalized Sierpinski gasket and Lindstrom’s snowflake

- $H \in \mathcal{L}_s(V_0)$ is called $\Theta$-invariant $\iff$
  \[ \forall \theta \in \Theta, \forall p, q \in V_0, H_{pq} = H_{\theta(p)\theta(q)}; \]

- A conductance matrix $c$ on $V_0$ is called $\Theta$-invariant $\iff$
  \[ \forall \theta \in \Theta, \forall p, q \in V_0, c(p, q) = c(\theta(p), \theta(q)); \]

- $r = (r_1, \ldots, r_N)$ with $r_i > 0, i = 1, N$ is called $\Theta$-invariant $\iff$
  \[ (\theta \in \Theta \text{ such that } \theta(\phi_i(V_0)) = \chi_j(V_0) \implies r_i = r_j). \]

- Denote by $F_{12}'\Theta$, $F_{123}'\Theta$ and $F_{123}'\Theta$ the $\mathbb{R}$-subcones of $\Theta$-invariant forms in $F_{12}'$, $F_{12}'s$ and $F_{123}'$ respectively.

Then

- There are one-to-one correspondences between $\Theta$-invariant forms, operators and conductance matrices.

- For affine nested fractals $\Theta$ is usually the symmetry group generated by the reflections in the hyperplanes bisecting the line segments which connects points in $V_0$ and shall be denoted by $G_s$. Denote, also $F_{12}'s := F_{12}'s$, $F_{12}'s := F_{12}'s$, $F_{12}'s := F_{12}'s$.

There are one-to-one correspondences between $\Theta$-invariant forms, operators and conductance matrices.

**Theorem 1.1.** For an affine nested fractal $\mathcal{S} := \{ F, \{ \phi_i \}_{i=1}^N \}$ and $r$ $G_s$-invariant:

1. ([9], [4], [16]) There exists $\alpha > 0$ and $A_0 \in F_{12}'$ (unique up to multiplication by a positive constant) such that $\Lambda(A_0) = \alpha A_0$.

2. ([11]) if $\gamma > 0$ is the eigenvalue for $F_{12}'$, then

\[ \forall A \in F_{12}' \left( \exists \alpha > 0 \left( \lim_{k \to \infty} ((1/\gamma)^k \Lambda)^k(A) = \alpha A \right) \right). \]

For the so called $N$-gaskets A. Teplyaev & others (2007) effectively determined this unique eigenform.

**VII. The scope of renormalization.** If we start with an irreducible eigenform for the renormalization map of a connected post critical finite self-similar structure $F$, then, applying the general theory of Dirichlet forms we can construct a local regular Dirichlet form on $L^2(F, \mu)$ (together with its associated diffusion process on $F$), where $\mu$ is a well chosen ”self-similar” probability measure on $F$ with respect to some ”good” weights (J. Kigami 1993, T. Kumagai 1993).
2. RENORMALIZATION OF A GENERALIZED SIERPINSKI GASKET AND LINDSTROM’S SNOWFLAKE

2.1. AN ALGORITHMIC METHOD FOR EXISTENCE OF EIGENFORMS

I developed an algorithmic method for obtaining the existence of the irreducible eigenforms for the renormalization map of a given connected post critical finite self-similar structure (for example coming from an affine nested fractal, but having a symmetry group smaller than the maximal one). The claimed method for performing the effective renormalization on particular fractals is based on the following facts (A. Nuică 2011):

- every $F_{12}'$-part is contained in a $F_{12}$-part;
- $\Lambda$’”acts” on $F_{12}'$-parts and also on $F_{12}$-parts (V-(4));
- there exists an eigenform in every closed convex $\Lambda$-invariant subcone (for example $F_{12}'$-part) of $F_{12}$ (V-(5));
- the eigenvalues of $\Lambda$ are unique on $F_{12}'$-parts (V-(6));
- the $\Lambda$ invariant $F_{12}'$-parts (which contain eigenforms) can be graphically determined via III-(3);
- if $\partial F_{12}$ is $\Lambda$-invariant and $\{F_k\}_k$ are the $\Lambda$-invariant $F_{12}'$-parts from $\partial F_{12}$, then we can try to obtain the existence of the irreducible eigenforms via Theorem 25 from [12] (V. Metz 2005).

2.2. RENORMALIZATION OF GENERALIZED SIERPINSKI GASKET

Consider $a_1, a_2, a_3$ the vertices of an unit triangle and then $a_4 = \frac{2}{5}a_1 + \frac{3}{5}a_2$, $a_5 = \frac{3}{5}a_2 + \frac{2}{5}a_3$, $a_6 = \frac{2}{5}a_3 + \frac{3}{5}a_1$, $a_7 = \frac{1}{3}(a_1 + a_2 + a_3)$, $a'_4 = \frac{2}{5}a_1 + \frac{3}{5}a_2$, $a'_5 = \frac{2}{5}a_2 + \frac{3}{5}a_3$, $a'_6 = \frac{2}{5}a_3 + \frac{3}{5}a_1$.

Consider the similitudes $\phi_i(x) = a_i + \frac{2}{5}(x - a_i)$, $i = 1, 2, 3$ and $\phi_{i+3}(x) = a_{i+3} + \frac{1}{5}(x - a_i)$, $i = 1, 2, 3$. $\phi_i(x) = a_i + \frac{1}{5}(x - a_i)$.

The attractor $F$ of $\left(\mathbb{R}^2, \| \cdot \|; \{\phi_i, r_i\}_{i=1,7}\right)$ is called generalised Sierpinski gasket (GSG).

The ramifying set is

\[ B = \{a_4, a'_4, a_5, a'_5, a_6, a'_6\}. \]
Fig. 1.: A generalized Sierpinski gasket

The critical, post critical and boundary set are given by (Figure 1)

$$\Gamma = \{(1\hat{2}), (4\hat{1}), (2\hat{1}), (4\hat{2}), (4\hat{3}), (7\hat{3}), (23), (5\hat{2}), (3\hat{2}), (5\hat{3}), (5\hat{1}), (7\hat{1}), (3\hat{1}), (6\hat{3}), (1\hat{3}), (6\hat{1}), (6\hat{2}), (7\hat{2})\},$$

where \(\hat{k} := kk \ldots k \ldots, k \in \mathbb{1,7}\);

$$P = \{(\hat{1}), (\hat{2}), (\hat{3})\}; \quad V_0 = \{a_1, a_2, a_3\}.$$

It is easy to see that GSG is an affine nested fractal with respect to the symmetry group \(G_s\).

In the following we will consider the trivial symmetry group \(\Theta = \{Id\}\) and the cones \(\mathbb{F}^\Theta_{12'} \supseteq \mathbb{F}_{12'}\) and \(\mathbb{F}^\Theta_{12'3} \supseteq \mathbb{F}_{12'3}\) (the "largest" possible).

With respect to this group the fractal \(F\) is a "non-nested" post critical finite self-similar structure. An arbitrary operator in \(\mathcal{L}_{12'3}\) is given by

$$E = E(\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} -\alpha_2 - \alpha_3 & \alpha_3 & \alpha_2 \\ \alpha_3 & -\alpha_1 - \alpha_3 & \alpha_1 \\ \alpha_2 & \alpha_1 & -\alpha_1 - \alpha_2 \end{pmatrix}, \quad \alpha_1, \alpha_2, \alpha_3 \geq 0;$$

The cone \(\mathbb{F}_{12'}\) is given by

$$\mathbb{F}_{12'} \cong \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \mid \alpha_1 + \alpha_2 + \alpha_3 \geq 0, \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 \geq 0\}$$

and

$$\mathbb{F}^1 := \mathbb{F}_{12} \cong \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1 + \alpha_2 + \alpha_3 > 0, \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 > 0\},$$

$$\mathbb{F}^2 := \partial \mathbb{F}_{12'} \cong \{(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1 + \alpha_2 + \alpha_3 \geq 0, \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 = 0\}$$

are the \(\mathbb{F}_{12'}\)-parts.
Of course $\mathcal{F}_{12}^3 \cong \mathbb{R}^3_+$ and the $\mathcal{F}_{12}^3$-parts are given by

$$\mathcal{F}_{(1,1,1)} := \mathcal{F}_{12}^3 \cong (0, \infty)^3,$$

$$\mathcal{F}_{(0,1,1)} \cong (0) \times (0, \infty)^2,$$

$$\mathcal{F}_{(1,0,0)} \cong (0, \infty) \times (0) \times (0, \infty),$$

$$\mathcal{F}_{(0,0,1)} \cong (0) \times (0, \infty) \times (0),$$

$$\mathcal{F}_{(1,1,0)} \cong (0, \infty)^2 \times (0).$$

The relations between $\mathcal{F}_{12}^3$-parts and $\mathcal{F}_{12}$-parts are:

$$\partial \mathcal{F}_{12}^3 = \mathcal{F}_{(0,0,0)} \cup \mathcal{F}_{(0,1,0)} \cup \mathcal{F}_{(0,0,1)}.$$

Because $\Theta$ is the trivial symmetry group, the weights $\mathbf{r} := (r_1, \ldots, r_7)$ can be considered taking arbitrarily $r_i > 0$.

In order to detect the $\Lambda$-invariant $\mathcal{F}_{12}^3$-parts, we apply Proposition 1.1: $\Lambda(\mathcal{F}_{(1,1,1)}) \subset \mathcal{F}_{(1,1,1)}$ (every two points in $V_0$ can be connected by a path in $\Gamma(\Psi(\mathcal{F}_{(1,1,1)}))$ avoiding $V_0$); $\Lambda(\mathcal{F}_{(0,0,0)}) \subset \mathcal{F}_{(0,0,0)}$; in the same way $\Lambda(\mathcal{F}_{(1,0,0)}) \subset \mathcal{F}_{(1,0,0)}$ and $\Lambda(\mathcal{F}_{(0,1,1)}) \subset \mathcal{F}_{(0,1,1)}$; $\Lambda(\mathcal{F}_{(0,1,0)}) \subset \mathcal{F}_{(0,1,0)}$ (just $a_1$ and $a_3$ can be connected by a path in $\Gamma(\Psi(\mathcal{F}_{(0,1,0)}))$ avoiding $V_0$); the same for $\Lambda(\mathcal{F}_{(0,0,1)}) \subset \mathcal{F}_{(0,0,1)}$ and $\Lambda(\mathcal{F}_{(0,0,0)}) \subset \mathcal{F}_{(0,0,0)}$.

Finally we conclude that $\mathcal{F}_{(1,1,1)}$, $\mathcal{F}_{(1,0,0)}$, $\mathcal{F}_{(0,1,0)}$, $\mathcal{F}_{(0,0,1)}$ are the $\Lambda$-invariant $\mathcal{F}_{12}^3$-parts.

The eigenvalue associated to $\mathcal{F}_{(1,1,1)}$ is hard to be computed in this general case (with $\Lambda$ defined on the "largest" cone $\mathcal{F}_{12}$ and with arbitrary weights). We determine it considering $\Theta = \Gamma_{\mathcal{G}}$ (the "maximal symmetry group") and the associated cones $\mathcal{F}_{12}^3 = \mathcal{F}_{12}^\prime \equiv \{ (\alpha, \alpha, \alpha) | \alpha \geq 0 \}$. We take the restriction of $\Lambda$ to the cone $\mathcal{F}_{12}^\prime$.

Consider the $\mathcal{G}_{\mathcal{F}}$-invariant irreducible operator given by

$$H = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix};$$

and the $\mathcal{G}_{\mathcal{F}}$-invariant family of weights $\mathbf{r} = (1, 1, 1, 1, 1, 1, t)$ ($t > 0$) (more general $\mathbf{r} = (r, r, r, r, s, s, s, t)$ ($r, s, t > 0$) can be considered).

The following Proposition gives the desired "highly-symmetric" eigenoperator:

**Propoziția 2.1.** $H$ is the unique $\mathcal{G}_{\mathcal{F}}$ irreducible eigenoperator (with eigenvalue $\gamma = \frac{3r+7}{7r+15}$) for the renormalization problem of the generalized Sierpinski gasket with weights $\mathbf{r} = (1, 1, 1, 1, 1, 1, t)$ ($t > 0$), or equivalently, $H$ is the unique fixed point for $\frac{3r+7}{7r+15} \Lambda$, or $\mathcal{F} = \mathcal{F}_{\mathcal{H}}$ is the unique (up to a positive constant) "highly-symmetric" eigenform (with eigenvalue $\gamma = \frac{3r+7}{7r+15}$).

**Proof.** We proceed on solving this renormalization problem by performing calculus with equivalent networks (using $Y - \Delta$ and $\Delta - Y$ transforms and Ohm’s law; see Figure 2):
(1)→(2): All the triangles from (1) have edges 1, except the one in the middle, with edges 1/t; they transform (by $\Delta - Y$) into $Y$-s with edges 3 and 3/t respectively;

(2)→(3): Ohm’s law for series resistances in (2) gives equivalent resistances in (3) with values 3/2 and 3/(t + 1) (for edges of $Y$ in the middle);

(3)→(4): A $Y - \Delta$ transform gives a middle triangle in (4) with edges 1/(t + 1);

(4)→(5): The three triangles from (4) transform, by $\Delta - Y$, in $Y$-s; the middle hexagon in (5) have edges with $3t+7$; the other edges from the three $Y$-s have values $\frac{3}{4}(3t+7)$;

(5)→(6): Ohm’s law gives a triangle in (6) with edges $\frac{3t+7}{4(t+1)}$;

(6)→(7): A $\Delta - Y$ transform gives a $Y$ in (7) with edges $\frac{3(3t+7)}{4(t+1)}$;

(7)→(8): Ohm’s law gives a $Y$ in (8) with edges $\frac{3(3t+7)}{4(t+15)}$;

(8)→(9): A $Y - \Delta$ transform gives the triangle in (9) with edges $\frac{3t+7}{4(t+15)}$. 

Fig. 2.: Renormalization of generalized Sierpinski gasket
So, finally, $D$ is the unique "highly-symmetric" irreducible eigenoperator (with eigenvalue $\gamma = \frac{3t+7}{7t+15}$) for the renormalization problem of the generalized Sierpinski gasket with weights $r = (1,1,1,1,1,t)$ ($t > 0$); or, equivalently, $D$ is the unique fixed point for $\frac{3t+7}{7t+15} \Lambda$.

2.3. RENORMALIZATION OF LINDSTROM’S SNOWFLAKE

Let $(z_i)_{i=1,7}$ the vertices of an unit regular hexagon, $z_7$ its center and $\phi_i(x) = z_i + \frac{1}{4}(x-z_i)\), $i = 1,7$. Consider the attractor $F$ of $\{(\mathbb{R}^2, || \cdot ||; \{\phi_i, r_i\}_{i=1,7})\}, r_i = 1/3$. See Figure 3.

It is easy to verify that the ramifying set is (Figure 3)

$$B = \{z_{12}, z_{23}, z_{34}, z_{45}, z_{56}, z_{61}, z_{72}, z_{27}, z_{37}, z_{47}, z_{57}, z_{67}\}.$$

The critical set is

$$\Gamma = \{(1\bar{3}), (2\bar{4}), (3\bar{5}), (4\bar{6}), (5\bar{1}), (6\bar{2}), (1\bar{4}), (2\bar{5}), (3\bar{6}), (4\bar{1}), (5\bar{2}), (6\bar{3}), (2\bar{6}), (3\bar{1}), (4\bar{2}), (5\bar{3}), (6\bar{4}), (1\bar{5}), (7\bar{1}), (7\bar{2}), (7\bar{3}), (7\bar{4}), (7\bar{5}), (7\bar{6})\}.$$

and the post critical set is

$$P = \{(1\bar{1}), (2\bar{2}), (3\bar{3}), (4\bar{4}), (5\bar{5}), (6\bar{6})\}.$$

The boundary set is $V_0 = \{z_i\}_{i=1,7}$.

It is easy to see that $\{F, \{\phi_i\}_{i=1,7}\}$ is an affine nested fractal with respect to the "maximal" symmetry group $S_3$. 

![Fig. 3: Lindstrom’s snowflake](image-url)
We consider a \( G_r \)-invariant operator, that is

\[
H = H(a, b, c) = \begin{pmatrix}
-s & a & b & c & b & a \\
a & -s & a & b & c & b \\
b & a & -s & a & b & c \\
c & b & a & -s & a & b \\
b & c & b & a & -s & a \\
a & b & c & b & a & -s
\end{pmatrix}
\]

(where \( s := 2a + 2b + c, a, b, c \geq 0 \)) and the weights \( r := (r_1, r_2, r_3, r_4, r_5, r_6) \). Due to almost obvious geometrical reasons \( r \) is \( G_r \)-invariant \( \iff r_1 = r_2 = r_3 = r_4 = r_5 = r_6 \).

In the following we effectively determine the irreducible eigenoperator using a Java application (proceeding as in [15] for the fractal called Pentakun) and validate Theorem 1.1-(2).

1. At the beginning we consider \( D = H(1, 1, 1) \) and \( r := (1, 1, 1, 1, 1, 1) \). After 3 iterations of \( \Lambda \) we get the operators (Figure 4):

\[
\Lambda(D) = H(0.880952, 0.452381, 0.380952),
\]

\[
\Lambda^2(D) = H(0.496924, 0.241773, 0.197933),
\]

\[
\Lambda^3(D) = H(0.270785, 0.131098, 0.107044).
\]

Because the sequence \( \Lambda^n(D) \) "is approaching" zero matrix very quick (see Figure 4) we conclude that the choice of \( r = (1, 1, 1, 1, 1, 1) \) was not very good. Dividing the corresponding entries for the above matrices and taking an "average" we get \( \gamma = 0.545 \). So we found a good estimation of the eigenvalue \( \gamma \) for \( \mathbb{P}_{123}^x \).

2. Consider \( H(1, 1, 1), r = (r, r, r, r, r, r), r = 0.545 \), and take again an "average" of the entries of the above operators we get \( H(1.0, 0.5, 0.4) \). Also, after 2121 iterations we get \( H(0.000826, 0.000400, 0.000326) \), which looks ok, because we imposed calculus with error less then 0.00001.

3. Starting with \( H(1.0, 0.5, 0.4) \) and \( r = 0.545 \) we get after "just" 1979 iterations the same "suspected" operator from (2), which multiplied by a suitable positive constant, becomes \( H(8.26, 4, 3.26) \). This looks a better estimation of the eigenoperator than \( 8 \cdot H(1.0, 0.5, 0.4) = H(8.0, 4.0, 3.2) \) and \( 8 \cdot H(1.1, 1) = H(8, 8, 8) \).

So, starting with \( H(1, 1, 1) \) or \( H(1.0, 0.5, 0.4) \) for \( r = 0.545 \), with an error less then 0.00001 we get after enough iterations the same (up to a positive constant) operator \( (H(8.26, 4, 3.26)) \), the desired eigenoperator.

4. For validating Theorem 1.1-(2), we can iterate any operator in \( \mathcal{E}_{12}^1 \) to get the same result \( (H(8.26, 4, 3.26)) \) after enough iterations: for example, the operator \( H(0, -1, 1) (\mathcal{E}_{12}^1(0, 0, -1, 1)) \) approximates (with an error less then 0.00001) the eigenoperator (multiplied by a suitable constant) after 1706 iterations.
Fig. 4.: The first three iterations for \( r := (1, 1, 1, 1, 1, 1) \) and \( D = H(1, 1, 1) \) and for \( r = (r, r, r, r, r, r) \), with \( r = 0.545 \) and \( D = H(1, 1, 1) \)

References


THE SYNCHRONIZATION OF TWO CHAOTIC MODELS OF CHEMICAL REACTIONS

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Abstract
The main goal of this paper is to study the synchronization of two identical chemical chaotic systems proposed by Peng and coworkers, which is based on six reactions and three intermediary species, using an adaptive feedback method. The transient time until synchronization depends on initial conditions of two systems, the strength and the number of the controllers. To achieve the synchronization is absolutely necessary to use two controllers, for example in the second and in the third equation from the slave system; if the time of synchronization might be longer than the previous case we can use only one controller, applied in the second equation of the tridimensional system.

Keywords: domains with perturbed boundaries; Laplace operator; boundary eigenvalue problems; perturbation and pseudoperturbediteration methods, regularization.


1. INTRODUCTION

Many physical, chemical, biological processes exhibit nonlinear dynamic characteristics, in the sense of the dependence between the magnitude of the driving forces and the intensity of the resulting flows. Chemical systems can exhibit specific behaviour and this fact is very important for chemical processes and for biological structures. Among such chemical reactions, the classical Belousov - Zhabotinsky (BZ) reaction is one of the best known systems. The BZ reaction is a family of oscillating chemical reactions; in these reactions bromate ions are reduced in an acidic medium by an organic compound (usually malonic acid) with or without a catalyst (usually cerous and/or ferrous ions). Due to the fact that these reactions are auto-catalytic, the rate equations are fundamentally nonlinear. This nonlinearity can lead to the spontaneous generation of order and chaos. Schmitz, Graziani, and Hudson were the first to report observations of chaos in a chemical reaction. They conducted an experiment on the BZ reaction in a continuous flow stirred tank reactor (CSTR), where the flow rate of the feed chemicals was maintained constant and the reaction behavior was monitored with bromide ion specific and platinum wire electrodes. Ilya Prigogine argued that, far from thermodynamic equilibrium, qualitatively new behav-
Bishops appear as the system enters new dynamical regimes. From this point of view the deliberate control of these phenomena have a great practical impact despite the fact that it is very difficult; this is the reason the theoretical models are useful in these situations. In addition, the control using these models can give the informations about the selfcontrol inside the biological structures where the behavior of the dynamic systems is realized by a feedback mechanism. Over the last decade, there has been considerable progress in generalizing the concept of synchronization to include the case of coupled chaotic oscillators especially from technical reasons. When the complete synchronization is achieved, the states of both systems become practically identical, while their dynamics in time remains chaotic. Many examples of synchronization have been documented in the literature, but currently theoretical understanding of the phenomena lags behind experimental studies [1-6]. The main goal of this paper is to study the synchronization of two chemical chaotic systems based on the adaptive feedback method of control. One of these chemical models was proposed by Peng et al. [7] and it is based on six reactions and three intermediary species. This system consists in the following elementary steps:

1. $P \rightarrow A$
2. $P + C \rightarrow A + C$
3. $A \rightarrow B$
4. $A + 2B \rightarrow 3B$
5. $B \rightarrow C$
6. $C \rightarrow D$

The rate equations for these autocatalytic reactions, having three intermediary species $A, B$ and $C$ are:

$$\frac{dA}{dt} = k_1PC - k_3A - k_4AB^2$$
$$\frac{dB}{dt} = k_3A + k_4AB^2 - k_3B$$
$$\frac{dC}{dt} = k_4B - k_3C$$

This system was simplified by Andrievskii and Fradkov [8] and the time evolution of the intermediary species $x_1, x_2$ and $x_3$ is given by the nonlinear system of equations into dimensionless form:

$$\frac{dx_1}{dt} = \mu(k + x_3) - x_1x_2^2 - x_1$$
$$\frac{dx_2}{dt} = \frac{1}{\sigma}(x_1x_2^2 + x_1 - x_2)$$
$$\frac{dx_3}{dt} = -\delta(x_2 - x_3).$$

This system has a chaotic behavior, for the following constants: $\sigma = 0.015$, $\delta = 1$, $\mu = 0.301$ and $\kappa = 2.5$.

2. CHAOTIC DYNAMICS OF CHEMICAL SYSTEM

The strange attractor for this system is given in the Figure 1. The dynamics of this...
The synchronization of two chaotic models of chemical reactions

Fig. 1.: The 2D attractor with initial conditions $x_1(0) = x_2(0) = x_3(0) = 1$

Fig. 2.: a- $x_1(t)$; b- $x_2(t)$ for initial conditions $x_1(0) = x_2(0) = x_3(0) = 1$
The chaotic chemical system is given in Figure 2. The chaotic behavior is sustained by Lyapunov exponents from Figure 3. From Figure 3 we can see that one of Lyapunov exponents is positive; that means this system is chaotic for given constants.

3. THE SYNCHRONIZATION OF TWO CHAOTIC SYSTEMS

To synchronize two identical chemical systems we followed the method proposed by D. Huang [9], Hu and Xu [10], Guo et al. [11], Oancea et al. [12] based on Lyapunov-Lasalle theory.

Let be a chaotic non-autonomous:

\[ \dot{x} = f(t, x) \quad \text{where} \quad x = (x_1, x_2, \ldots)^T \in \mathbb{R}^n \quad (2) \]

is the state vector of the system and \( f = (f_1, f_2, \ldots)^T \in \mathbb{R}^n \) is the non-linear vector field of the system, which is considered as a driving system. For any \( x = (x_1, x_2, \ldots)^T \in \mathbb{R}^n \) and \( y = (y_1, y_2, \ldots)^T \in \mathbb{R}^n \) there exists a positive constant \( \ell \) such that:

\[ |f(x, t) - f(y, t)| \leq \ell \max |x_i - y_j| \quad i, j = 1, 2, \ldots, n. \]

The slave system will be:

\[ \dot{y} = f(y, t) + z(z_1, z_2, \ldots) \quad (3) \]

where \( z(z_1, z_2, \ldots) \) is the controller. If the error vector is \( e = y - x \), the objective of synchronization is to make

\[ \lim |e(t)| \rightarrow 0 \quad \text{for} \quad t \rightarrow +\infty \]
The synchronization of two chaotic models of chemical reactions

Fig. 4.: a- $x_1(t)$- black; $y_1(t)$- green; b- Synchronization errors between master and slave systems $[x_1(0) = 1, x_2(0) = 1, x_3(0) = 1; y_1(0) = -1; y_2(0) = -1, y_3(0) = -1; z_1(0) = 1; z_2(0) = 1; z_3(0) = 1]$.

The controller is of the form:

$$z_i = e_i(x_i - y_i)$$  \hspace{1cm} (4)

and

$$\dot{e}_i = -\gamma_i e_i^2, \hspace{1cm} i = 1, 2, ..., n \hspace{1cm} \text{and} \hspace{1cm} \gamma_i, \hspace{0.2cm} i = 1, 2, ..., n$$  \hspace{1cm} (5)

are arbitrary positive constants.

4. THE SYNCHRONIZATION OF TWO CHAOTIC CHEMICAL SYSTEMS

According this method of synchronization, the slave system for the system (1) will be:

$$\frac{dy_1}{dt} = 0.30\ell(k + y_3) - y_1y_2^2 - x_1 + z_1(y_1 - x_1)$$
$$\frac{dy_2}{dt} = \frac{\ell}{0.015}(y_1y_2^2 + y_1 - y_2 + z_2(y_2 - x_2))$$
$$\frac{dy_3}{dt} = (y_2 - y_3) + z_3(y_3 - x_3).$$  \hspace{1cm} (6)

and the control strength:

$$\dot{z}_1 = -(y_1 - x_1)^2$$
$$\dot{z}_2 = -(y_2 - x_2)^2$$
$$\dot{z}_3 = -(y_3 - x_3)^2.$$  \hspace{1cm} (7)

Figures 4-6 demonstrate the synchronization of the two chemical systems.

From practical point of view, the synchronization using a single controller is of interest. We obtained such synchronization for Lorenz system adding one controller in any equation of the three-dimensional system [6]. D. Huang [9], by testing the chaotic systems including the Lorenz system, Rossler system, Chua’s circuit, and the Sprott’s collection of the simplest chaotic flows found that it can use a single controller to achieve identical synchronization of a three-dimensional system. For Lorenz system this is possible only by adding the controller in the second equation.
Fig. 5.: Phase portrait a) - $(x_1, x_2$-black) and $(x_1, y_1$-red); b) - $(x_1, x_2$-black) and $(x_2, y_2$-green) for two systems $[x_1(0) = 1, x_2(0) = 1, x_3(0) = 1; y_1(0) = 1.1; y_2(0) = 1.1, y_3(0) = 1.1; z_1(0) = 1; z_2(0) = 1; z_3(0) = 1]$

Fig. 6.: The control strength $z_1(t) | x_1(0) = 1, x_2(0) = 1, x_3(0) = 1; y_1(0) = 1.1; y_2(0) = 1.1, y_3(0) = 1.1; z_1(0) = 1; z_2(0) = 1; z_3(0) = 1]$
The synchronization of two chaotic models of chemical reactions

Fig. 7.: a - Synchronization errors between master and slave for chemical systems with one controller $z_2$; b - Synchronization errors between master and slave for chemical systems with one controller $z_3$; $[x_1(0) = 1, x_2(0) = 1, x_3(0) = 1; y_1(0) = -1; y_2(0) = -1, y_3(0) = -1; z_2(0) = 1, z_3(0) = 1]$

For the systems (1), (6) and (7) we achieved the synchronization if one controller is applied only in the second equation (Figure 7a). From Figures 4b and 7a we can see that the synchronization is obtained two times later if one controller was applied in second equation of the slave system. Despite the fact that we used the very closed initial conditions for master and slave, the synchronization is not achieved when the one controller is applied in the first or the third equation of the slave system (Figure 7b). Using two controllers, in the second and in the third equations, the synchronization is obtained very fast (2 unities of time), as in the case when all the controllers are used in the slave system (Figure 8).

5. CONCLUSIONS

In this work we analyzed the dynamics of Peng and coworkers system which is based on six reactions and three intermediary species. We performed the synchronization of two systems using an adaptive feedback method. The transient time until synchronization depends on initial conditions of two systems, the strength and the number of the controllers. From practical point of view, the synchronization using a single controller is of interest. To meet this goal we tried to achieve the synchronization using a single controller in any equation of the system but this isn’t possible; only one controller is applied in the second equation of the 3-dimensional system the synchronization is achieved. In this case the time until synchronization is longer than the all the controllers are applied. Then is absolutely necessary to use even two controllers (in the second and in the third equation from the slave system); if the time of synchronization might be longer than the previous case we can use only one controller, applied in the second equation of the 3-dimensional system.
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Fig. 8.: Synchronization errors between master and slave for chemical systems with two controllers \( x_1(0) = 1, x_2(0) = 1, x_3(0) = 1; y_1(0) = -1; y_2(0) = -1; y_3(0) = -1; z_2(0) = 1; z_3(0) = 1 \)

References


THE THEORETICAL APPROACH OF FACE SEALS PRESSURE FOR HYDRODYNAMIC OPERATING MODEL

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Abstract
The power series approach is used to solve the Reynolds equation for hydrodynamic face seal lubrication. The angular misalignment of the stator and rotor is considered. Obtaining another set of partial differential equations, the Fourier series approaches the pressure components. Solving the particulars non-homogenous Euler-Cauchy equations, the analytical expression of hydrodynamic film pressure between the seal rings is computed.

Keywords: face seal, Reynolds equation, power series, Fourier series.

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1. INTRODUCTION

The face seals are used within the mechanical systems where efficient operation is required. This is the reason for that physicists, mathematicians and engineers are interested by the face seals study.

The face seal operation is conditioned by minimum friction losses, that means the full separation of the ring surfaces. By the other way, a great interspace supposes great seal leakage. The pressure distribution within the fluid supposes a separation field model [1]. The simplest model is considered the model with flat, smooth and parallel surfaces, with uniform film and linear pressure distribution. The nature of the film was the subject of many researches, most of authors considering hydrodynamic behaviour [1, 2]. The thin film hydrodynamic approach was introduced by O. Reynolds and this is the fundamental basis for the theoretical studies [4].

To solve Reynolds equation the numerical methods are widely used, as shown in [3], but the analytical solution is important from seal optimisation point of view. In [5] the pressure was approached by Fourier series and was obtained a set of ordinary differential equations, easy to solve. Pressure, load capacity and leakage rate are plotted depending by the tilt parameter. To solve analytically the Reynolds equation in cylindrical coordinates, two approaches are considered: power series and Fourier series. The theory behind the power series method to solve differential equations is rather simple but the algebraic procedures involved could be quite complex, partic-
ularly when the solution cannot be expressed in terms of elementary functions. The Fourier series are suitable due to rotation motion of the rotor.

2. THE REYNOLDS EQUATION FOR FACE SEAL

Most of the studies considers that the thickness between the seal rings centers is constant. For the viscous fluid movement between the surfaces of a mechanical face seal, the following hypothesis are assumed:
- the fluid is Newtonian and the flow is laminar;
- the exterior gravity forces and inertia forces are neglected;
- the film thickness is very small related to other dimensions of seal interface;
- the fluid velocity along the film height is very small related to the velocity components along the other direction;
- the velocity ratio is the same with the dimensions ratio;
- the height roughness are small related to film thickness.

On the $S_1$ and $S_2$ surfaces (with $z = H_1$ and $z = H_2$), the following conditions are considered:

$$
v^r = v^r_1, v^θ = v^θ_1, v^z = v^z_1, \text{ for } z = H_1 (r, θ, t)
$$

$$
v^r = v^r_2, v^θ = v^θ_2, v^z = v^z_2, \text{ for } z = H_2 (r, θ, t)
$$

(1)

where: $r$, $θ$, $z$ are the cylindrical co-ordinates; $v^r$, $v^z$, $v^θ$ are the velocity components in cylindrical co-ordinates; $p$ is the pressure, $ρ$ is the specific density and $μ$ is the fluid viscosity.

The fluid pressure depends by seal ring radius and angle, $p = p(r, θ, t)$ and the Reynolds equation in cylindrical coordinates is given below [4]:

$$
\frac{∂}{∂r} \left[ \frac{r}{μ} (H_2 - H_1)^3 \frac{∂p}{∂r} \right] + \frac{∂}{∂θ} \left[ \frac{1}{μr} (H_2 - H_1)^3 \frac{∂p}{∂θ} \right] = 6r (H_2 - H_1) \frac{∂(v^r + v^z)}{∂r} + 6r (v^r_1 - v^r_2) \frac{∂(H_2 - H_1)}{∂r} + 6 (v^θ_1 - v^θ_2) \frac{∂(H_2 - H_1)}{∂θ} + 12r (v^r_1 - v^r_2). 
$$

(2)

We suppose that $S_1$ and $S_2$ are circular and coaxial rings, where $S_1$ is stationary and $S_2$ mobile with $ω$ as angular velocity around OZ axis, as shown in fig.1.

The height $H_1 = H(r, θ)$ isn’t time dependent because defines the stationary ring position and to find it the point $N_1$ is projected over an horizontal reference surface and for $θ = 0$ as shown in fig.1 and fig.2, it obtains:

$$
z(r) = NN_1 = OO_1 ± r \tan(N_1O_1P) = OO_1 ± r \tan χ_1
$$

Considering $M_1$ another point of the $r$ radius circle and $θ$ the angle between the $N_1$ and $M_1$ points, using the fig.1 the following relation can be deduced:

$$
z(r, θ) = NN_1 = OO_1 ± r \tan χ_1 \sin θ
$$
The theoretical approach of face seals for hydrodynamic operating model

Fig. 1.: Face seal model.

Fig. 2.: Seal rings, planar section.
Because parameter $\chi_1$ is small, we approach $\tan \chi_1 \approx \sin \chi_1 \approx \chi_1$, so:

$$H_1(r, \theta) = OO_1 \pm r\chi_1 \sin \theta$$  \hspace{1cm} (3)

Similarly, for $H_2$ (but replacing $\theta$ with $\theta - \omega t$ due to $S_2$ motion) we obtain:

$$H_2(r, \theta, t) = OO_1 + h_0 \pm r\chi_2 \sin(\theta - \omega t)$$

Because the fluid is viscous, it adheres to the surfaces, so:

$v_1^0 = 0, v_1' = 0, v_2' = r\omega, v_2'' = 0$.

Considering $h_0$ constant we suppose that $v_2'' = 0$. The equation (2) becomes:

$$\frac{\partial}{\partial r} \left( \frac{r}{\mu} (H_2 - H_1)^3 \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{1}{\mu r} (H_2 - H_1)^3 \frac{\partial p}{\partial \theta} \right) = -6r\chi_2 \frac{\partial (H_2 - H_1)}{\partial \theta}$$  \hspace{1cm} (4)

We denote $h = H_2 - H_1$ and considering $\mu$ constant the equation (4) becomes:

$$\frac{\partial}{\partial r} \left( rh^3 \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( h^3 \frac{\partial p}{\partial \theta} \right) = -6\mu \omega r^2 [\chi_2 \cos(\theta - \omega t) \pm \chi_1 \cos \theta]$$  \hspace{1cm} (5)

Following non-dimensional parameters are defined: thickness $\tilde{h} = \frac{h}{h_0}$; radiuses $\tilde{r} = \frac{r}{R_0}$ and $\tilde{t} = \frac{t}{R_0}$; relative tilt $\tilde{\chi} = \chi_1 \chi_2$; film tilt-thickness parameter $\tilde{e}_2 = \frac{R_0}{h_0} \theta$ and pressure $\tilde{p} = \frac{h_0^2}{6\mu \omega \tilde{e}_2 r} p$.

Then, considering $r \in [R_t, R_e], \tilde{r} \in [\tilde{r}_t, 1]$ and $\frac{\partial}{\partial \tilde{r}} = \frac{\partial}{\partial r} \frac{\partial r}{\partial \tilde{r}}$ so $\frac{\partial}{\partial \tilde{r}} = \frac{1}{\tilde{r}} \frac{\partial}{\partial r}$ and the same sign for the last members of the $H1$ and $H2$, the thickness film equation is presented below:

$$h = H_2 - H_1 = h_0 + r\chi_2 \left[ \sin(\theta - \omega t) - \frac{\chi_1}{\chi_2} \sin \theta \right]$$

Dividing the equation above by $h_0$ gives:

$$\tilde{h} = 1 + \tilde{r}\tilde{e}_2 [\sin(\theta - \Omega) - \chi \sin \theta]$$  \hspace{1cm} (6)

Using the non-dimensional parameters, the equation (5) can be written as,

$$\frac{\partial}{\partial \tilde{r}} \left( r\tilde{h}^3 \frac{\partial \tilde{p}}{\partial \tilde{r}} \right) + \frac{\partial}{\partial \tilde{\theta}} \left( \tilde{h}^3 \frac{\partial \tilde{p}}{\partial \tilde{\theta}} \right) = -e_2 \tilde{r}^2 \left[ \cos(\theta - \Omega) - \chi \cos \theta \right]$$  \hspace{1cm} (7)

### 3. REYNOLDS EQUATION APPROACH

To simplify the notations we use instead of $\tilde{h}, \tilde{r}, \tilde{p}$, the symbols $h, r, p$. Because $e_2 < 1$, (from $R_e(\chi_1 + \chi_2) < h_0$ so $e_2(1 + \chi) < 1$) we develop $p$ as powers series after $e_2$.

$$h = 1 + e_2 h_1(r, \theta), \text{ where } h_1(r, \theta) = r \left[ \sin(\theta - \Omega) - \chi \sin \theta \right]$$

$$p = p_0(r, \theta) + e_2 p_1(r, \theta) + e_2^2 p_2(r, \theta) + \cdots$$  \hspace{1cm} (8)
Replacing the expression of pressure given by the relation (8), the equation (7) becomes:

\[
\frac{\partial}{\partial r} \left[ r(1 + e_2 h_1)^3 \frac{\partial}{\partial r} \left( \sum_{k=0}^{m} e_k p_k \right) \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{r} (1 + e_2 h_1)^3 \frac{\partial}{\partial \theta} \left( \sum_{k=0}^{m} e_k p_k \right) \right] =
\]

\[
= -e_2 r^2 [\cos(\theta - \Omega) - \chi \cos \theta]
\]

Applying the partial derivation, the following relation is obtained:

\[
\left[ (1 + e_2 h_1)^3 + 3 e_2 r (1 + e_2 h_1)^2 \frac{\partial h_1}{\partial r} \right] \sum_{k=0}^{m} e_k \frac{\partial p_k}{\partial r} + r (1 + e_2 h_1)^3 \sum_{k=0}^{m} e_k \frac{\partial^2 p_k}{\partial r^2} +
\]

\[
+ \frac{3 e_2}{r} (1 + e_2 h_1)^2 \frac{\partial h_1}{\partial \theta} \sum_{k=0}^{m} e_k \frac{\partial p_k}{\partial \theta} + \frac{1}{r} (1 + e_2 h_1)^3 \sum_{k=0}^{m} e_k \frac{\partial^2 p_k}{\partial \theta^2} =
\]

\[
= -e_2 r^2 [\cos(\theta - \Omega) - \chi \cos \theta]
\]

Using the identity of \(e_k\) coefficients gives:

- for \(e_0\):

\[
\frac{\partial p_0}{\partial r} + \frac{\partial^2 p_0}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 p_0}{\partial r \partial \theta} = 0
\]

The limit conditions \(p_0(r, \theta) = p_0(1, \theta) = 0\) and \(p_0(r, \theta) = p_0(r, \theta + 2\pi) = 0\) will give \(p_0 = 0\).

- for \(e_1\):

\[
\frac{\partial p_1}{\partial r} + \frac{\partial^2 p_1}{\partial \theta^2} + \frac{1}{r} \frac{\partial^2 p_1}{\partial r \partial \theta} = -r^2 [\cos(\theta - \Omega) - \chi \cos \theta].
\]

- for \(e_2\):

\[
\frac{\partial p_2}{\partial r} + \left(3 h_1 + 3 e_2 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_1}{\partial r} + \frac{\partial^2 p_2}{\partial \theta^2} + 3 r h_1 \frac{\partial^2 p_1}{\partial r^2} + \frac{3}{r} \frac{\partial h_1}{\partial \theta} \frac{\partial p_1}{\partial \theta} + \frac{1}{r} \frac{\partial^2 p_1}{\partial r \partial \theta} +
\]

\[
+ 3 h_1 \frac{\partial^2 p_1}{\partial r \partial \theta} = 0
\]

- for \(e_3\):

\[
\frac{\partial p_3}{\partial r} + \left(3 h_1 + 3 e_2 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_2}{\partial r} + \left(3 h_1^2 + 6 e_2 h_1 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_1}{\partial r} + \frac{\partial^2 p_3}{\partial \theta^2} + 3 r h_1 \frac{\partial^2 p_2}{\partial r^2} +
\]

\[
+ 3 r h_1 \frac{\partial^2 p_1}{\partial r \partial \theta} + 3 \frac{\partial h_1}{\partial \theta} \frac{\partial p_2}{\partial \theta} + 6 h_1 \frac{\partial h_1}{\partial \theta} \frac{\partial p_1}{\partial \theta} + \frac{1}{r} \frac{\partial^2 p_2}{\partial r \partial \theta} + 3 h_1 \frac{\partial^2 p_2}{\partial r \partial \theta} + \frac{3}{r} h_1 \frac{\partial^2 p_1}{\partial r \partial \theta} = 0
\]

For \(e^m\) member where \(m \geq 4\):

\[
\frac{\partial p_m}{\partial r} + \left(3 h_1 + 3 e_2 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_{m-1}}{\partial r} + \left(3 h_1^2 + 6 e_2 h_1 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_{m-2}}{\partial r} +
\]

\[
+ \left(h_1^3 + 3 r h_1^2 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_{m-3}}{\partial r} + r \frac{\partial^2 p_{m-2}}{\partial \theta^2} + 3 r h_1 \frac{\partial^2 p_{m-2}}{\partial r^2} + 3 r h_1 \frac{\partial^2 p_{m-2}}{\partial r \partial \theta} +
\]

\[
+ 3 r h_1 \frac{\partial^2 p_{m-2}}{\partial r \partial \theta} + 3 \frac{\partial h_1}{\partial \theta} \frac{\partial p_{m-1}}{\partial \theta} + 6 h_1 \frac{\partial h_1}{\partial \theta} \frac{\partial p_{m-2}}{\partial \theta} + \frac{1}{r} \frac{\partial^2 p_{m-2}}{\partial r \partial \theta} + 3 h_1 \frac{\partial^2 p_{m-2}}{\partial r \partial \theta} + \frac{3}{r} h_1 \frac{\partial^2 p_{m-2}}{\partial r \partial \theta} = 0.
\]
The equations of the coefficients \( e_1^2 \) and \( e_2^2 \) are presented below and will be analysed:

\[
\begin{align*}
\frac{\partial^2 p_1}{\partial r^2} + \frac{1}{r} \frac{\partial p_1}{\partial r} + \frac{\partial p_1}{\partial \theta^2} &= -r^2 [\cos(\theta - \Omega) - \chi \cos \theta] \\
r \frac{\partial^2 p_2}{\partial r^2} + \frac{1}{r} \frac{\partial p_2}{\partial r} + \frac{\partial p_2}{\partial \theta^2} &= -(3h_1 + 3\theta \frac{\partial h_1}{\partial \theta}) \frac{\partial p_1}{\partial r} - 3r h_1 \frac{\partial^2 p_1}{\partial \theta^2} - \frac{3}{r} \frac{\partial h_1}{\partial \theta} \frac{\partial p_1}{\partial r} - \frac{3}{r} \frac{\partial^2 p_1}{\partial \theta^2}.
\end{align*}
\]

(9) \quad (10)

A. Solving the equation (9)

Because \( p(r, \theta) = p(r, \theta + 2\pi) \) then \( p_1(r, \theta) = p_1(r, \theta + 2\pi) \). To solve the differential equation (9), the Fourier series is used to develop pressure \( p_1(r, \theta) \):

\[
p_1(r, \theta) = p_1^0(r) + \sum_{k \geq 1} \left[ p_1^{1k}(r) \cos k\theta + p_1^{2k}(r) \sin k\theta \right]
\]

(11)

Substitution of the form (11) into equation (9) gives:

\[
r \left[ \frac{\partial^2 p_1^0}{\partial r^2} + \sum_{k \geq 1} \left( \frac{\partial^2 p_1^{1k}}{\partial r^2} \cos k\theta + \frac{\partial^2 p_1^{2k}}{\partial r^2} \sin k\theta \right) \right] - \frac{1}{r} \left[ k^2 \sum_{k \geq 1} \left( p_1^{1k} \cos k\theta + p_1^{2k} \sin k\theta \right) \right] + \frac{dp_1^0}{dr} + \sum_{k \geq 1} \left( \frac{dp_1^{1k}}{dr} \cos k\theta + \frac{dp_1^{2k}}{dr} \sin k\theta \right) = -r^2 \left[ (\cos \Omega - \chi) \cos \theta + \sin \Omega \sin \theta \right]
\]

Identification of the coefficients of equal trigonometric functions gives the following set of differential equations:

\[
\begin{align*}
- \text{ for the free terms: } & \quad r \frac{dp_0^0}{dr} + \frac{dp_0^0}{dr} = 0, \\
- \text{ for } \cos \theta: & \quad r \frac{d^2 p_1^0}{dr^2} - \frac{1}{r} p_1^0 + \frac{dp_1^0}{dr} = -r^2 (\cos \Omega - \chi), \\
- \text{ for } \sin \theta: & \quad r \frac{d^2 p_1^0}{dr^2} - \frac{1}{r} p_1^0 + \frac{dp_1^0}{dr} = -r^2 (\sin \Omega), \\
- \text{ for } \cos k\theta, k \geq 2: & \quad r \frac{d^2 p_1^{1k}}{dr^2} - \frac{k^2}{r} p_1^{1k} + \frac{dp_1^{1k}}{dr} = 0, \\
- \text{ for } \sin k\theta, k \geq 2: & \quad r \frac{d^2 p_1^{2k}}{dr^2} - \frac{k^2}{r} p_1^{2k} + \frac{dp_1^{2k}}{dr} = 0.
\end{align*}
\]

(12)

Because \( p_r = p_1 = 0 \), we write the limit conditions for \( r = r_1 \) and \( r = 1 \). The equations \((12)_1\) and \((12)_{4,5}\) have trivial solutions, so:

\[
p_1^0 = 0, p_1^{1k} = 0, p_1^{2k} = 0, \text{ for } k > 2
\]

(13)

The equations \((12)_{2,3}\), which differ only in polynomial terms, are particular cases of non-homogenous Euler-Cauchy equation [7]:

\[
r^2 \frac{d^2 x}{dt^2} + \frac{dx}{dt} - x = \alpha r^3
\]

(14)
To solve the equation (14), setting $t = e^s (s = \ln t)$ and $x(t) = y(s)$ we have,

$$\frac{ds}{dt} = \frac{dy}{ds} \frac{dt}{ds} = \frac{1}{t} \frac{dy}{ds} \frac{1}{t} - \frac{dy}{ds} \frac{1}{t^2} = \frac{d^2y}{ds^2} \frac{1}{t^2} - \frac{dy}{ds} \frac{1}{t^2}$$

We return to relation (14) and the following equation is obtained:

$$\frac{d^2y}{ds^2} - y = ae^{3s}$$

The differential equation presented above has a particular solution as $y_0(s) = be^{3s}$.

Substitution of particular solution into equation (15) gives:

$$9be^{3s} - be^{3s} = ae^{3s}$$

where $b = \frac{a}{8}$ and $y_0(s) = \frac{a}{8}e^{3s}$.

The homogenous equation attached to the equation (15) is given below:

$$\frac{d^2y}{ds^2} - \bar{y} = 0$$

The characteristic equation is: $\lambda^2 - 1 = 0$ with $\lambda_{1,2} = \pm 1$. So, the solution of homogenous equation is:

$$\bar{y}(s) = C_1e^{\lambda_1s} + C_2e^{\lambda_2s} = C_1e^s + C_2e^{-s}$$

The solution of the differential equation (15) is given as:

$$y(s) = y_0(s) + \bar{y}(s) = \frac{a}{8}e^{3s} + C_1e^s + C_2e^{-s}$$

and the solution of the equation (14) is:

$$x(t) = \frac{a}{8}t^3 + C_1t + C_2\frac{1}{t}$$

The general solution of equation $(12)_2$ is:

$$p_1^1(r) = -\frac{1}{8} (\cos \Omega - \chi) r^3 + C_1r + C_2\frac{1}{r}$$

Using the initial conditions $p_1^1(r_i) = 0, p_1^1(1) = 0$ we obtain a linear equation system with $C_1$ and $C_2$ as unknowns.

Solving the linear equation system, the coefficients $C_1$ and $C_2$ are computed as,

$$C_1 = \frac{1}{8} (\cos \Omega - \chi) (r_i^2 + 1); C_2 = -\frac{1}{8} (\cos \Omega - \chi) r_i^2$$

then:
\[ p_1^1(r) = -\frac{1}{8} (\cos \Omega - \chi) \left[ r^3 - (r_i^2 + 1)r + r_i^2 \frac{1}{r} \right] \]  \hspace{1cm} (16)

Replacing \((\cos \Omega - \chi)\) with \(\sin \Omega\) into relation (16), the solution of the equation (12) is,

\[ p_1^1(r) = -\frac{1}{8} \sin \Omega \left[ r^3 - (r_i^2 + 1)r + r_i^2 \frac{1}{r} \right] \]  \hspace{1cm} (17)

From relations (11), (13), (16) and (17) is deduced pressure \(p_1\) as,

\[ p_1(r, \theta) = -\frac{1}{8} \sin \Omega \left[ r^3 - (r_i^2 + 1)r + r_i^2 \frac{1}{r} \right] \cdot \] 
\[ \cdot [(\cos \Omega - \chi) \cos \theta + \sin \Omega \sin \theta] \]  \hspace{1cm} (18)

**B. Solving the equation (10)**

Because \(p(r, \theta) = p(r, \theta + 2\pi)\) then \(p_2(r, \theta) = p_2(r, \theta + 2\pi)\). To solve the differential equation (10), the Fourier series is used to develop pressure \(p_2(r, \theta)\):

\[ p_2(r, \theta) = p_0^2(r) + \sum_{k \geq 1} \left[ p_{2k}^2(r) \cos k\theta + p_{2k}^2(r) \sin k\theta \right] \]  \hspace{1cm} (19)

Substitution of the relation (19) into the equation (10) gives:

\[ r \left[ \frac{d^2 p_0^2}{dr^2} + \sum_{k \geq 1} \left( \frac{d^2 p_{2k}^2}{dr^2} \cos k\theta + \frac{d^2 p_{2k}^2}{dr^2} \sin k\theta \right) \right] - \frac{1}{r} \left[ \sum_{k \geq 1} k^2 (p_{2k}^2 \cos k\theta + p_{2k}^2 \sin k\theta) \right] + \frac{dp_0^2}{dr} + \sum_{k \geq 1} \left( \frac{dp_{2k}^2}{dr} \cos k\theta + \frac{dp_{2k}^2}{dr} \sin k\theta \right) = \frac{3}{4} \left( 5r^3 - r \right) \cdot E_1 \]  \hspace{1cm} (20)

where \(E_1\) is denoted as the following expression:

\[ E_1 = \frac{1}{2} \sin 2\theta \cdot [(\cos \Omega - \chi)^2 - \sin^2 \Omega] - \cos 2\theta \cdot (\cos \Omega - \chi) \cdot \sin \Omega \]  \hspace{1cm} (21)
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The coefficients identity for equal trigonometric functions from relation (20) gives the following set of differential equations:

- for the free terms: \( r \frac{d^2 p_0}{dr^2} + \frac{dp_0}{dr} = 0 \)
- for \( \cos \theta \): \( r \frac{d^2 p_1}{dr^2} - \frac{1}{r} p_1 + \frac{dp_1}{dr} = 0 \)
- for \( \sin \theta \): \( r \frac{d^2 p_2}{dr^2} - \frac{1}{r} p_2 + \frac{dp_2}{dr} = 0 \)
- for \( \cos 2\theta \): \( r \frac{d^2 p_3}{dr^2} - \frac{2}{r} p_3 + \frac{dp_3}{dr} = 0 \)

\begin{equation}
\frac{d^2 p_0}{dr^2} + \frac{dp_0}{dr} = 0
\end{equation}

\[ - \frac{3}{4} \left( 5r^3 - \frac{r^2}{r} \right) \cdot \sin(\Omega(\cos \Omega - \chi)) \]

\begin{equation}
\frac{d^2 p_1}{dr^2} - \frac{1}{r} p_1 + \frac{dp_1}{dr} = 0
\end{equation}

\begin{equation}
\frac{d^2 p_2}{dr^2} - \frac{1}{r} p_2 + \frac{dp_2}{dr} = 0
\end{equation}

\begin{equation}
\frac{d^2 p_3}{dr^2} - \frac{2}{r} p_3 + \frac{dp_3}{dr} = 0
\end{equation}

Considering the limit conditions for the unknown functions \( r = r_i \) and \( r = 1 \), the following results are obtained:

\( p_0^0 = 0, p_1^1 = p_2^1 = 0, p_3^k = p_4^k = 0, k > 3 \) \hspace{1cm} (23)

The equations (22) are particular cases of non-homogenous Euler-Cauchy equation [7]:

\[ t^2 \frac{d^2 x}{dt^2} + t \frac{dx}{dt} - 4x = a_0 + a_4 t^4 \] \hspace{1cm} (24)

Setting \( t = e^s (s = \ln t) \) and \( x(t) = y(s) \) we have,

\[ \frac{dx}{dt} = \frac{dy}{ds} \frac{1}{t} \]

The equation (24) becomes:

\[ \frac{d^2 y}{ds^2} - 4y = a_0 + a_4 e^{4s} \] \hspace{1cm} (25)

For this kind of equation it searches a particular solution as below:

\[ y_0(s) = b_0 + b_4 e^{4s} \]

Substitution of the particular solution into the equation (25) gives:

\[ b_0 = -\frac{a_0}{4} \] and \( b_4 = \frac{a_4}{16} \).

The homogenous equation attached to the equation (25) is given below:

\[ \frac{d^2 \tilde{y}}{ds^2} - 4\tilde{y} = 0 \]
This equation has \( \lambda^2 - 4 = 0 \) as characteristic equation, with \( \lambda_{1,2} = \pm 2 \) as solutions, so the solution of the homogenous equation is:

\[
\tilde{y}(s) = C_0 e^{A_1 s} + C_1 e^{A_2 s} = C_0 e^{2s} + C_1 e^{-2s}
\]

The general solution of equation (25) is presented below:

\[
y(s) = y_0(s) + \tilde{y}(s) = -\frac{a_0}{4} + \frac{a_4}{12} e^{4s} + C_0 e^{2s} + C_1 e^{-2s}
\]

Replacing the function \( t = e^s \), is obtained the general solution for the equation (25):

\[
x(t) = C_0 t^2 + C_1 \frac{1}{t^2} - \frac{a_0}{4} + \frac{a_4}{12} t^4
\]  \hspace{1cm} (26)

The integration constants \( C_0 \) and \( C_1 \) are determined using the boundary conditions:

\( p^C_2(r_i) = 0 \) and \( p^C_2(1) = 0 \).

The integration constants are given below:

\[
C_0 = \frac{1}{1 + r_i^2} \left[ \frac{a_0}{4} - \frac{a_4}{12} (r_i^4 + r_i^2 + 1) \right] = \frac{15r_i^4 + 27r_i^2 + 15}{48(r_i^2 + 1)} \cdot \sin \Omega (\cos \Omega - \chi)
\]

\[
C_1 = \frac{r_i^2}{1 + r_i^2} \left( \frac{a_0}{4} + \frac{a_4}{12} r_i^2 \right) = -\frac{1}{8} \frac{r_i^2}{r_i^2 + 1} \cdot \sin \Omega (\cos \Omega - \chi)
\]

So,

\[
p^C_2(r) = \left( \frac{1}{48} \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 - \frac{1}{8} \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} - \frac{3r_i^2}{16} - \frac{15r_i^4}{48} r_i^4 \right) \cdot \sin \Omega (\cos \Omega - \chi)
\]  \hspace{1cm} (27)

Replacing \( \sin \Omega (\cos \Omega - \chi) \) with \( \frac{1}{2} \left[ (\cos \Omega - \chi)^2 - \sin^2 \Omega \right] \) into relation (27) the solution of the equation (22) is computed,

\[
p^C_2(r) = \frac{1}{2} \left[ \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 + \frac{1}{8} \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} + \frac{3r_i^2}{16} + \frac{15r_i^4}{48} r_i^4 \right] \cdot (\cos \Omega - \chi)^2 - \sin^2 \Omega
\]  \hspace{1cm} (28)

The pressure \( p_2(r, \theta) \) is deduced by substitution of the relations (27) and (28) into (19):

\[
p_2(r, \theta) = p^C_2 + p^C_2 = \left( -\frac{1}{48} \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 + \frac{1}{8} \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} + \frac{3r_i^2}{16} + \frac{15r_i^4}{48} r_i^4 \right) \cdot \frac{1}{2} \sin 2\theta ((\cos \Omega - \chi)^2 - \sin^2 \Omega - \cos 2\theta \sin \Omega (\cos \Omega - \chi))
\]  \hspace{1cm} (29)

It returns to initial notations as \( \tilde{p}, \tilde{r}, \tilde{r}_i \) and non-dimensional pressure becomes:

\[
\tilde{p} = e_2 \tilde{p}_1(r, \theta) + e_2^2 \tilde{p}_2(r, \theta) =
\]

\[
= -\frac{c_1^2}{8} \left[ \frac{-c_1^2}{8} \left[ \frac{r_i^4}{r_i^2 + 1} \frac{1}{r_i^2 + 1} + \frac{c_1^2}{8} \left( \frac{-1}{6} \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 + \frac{1}{8} \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} + \frac{3r_i^2}{16} + \frac{15r_i^4}{48} r_i^4 \right) \cdot \left[ \frac{1}{2} \sin 2\theta ((\cos \Omega - \chi)^2 - \sin^2 \Omega - \cos 2\theta \sin \Omega (\cos \Omega - \chi)) \right] \right]
\]  \hspace{1cm} (30)
4. CONCLUSION

The proposed method using power series and Fourier series gives analytical solution for the pressure distribution within face seal interface. In this way, the influence of the constructive parameters on the interface pressure is shown.

References


CONVERGENCE THEOREMS FOR TWO
FINITE FAMILIES OF ASYMPTOTICALLY
QUASI-NONEXPANSIVE MAPPINGS
IN BANACH SPACES
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Abstract
In this paper, we study the weak and strong convergence of finite step iteration process
with errors for two finite families of asymptotically quasi-nonexpansive mappings in
the framework of uniformly convex Banach spaces. Also, we establish some strong
convergence theorems using semi-compactness and condition (B) which are weaker than
completely continuous condition and some weak convergence theorems using Opial’s
condition and Kadec-Klee property. The results established in this paper improve and
extend some results in [1, 17] and many others from the existing literature.

Keywords: Asymptotically quasi-nonexpansive mapping, finite-step iteration process with errors, common fixed point, strong convergence, weak convergence, uniformly convex Banach space.

2010 MSC: 47H09, 47H10, 47J25.

1. INTRODUCTION AND PRELIMINARIES

Let $E$ be a real Banach space and $K$ be its nonempty subset. Let $T: K \rightarrow K$ be a
mapping, then we denote the set of all fixed points of $T$ by $F(T)$. The set of common
fixed points of two mappings $S$ and $T$ will be denoted by $F = F(S) \cap F(T)$. A mapping $T: K \rightarrow K$ is said to be:

1) nonexpansive if
$$\|Tx - Ty\| \leq \|x - y\|$$
for all $x, y \in K$.

2) quasi-nonexpansive if $F(T) \neq \emptyset$ and
$$\|Tx - p\| \leq \|x - p\|$$
for all $x \in K, p \in F(T)$.

3) asymptotically nonexpansive [5] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with
$\lim_{n \to \infty} k_n = 1$ such that
$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

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for all $x, y \in K$ and $n \geq 1$.

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$\|T^nx - p\| \leq k_n\|x - p\|$$

(4)

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

(5) uniformly $L$-Lipschitzian if there exists a positive constant $L$ such that

$$\|T^nx - T^ny\| \leq L\|x - y\|$$

(5)

for all $x, y \in K$ and $n \geq 1$.

**Remark 1.1.** It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian with $L = \sup_{n \geq 1} |k_n| \geq 1$. Also, if $F(T) \neq \emptyset$, then a nonexpansive mapping is a quasi-nonexpansive mapping and an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive mapping. But the converse is not true in general.

In 1972, Goebel and Kirk [5] proposed and analyzed a concept of asymptotically nonexpansive as an important generalization of nonexpansive mappings. They proved that if $K$ is a nonempty closed convex subset of a real uniformly convex Banach space and $T$ is an asymptotically nonexpansive self-mapping of $K$, then $T$ has a fixed point. Since then, the study of approximation theory of fixed points of asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings in Banach spaces have been studied extensively by many authors (see [2],[4],[6]-[8],[10]-[16]).


Recently, Chen and Guo [1] introduced and studied a new finite-step iteration scheme with errors for two finite families of asymptotically nonexpansive mappings in the framework of Banach spaces. The iteration scheme is as follows:
Let $K$ be a nonempty convex subset of a Banach space $E$ with $K + K \subset K$. Let $\{S_i\}_{i=1}^{N}, \{T_i\}_{i=1}^{N} : K \to K$ be $2N$ asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ defined by

\[
\begin{align*}
  x_{1} &= x \in K, \\
  x_{n}^{(0)} &= x_{n}, \\
  x_{n}^{(1)} &= \alpha_{n}^{(1)} T_{1}^{n} x_{n}^{(0)} + (1 - \alpha_{n}^{(1)}) S_{1}^{n} x_{n} + u_{n}^{(1)}, \\
  x_{n}^{(2)} &= \alpha_{n}^{(2)} T_{2}^{n} x_{n}^{(1)} + (1 - \alpha_{n}^{(2)}) S_{2}^{n} x_{n} + u_{n}^{(2)}, \\
  &\vdots \\
  x_{n}^{(N-1)} &= \alpha_{n}^{(N-1)} T_{N-1}^{n} x_{n}^{(N-2)} + (1 - \alpha_{n}^{(N-1)}) S_{N-1}^{n} x_{n} + u_{n}^{(N-1)}, \\
  x_{n}^{(N)} &= \alpha_{n}^{(N)} T_{N}^{n} x_{n}^{(N-1)} + (1 - \alpha_{n}^{(N)}) S_{N}^{n} x_{n} + u_{n}, \\
  x_{n}^{N+1} &= x_{n}, \forall n \geq 1,
\end{align*}
\]

where $\{\alpha_{n}^{(i)}\} \subset [0, 1]$ and $\{u_{n}^{(i)}\}$ are bounded sequences in $K$ for all $i \in I = \{1, 2, \ldots, N\}$, and the weak and strong convergence theorems are proved, which improve and generalize some results in [17].

If we take $u_{n}^{(i)} = 0$ for all $n \geq 1$, $i \in I$ in (6), then we obtain the following iteration scheme:

\[
\begin{align*}
  x_{1} &= x \in K, \\
  x_{n}^{(0)} &= x_{n}, \\
  x_{n}^{(1)} &= \alpha_{n}^{(1)} T_{1}^{n} x_{n}^{(0)} + (1 - \alpha_{n}^{(1)}) S_{1}^{n} x_{n}, \\
  x_{n}^{(2)} &= \alpha_{n}^{(2)} T_{2}^{n} x_{n}^{(1)} + (1 - \alpha_{n}^{(2)}) S_{2}^{n} x_{n}, \\
  &\vdots \\
  x_{n}^{(N-1)} &= \alpha_{n}^{(N-1)} T_{N-1}^{n} x_{n}^{(N-2)} + (1 - \alpha_{n}^{(N-1)}) S_{N-1}^{n} x_{n}, \\
  x_{n}^{(N)} &= \alpha_{n}^{(N)} T_{N}^{n} x_{n}^{(N-1)} + (1 - \alpha_{n}^{(N)}) S_{N}^{n} x_{n}, \\
  x_{n}^{N+1} &= x_{n}, \forall n \geq 1.
\end{align*}
\]

where $\{\alpha_{n}^{(i)}\} \subset [0, 1]$ for all $i \in I$ and the author [1] proved weak convergence theorem of iteration scheme (7).

The purpose of this article is to study the weak and strong convergence of iteration scheme (6) and (7) to converge to common fixed points for two finite families of uniformly $L$-Lipschitzian and asymptotically quasi-nonexpansive mappings in the framework of uniformly convex Banach spaces. The results established in this paper improve and extend the corresponding results of [1, 17] and many others from the existing literature.
In order to prove the main results of this paper, we need the following concepts and lemmas.

Let $E$ be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of $E$ is the function $\delta_E(\varepsilon): (0, 2) \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{1 - \frac{1}{2} \|x + y\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x - y\|\right\}.$$  

A Banach space $E$ is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Recall that a Banach space $E$ is said to satisfy Opial’s condition [9] if, for any sequence $(x_n)$ in $E$, $x_n \to x$ weakly implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

A Banach space $E$ has the Kadec-Klee property [17] if for every sequence $(x_n)$ in $E$, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

A mapping $T: K \to K$ is said to be semi-compact [2] if for any bounded sequence $(x_n)$ in $K$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $(x_{n_k}) \subset (x_n)$ such that $x_{n_k} \to x^* \in K$ strongly.

**Lemma 1.1.** (See [18]) Let $(\alpha_n)_{n=1}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n, \forall n \geq 1.$$  

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n \to \infty} \alpha_n$ exists. In particular, $(\alpha_n)_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n \to \infty} \alpha_n = 0$.

**Lemma 1.2.** (See [14]) Let $E$ be a uniformly convex Banach space and $0 < \alpha \leq t_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Suppose further that $(x_n)$ and $(y_n)$ are sequences of $E$ such that $\limsup_{n \to \infty} \|x_n\| \leq a$, $\limsup_{n \to \infty} \|y_n\| \leq a$ and $\lim_{n \to \infty} \|t_n x_n + (1 - t_n)y_n\| = a$ hold for some $a \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

**Lemma 1.3.** (See [17]) Let $E$ be a real reflexive Banach space with its dual $E^*$ has the Kadec-Klee property. Let $(x_n)$ be a bounded sequence in $E$ and $p, q \in w^*_n(x_n)$ (where $w^*_n(x_n)$ denotes the set of all weak subsequential limits of $(x_n)$). Suppose $\lim_{n \to \infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0, 1]$. Then $p = q$.

**Lemma 1.4.** (See [17]) Let $K$ be a nonempty convex subset of a uniformly convex Banach space $E$. Then there exists a strictly increasing continuous convex function
\[ \phi : [0, \infty) \to [0, \infty) \text{ with } \phi(0) = 0 \text{ such that for each Lipschitzian } T : K \to K \text{ with the Lipschitz constant } L, \]

\[ \|tT x + (1 - t)T y - T(tx + (1 - t)y)\| \leq L\phi^{-1}\left(\frac{\|x - y\|}{L}\right) \]

for all \( x, y \in K \) and all \( t \in [0, 1] \).

**Proposition 1.1.** Let \( K \) be a nonempty subset of a Banach space \( E \) and \( \{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K \) be \( 2N \) asymptotically quasi-nonexpansive mappings. Then there exist sequences \( \{k_n\}, \{h_n\} \subset [1, \infty) \) with \( k_n \to 1 \) and \( h_n \to 1 \) as \( n \to \infty \) such that

\[ \|S_n^ix - S_n^iy\| \leq k_n\|x - y\|, \forall n \geq 1, \]

and

\[ \|T_n^ix - T_n^iy\| \leq h_n\|x - y\|, \forall n \geq 1, \]

for all \( x, y \in K \) and each \( i = 1, 2, \ldots, N \). This completes the proof.

**Proof.** Since for each \( i = 1, 2, \ldots, N \), \( S_i, T_i : K \to K \) are asymptotically quasi-nonexpansive mappings, there exist sequences \( \{k_n^{(i)}\}, \{h_n^{(i)}\} \subset [1, \infty) \) with \( k_n^{(i)} \to 1 \) and \( h_n^{(i)} \to 1 \) as \( n \to \infty \) such that

\[ \|S_n^ix - S_n^iy\| \leq k_n^{(i)}\|x - y\|, \forall n \geq 1, \]

and

\[ \|T_n^ix - T_n^iy\| \leq h_n^{(i)}\|x - y\|, \forall n \geq 1, \]

for all \( x, y \in K \) and each \( i = 1, 2, \ldots, N \).

Letting

\[ k_n = \max\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(N)}\}, \quad h_n = \max\{h_n^{(1)}, h_n^{(2)}, \ldots, h_n^{(N)}\}, \]

then we have that \( \{k_n\}, \{h_n\} \subset [1, \infty) \) with \( k_n \to 1 \), \( h_n \to 1 \) as \( n \to \infty \) and

\[ \|S_n^ix - S_n^iy\| \leq k_n^{(i)}\|x - y\| \leq k_n\|x - y\|, \forall n \geq 1, \]

and

\[ \|T_n^ix - T_n^iy\| \leq h_n^{(i)}\|x - y\| \leq h_n\|x - y\|, \forall n \geq 1, \]

for all \( x, y \in K \) and each \( i = 1, 2, \ldots, N \). This completes the proof. \( \blacksquare \)
2. STRONG CONVERGENCE THEOREMS

In this section, we first prove the following lemmas in order to prove our main theorems.

Lemma 2.1. Let $E$ be a real Banach space and $K$ be a nonempty closed convex subset of $E$ with $K + K \subset K$. Let $(S_i)_{i=1}^{N}$, $(T_i)_{i=1}^{N}$ : $K \to K$ be $2N$ asymptotically quasi-nonexpansive mappings with sequences $\{h_n\}$, $\{h_n\} \subset [1, \infty)$ given in proposition 1.1 and $F = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (6), where $\{\alpha_n^{(i)}\} \subset [0, 1]$ for all $i \in I$ with the following restrictions:

(i) $\sum_{n=1}^{\infty} (k_n h_n - 1) < \infty$;

(ii) $\sum_{n=1}^{\infty} ||h_n^{(i)}|| < \infty$ for all $i \in I$.

Then the limit $\lim_{n \to \infty} ||x_n - q||$ exists for all $q \in F$.

Proof. Let $q \in F$. Then from (6), we have

$$||x_n^{(1)} - q|| = ||\alpha_n^{(1)} T_1^n x_n + (1 - \alpha_n^{(1)}) S_1^n x_n + u_n^{(1)} - q||$$

$$\leq \alpha_n^{(1)} ||T_1^n x_n - q|| + (1 - \alpha_n^{(1)}) ||S_1^n x_n - q|| + ||u_n^{(1)}||$$

$$\leq \alpha_n^{(1)} h_n ||x_n - q|| + (1 - \alpha_n^{(1)}) k_n ||x_n - q|| + ||u_n^{(1)}|| + k_n h_n ||u_n^{(1)}||$$

$$\leq k_n h_n ||x_n - q|| + k_n h_n ||u_n^{(1)}||. \quad (8)$$

Again using (6) and (8), we obtain

$$||x_n^{(2)} - q|| = ||\alpha_n^{(2)} T_2^n x_n^{(1)} + (1 - \alpha_n^{(2)}) S_2^n x_n^{(1)} + u_n^{(2)} - q||$$

$$\leq \alpha_n^{(2)} ||T_2^n x_n^{(1)} - q|| + (1 - \alpha_n^{(2)}) ||S_2^n x_n^{(1)} - q|| + ||u_n^{(2)}||$$

$$\leq \alpha_n^{(2)} k_n h_n ||x_n^{(1)} - q|| + (1 - \alpha_n^{(2)}) k_n ||x_n - q|| + ||u_n^{(2)}|| + k_n h_n ||u_n^{(2)}||$$

$$\leq \alpha_n^{(2)} k_n h_n ||x_n^{(1)} - q|| + k_n h_n ||x_n - q|| + k_n h_n ||u_n^{(1)}|| + k_n h_n ||u_n^{(2)}||$$

$$\leq k_n h_n^2 ||x_n - q|| + k_n h_n^2 ||u_n^{(1)}|| + ||u_n^{(2)}||. \quad (9)$$

Continuing the above process, we get that

$$||x_n^{(i)} - q|| \leq k_n^i h_n^i ||x_n - q|| + k_n^i h_n^i \sum_{k=1}^{i} ||u_n^{(k)}|| \quad (10)$$
Let \( E \) be a real uniformly convex Banach space and \( K \) be a nonempty closed convex subset of \( E \) with \( K + K \subset K \).

Let \( \{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K \) be \( 2N \) uniformly \( L \)-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences \( \{k_n\}, \{h_n\} \subset [1, \infty) \) given in proposition 1.1 and \( F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset \). Let \( \{x_n\} \) be the sequence defined by (6), where \( \{a_n^{(i)}\} \subset [a, 1 - a] \) for some \( a \in (0, 1) \) and all \( i \in I \) with the following restrictions:

\( i \) \( \sum_{n=1}^\infty (k_n h_n - 1) < \infty; \)

\( ii \) \( \sum_{n=1}^\infty |a_n^{(i)}| < \infty \) for all \( i \in I \).

Then \( \lim_{n \to \infty} \|S_i^n x_n - T_i^n x_n^{(i-1)}\| = 0 \) for all \( i \in I \).

**Proof.** By Lemma 2.1, we know that \( \lim_{n \to \infty} \|x_n - q\| \) exists. So we can assume that

\[ \lim_{n \to \infty} \|x_n - q\| = d \tag{12} \]

for all \( q \in F \), where \( d \geq 0 \) is nonnegative number. It follows from condition \( ii \), (10), (12) and \( \lim_{n \to \infty} k_n h_n = 1 \) that

\[ \limsup_{n \to \infty} \|x_n^{(N-1)} - q\| \leq d \tag{13} \]

and so

\[ \limsup_{n \to \infty} \|T_i^n x_n^{(N-1)} - q + u_n^{(N)}\| \leq d. \tag{14} \]

Also,

\[ \limsup_{n \to \infty} \|S_i^n x_n - q + u_n^{(N)}\| \leq d. \tag{15} \]

Further, from (6) and (12), we have

\[ d = \lim_{n \to \infty} \|x_n^{(N)} - q\| \]

\[ = \lim_{n \to \infty} \|a_n^{(N)} (T_i^n x_n^{(N-1)} - q + u_n^{(N)}) + (1 - a_n^{(N)}) (S_i^n x_n - q + u_n^{(N)}) \|. \]
By Lemma 1.2, we get that

\[
\lim_{n \to \infty} \| S_n^{n} x_n - T_n^{n} x_n^{(N-1)} \| = 0
\]

and

\[
\lim_{n \to \infty} \| T_n^{n} x_n^{(N-1)} - q + u_n^{(N)} \| = d.
\]

From (13), we have

\[
d = \liminf_{n \to \infty} \| T_n^{n} x_n^{(N-1)} - q + u_n^{(N)} \| \\
\leq \liminf_{n \to \infty} h_n \| x_n^{(N-1)} - q \| + \lim_{n \to \infty} \| u_n^{(N)} \| \\
= \liminf_{n \to \infty} \| x_n^{(N-1)} - q \| \leq \limsup_{n \to \infty} \| x_n^{(N-1)} - q \| \leq d
\]

and so

\[
\lim_{n \to \infty} \| x_n^{(N-1)} - q \| = d. \tag{16}
\]

It follows from the condition (ii), (10), (12) and \( \lim_{n \to \infty} k_n h_n = 1 \) that

\[
\limsup_{n \to \infty} \| x_n^{(N-2)} - q \| \leq d.
\]

Further, we know that

\[
\limsup_{n \to \infty} \| T_{n-1}^{n} x_n^{(N-2)} - q + u_n^{(N-1)} \| \leq d \tag{17}
\]

and

\[
\limsup_{n \to \infty} \| S_{n-1}^{n} x_n - q + u_n^{(N-1)} \| \leq d. \tag{18}
\]

From (6) and (16), we have

\[
d = \lim_{n \to \infty} \| x_n^{(N-1)} - q \| \\
= \lim_{n \to \infty} \| \alpha_n^{(N-1)} (T_{n-1}^{n} x_n^{(N-2)} - q + u_n^{(N-1)}) \| \\
+ (1 - \alpha_n^{(N-1)}) (S_{n-1}^{n} x_n - q + u_n^{(N-1)}) \|. \tag{19}
\]

It follows from (17)-(19) and Lemma 1.2 that

\[
\lim_{n \to \infty} \| S_{n-1}^{n} x_n - T_{n-1}^{n} x_n^{(N-2)} \| = 0.
\]

Continuing the above process, we obtain the result of Lemma 2.2. This completes the proof.
Lemma 2.3. Under the assumptions of Lemma 2.2, if
\[ \lim_{n \to \infty} ||x_n - S^n_i x_n|| = 0 \]  
(20)
for all \( i \in I \). Then
\[ \lim_{n \to \infty} ||x_n - T_i x_n|| = 0, \quad \forall i \in I. \]

Proof. Since \( \lim_{n \to \infty} ||S^n_i x_n - T^n_i x^{(i-1)}|| = 0 \) for all \( i \in I \) by Lemma 2.2. It follows from (20) that
\[ \lim_{n \to \infty} ||x_n - T^n_i x^{(i-1)}|| = 0 \]  
(21)
for all \( i \in I \). Next, from (6), we have
\[ ||x_n - x_{n+1}|| \leq \alpha^{(N)}_n ||x_n - T^n_N x^{(N-1)}|| + (1 - \alpha^{(N)}_n) ||x_n - S^n_N x_n|| + ||\mu^{(N)}_n||. \]
Using (20), (21) and \( \lim_{n \to \infty} ||\mu^{(N)}_n|| = 0 \), we have
\[ \lim_{n \to \infty} ||x_n - x_{n+1}|| = 0. \]  
(22)
Since \( \lim_{n \to \infty} ||x_n - T^n_i x_n|| = 0 \) by (21) and
\[ ||x_n - T^n_i x_n|| \leq ||x_n - T^n_i x^{(i-1)}|| + ||T^n_i x^{(i-1)} - T^n_i x_n|| \]
\[ \leq ||x_n - T^n_i x^{(i-1)}|| + L||x_n^{(i-1)} - x_n|| \]
\[ \leq ||x_n - T^n_i x^{(i-1)}|| + L\alpha^{(i-1)}_n ||T^n_{i-1} x^{(i-2)} - x_n|| \]
\[ + L(1 - \alpha^{(i-1)}_n) ||S^n_{i-1} x_n - x_n|| \]
\[ + L||\mu^{(i-1)}_n|| \]  
(23)
for all \( i = 1, 2, \ldots, N \). From (20), (21), (23) and \( \lim_{n \to \infty} ||\mu^{(i-1)}|| = 0 \), we have
\[ \lim_{n \to \infty} ||x_n - T^n_i x_n|| = 0 \]  
(24)
for all \( i \in I \). It follows from (22) and (24) that
\[ ||x_n - T_i x_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}_i x_{n+1}|| \]
\[ + ||T^{n+1}_i x_{n+1} - T^n_i x_n|| + ||T^n_i x_n - T_i x_n|| \]
\[ \leq ||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}_i x_{n+1}|| \]
\[ + L||x_{n+1} - x_n|| + L||T^n_i x_n - x_n|| \]
\[ \leq (1 + L)||x_n - x_{n+1}|| + ||x_{n+1} - T^{n+1}_i x_{n+1}|| + L||T^n_i x_n - x_n||. \]
Using (22) and (24), we get that
\[ \lim_{n \to \infty} ||x_n - T_i x_n|| = 0. \]
for all \( i \in I \). This completes the proof. \(\blacksquare\)

**Lemma 2.4.** Under the assumptions of Lemma 2.2, if

\[
\|x - T_i y\| \leq \|S_i x - T_i y\| \tag{26}
\]

for all \( x, y \in K \) and \( i \in I \). Then

\[
\lim_{n \to \infty} \|x_n - S_i x_n\| = \lim_{n \to \infty} \|x_n - T_i x_n\| = 0, \quad \forall \ i \in I.
\]

**Proof.** By (26), we obtain that

\[
0 \leq \|x_n - T_i^n x_n^{(i-1)}\| \leq \|S_i x_n - T_i^n x_n^{(i-1)}\| \leq \|S_i^n x_n - T_i^n x_n^{(i-1)}\| \tag{27}
\]

for all \( i \in I \). It follows from (27) and Lemma 2.2 that

\[
\lim_{n \to \infty} \|S_i x_n - T_i^n x_n^{(i-1)}\| = \lim_{n \to \infty} \|x_n - T_i^n x_n^{(i-1)}\| = 0. \tag{28}
\]

Since

\[
\|x_n - S_i x_n\| \leq \|x_n - T_i^n x_n^{(i-1)}\| + \|T_i^n x_n^{(i-1)} - S_i x_n\|. \tag{29}
\]

Using (28) in (29), we obtain

\[
\lim_{n \to \infty} \|x_n - S_i x_n\| = 0 \tag{30}
\]

for all \( i \in I \). Also,

\[
\|x_n - S_i^n x_n\| \leq \|x_n - T_i^n x_n^{(i-1)}\| + \|T_i^n x_n^{(i-1)} - S_i^n x_n\|. \tag{31}
\]

Using (28) and Lemma 2.2 in (31), we obtain

\[
\lim_{n \to \infty} \|x_n - S_i^n x_n\| = 0 \tag{32}
\]

for all \( i \in I \). Thus \( \lim_{n \to \infty} \|x_n - T_i x_n\| = 0 \) for all \( i \in I \) by Lemma 2.3. This completes the proof. \(\blacksquare\)

**Theorem 2.1.** Let \( E \) be a real Banach space and \( K \) be a nonempty closed convex subset of \( E \) with \( K + K \subset K \). Let \( \{S_i\}_{i=1}^N, \{T_i\}_{i=1}^N : K \to K \) be 2N asymptotically quasi-nonexpansive mappings with sequences \( \{k_n\}, \{h_n\} \subset [1, \infty) \) and \( F = \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset \). Let \( \{x_n\} \) be the sequence defined by (6), where \( \{\alpha_n^{(i)}\} \subset [0, 1] \) for all \( i \in I \) with the following restrictions:

(i) \( \sum_{n=1}^\infty (k_n h_n - 1) < \infty \);

(ii) \( \sum_{n=1}^\infty \|u_n^{(i)}\| < \infty \) for all \( i \in I \).
The necessity of Theorem 2.1 is obvious. So, we will prove the sufficiency. Assume that \( \lim \inf_{n \to \infty} d(x_n, F) = 0 \). Taking the infimum over all \( q \in F \) in (11), we have

\[
d(x_{n+1}, F) \leq [1 + (k^N_n h^N_n - 1)]d(x_n, F) + Q \sum_{k=1}^{N} \|u^{(k)}_n\|.
\]

By using the conditions (i), (ii) and Lemma 1.1, we know that \( \lim_{n \to \infty} d(x_n, F) \) exists and so \( \lim_{n \to \infty} d(x_n, F) = 0 \).

Now, we show that \( \{x_n\} \) is a Cauchy sequence in \( K \). In fact, letting \( B_n = (k^N_n h^N_n - 1) \), \( C_n = Q \sum_{k=1}^{N} \|u^{(k)}_n\| \) in (11). For any positive integers \( m, n, m > n \), from \( 1 + x \leq e^x \) for all \( x \geq 0 \) and (11), we have

\[
\|x_m - q\| \leq (1 + B_{m-1})\|x_{m-1} - q\| + C_{m-1}
\]

\[
\leq e^{B_{m-1}}\|x_{m-1} - q\| + C_{m-1}
\]

\[
\leq e^{B_{m-1}}(e^{B_{m-2}}\|x_{m-2} - q\| + C_{m-2}) + C_{m-1}
\]

\[
\leq e^{B_{m-1} + B_{m-2}}\|x_{m-2} - q\| + e^{B_{m-1}}C_{m-2} + C_{m-1}
\]

\[
\leq e^{B_{m-1} + B_{m-2}}\|x_{m-2} - q\| + e^{B_{m-1}}(C_{m-2} + C_{m-1})
\]

\[
\leq \ldots
\]

\[
\leq (\sum_{k=n}^{m-1} e^{B_k})\|x_n - q\| + \left( \sum_{k=n}^{m-2} e^{B_k} \right) \sum_{k=n}^{m-1} C_k
\]

\[
\leq W\|x_n - q\| + W \sum_{k=n}^{\infty} C_k
\]

where \( W = \sum_{n=1}^{\infty} e^{B_n} \). Thus for any \( q \in F \), we have

\[
\|x_m - x_n\| \leq \|x_m - q\| + \|x_n - q\|
\]

\[
\leq (1 + W)\|x_n - q\| + W \sum_{k=n}^{\infty} C_k.
\]

Taking the infimum over all \( q \in F \), we obtain that

\[
\|x_m - x_n\| \leq (1 + W)d(x_n, F) + W \sum_{k=n}^{\infty} C_k.
\]

It follows from \( \sum_{n=1}^{\infty} C_n < \infty \) and \( \lim_{n \to \infty} d(x_n, F) = 0 \) that \( \{x_n\} \) is a Cauchy sequence, \( K \) is a closed subset of \( E \) and so \( \{x_n\} \) converges strongly to \( q_0 \in K \). Further, \( F(T_i) \) and \( F(S_j) \) \( (i = 1, 2, \ldots, N) \) are closed sets, and so \( F \) is a closed subset of \( K \). Therefore, \( q_0 \in F \). that is, \( \{x_n\} \) converges strongly to a common fixed point of the mappings \( \{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\} \) in \( K \). This completes the proof. \( \blacksquare \)
A family $\{T_i : 1, 2, \ldots, m\}$ of $m$ self-mappings of $K$ with $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ is said to satisfy condition (B) [1] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\max_{1 \leq i \leq m} \{\|x - T_i x\|\} \geq f(d(x, F))$ for all $x \in K$.

**Theorem 2.2.** Under the assumptions of Lemma 2.4, if the family $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ satisfies condition (B), then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$.

**Proof.** By Lemma 2.4, we know that $\lim_{n \to \infty} ||x_n - S_i x_n|| = 0$ and $\lim_{n \to \infty} ||x_n - T_i x_n|| = 0$ for all $i \in I$, and so $\max_{1 \leq i \leq N} \{||x_n - S_i x_n||, ||x_n - T_i x_n||\} \to 0$ ($n \to \infty$). It follows from the condition (B) that $\lim_{n \to \infty} f(d(x_n, F)) = 0$. By the proof of Theorem 2.1, we know that $\lim_{n \to \infty} d(x_n, F)$ exists. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $f(0) = 0$ and so $\lim_{n \to \infty} d(x_n, F) = 0$. By Theorem 2.1, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$. This completes the proof. $\blacksquare$

**Remark 2.1.** Since a nonexpansive and an asymptotically nonexpasive mapping with $F(T) \neq \emptyset$ are asymptotically quasi-nonexpasive mappings. Theorem 2.2 improves and generalizes Theorem 2.2 in [1] and Theorem 1 in [17].

**Theorem 2.3.** Under the assumptions of Lemma 2.4, if there exists a $T_i$ or $S_i$, $i \in I$, which is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$.

**Proof.** Without loss of generality, we can assume that $T_1$ is semi-compact. From Lemma 2.1 we know that the sequence $\{x_n\}$ is bounded and $\lim_{n \to \infty} ||x_n - S_i x_n|| = 0$ and $\lim_{n \to \infty} ||x_n - T_i x_n|| = 0$ for all $i \in I$ by Lemma 2.4. Since $T_1$ is semi-compact and $\lim_{n \to \infty} ||x_n - T_1 x_n|| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ as $i \to \infty$. Thus

$$||x^* - T_i x^*|| = \lim_{i \to \infty} ||x_{n_i} - T_i x_{n_i}|| = 0$$

and

$$||x^* - S_i x^*|| = \lim_{i \to \infty} ||x_{n_i} - S_i x_{n_i}|| = 0$$

for all $i \in I$. This implies that $x^* \in F = \bigcap_{i=1}^{N} F(S_i) \cap F(T_1)$ and so $\lim \inf_{n \to \infty} d(x_n, F) \leq \lim \inf_{n \to \infty} d(x_n, F) \leq \lim \inf_{n \to \infty} ||x_n - x^*|| = 0$. It follows from Theorem 2.1 that $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\}$ in $K$. This completes the proof. $\blacksquare$

**Remark 2.2.** Since an asymptotically nonexpasive mappings with $F(T) \neq \emptyset$ is an asymptotically quasi-nonexpasive mapping. Theorem 2.3 improves and generalizes Theorem 2.3 in [1].
3. WEAK CONVERGENCE THEOREMS

In this section, we prove weak convergence theorems of the iteration scheme (6) and (7) in uniformly convex Banach spaces.

**Theorem 3.1.** Under the assumptions of Lemma 2.4, if *E* satisfying Opial’s condition and assume that the mappings *J* = *S*₁ and *J* = *T*ᵢ for all *i* ∈ *I*, where *J* denotes the identity mapping, are demiclosed at zero, then {*ₓₙₖ*} converges weakly to a common fixed point of the mappings {*T*₁, *T*₂,..., *T*ₙ, *S*₁, *S*₂,..., *S*ₙ}.

**Proof.** Let *q* ∈ *F*, from Lemma 2.1 the sequence {∥*xₙ − q*∥} is convergent and hence bounded. Since *E* is uniformly convex, every bounded subset of *E* is weakly compact. Thus there exists a subsequence {*ₓₙₖ*} ⊂ {*ₓₙ*} such that {*ₓₙₖ*} converges weakly to *q*′ ∈ *K*. From Lemma 2.4, we get that

\[
\lim_{k \to \infty} ∥ₓₙₖ − *S*ᵢₓₙₖ∥ = 0 \quad \text{and} \quad \lim_{k \to \infty} ∥ₓₙₖ − *T*ᵢₓₙₖ∥ = 0
\]

for all *i* ∈ *I*. Since the mappings *J* = *S*₁ and *J* = *T*ᵢ for all *i* ∈ *I*, where *J* denotes the identity mapping, are demiclosed at zero, therefore *S*ᵢ{*q*′} = *q*′ and *T*ᵢ{*q*′} = *q*′, which means *q*′ ∈ *F*. Finally, let us prove that {*ₓₙ*} converges weakly to *q*′. Suppose on contrary that there is a subsequence {*ₓₙₖ*} ⊂ {*ₓₙ*} such that {*ₓₙₖ*} converges weakly to *p*′ ∈ *K* and *q*′ ≠ *p*′. Then by the same method as given above, we can also prove that *p*′ ∈ *F*. From Lemma 2.1 the limits limₙ→∞ ∥*xₙ − q*∥ and limₙ→∞ ∥*xₙ − p*′∥ exist. By virtue of the Opial condition of *E*, we obtain

\[
\lim_{n \to \infty} ∥*xₙ − q*′∥ = \lim_{nₖ \to \infty} ∥*xₙₖ − q*′∥ < \lim_{nₖ \to \infty} ∥*xₙₖ − p*′∥ = \lim_{n \to \infty} ∥*xₙ − p*′∥ = \lim_{n \to \infty} ∥*xₙ − p*′∥ < \lim_{n \to \infty} ∥*xₙ − q*′∥ = \lim_{n \to \infty} ∥*xₙ − q*′∥
\]

which is a contradiction so *q*′ = *p*′. Thus {*ₓₙ*} converges weakly to a common fixed point of the mappings {*T*₁, *T*₂,..., *T*ₙ, *S*₁, *S*₂,..., *S*ₙ}. This completes the proof. ■

**Lemma 3.1.** Let *E* be a real uniformly convex Banach space and *K* be a nonempty closed convex subset of *E* with *K + K* ⊂ *K*. Let {*S*ᵢ}ᵢ=₁ᴺ, {*T*ᵢ}ᵢ=₁ᴺ : *K* → *K* be 2*N* uniformly *L*-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences {*k*ₙᵢ}, {hₙᵢ} ∈ [1, ∞) given in Proposition 1.1 and *F* = ∩¹ᵢ=₁ᴺ *F*{*S*ᵢ} ∩ *F*{*T*ᵢ} ≠ ∅. Let {*ₓₙ*} be the sequence defined by (7), where {αₙ⁽ᵢ⁾} ∈ [a, 1 − a] for some *a* ∈ (0, 1) and all
\( i \in I \) with \( \sum_{n=1}^{\infty} (k_n h_n - 1) < \infty \). Then \( \lim_{n \to \infty} \|tx_n + (1-t)p - q\| \) exists for all \( p, q \in F \) and \( t \in [0, 1] \).

**Proof.** By Lemma 2.1, we know that \( \{x_n\} \) is bounded. Letting

\[
 a_n(t) = \|tx_n + (1-t)p - q\|
\]

for all \( t \in [0, 1] \). Then \( \lim_{n \to \infty} a_n(0) = \|p - q\| \) and \( \lim_{n \to \infty} a_n(1) = \|x_n - q\| \) exists by Lemma 2.1. It, therefore, remains to prove the Lemma 3.1 for \( t \in (0, 1) \). For all \( x \in K \), we define the mapping \( W_n : K \to K \) by

\[
 x_1^{(1)} = a_n^{(1)} T_n x_0, \quad x_2^{(2)} = a_n^{(2)} T_n x_1, \quad \vdots \\
 x_t^{(N-1)} = a_n^{(N-1)} T_n^{N-1} x_1, \quad W_n(x) = a_n^{(N)} T_n^N x_1 + (1 - a_n^{(N)}) S_n x.
\]

It is easy to prove

\[
 \|W_n x - W_n y\| \leq \mu_n \|x - y\|, \quad (33)
\]

for all \( x, y \in K \), where \( \mu_n = (1 + B_n) \) and \( B_n = (t_n N^{-1} - 1) \) with \( \sum_{n=1}^{\infty} B_n < \infty \) and \( \mu_n \to 1 \) as \( n \to \infty \). Setting

\[
 S_n m = W_{n+1} W_{n+2} \ldots W_n, \quad m \geq 1 \quad (34)
\]

and

\[
 b_{n,m} = \|S_n m (tx_n + (1-t)p) - (tS_n m x_n + (1-t)S_n m q)\|. \quad (35)
\]

From (33) and (34), we have

\[
 \|S_n m x - S_n m y\| \leq \mu_m \mu_{m+1} \cdots \mu_{m+n-1} \|x - y\| \\
 \leq \left( \prod_{j=n}^{n+m-1} \mu_j \right) \|x - y\| \\
 = L_n \|x - y\| \quad (36)
\]

for all \( x, y \in K \), where \( L_n = \prod_{j=n}^{n+m-1} \mu_j \) and \( S_n m x_n = x_{n+m}, S_n m p = p \) for all \( p \in F \). Thus

\[
 a_{n+m}(t) = \|tx_{n+m} + (1-t)p - q\| \\
 \leq b_{n,m} + \|S_n m (tx_n + (1-t)p) - q\| \\
 \leq b_{n,m} + L_n a_n(t). \quad (37)
\]

It follows from (36), (37) and Lemma 1.4 that

\[
 b_{n,m} \leq L_n \phi^{-1}(\|x_n - p\| - L_n^{-1} \|x_{n+m} - p\|).
\]

By Lemma 2.1 and \( \lim_{n \to \infty} L_n = 1 \), we have \( \lim_{n \to \infty} b_{n,m} = 0 \) and so

\[
 \lim_{m \to \infty} a_m(t) \leq \lim_{n \to \infty} b_{n,m} + \lim_{n \to \infty} \inf_{n \to \infty} a_n(t).
\]

By Lemma 2.1 and \( \lim_{n \to \infty} L_n = 1 \), we have \( \lim_{n \to \infty} b_{n,m} = 0 \) and so

\[
 \lim_{m \to \infty} a_m(t) \leq \lim_{n \to \infty} b_{n,m} + \lim_{n \to \infty} \inf_{n \to \infty} a_n(t).
\]
This shows that \( \lim_{n \to \infty} a_n(t) \) exists, that is,

\[
\lim_{n \to \infty} \|tx_n + (1 - t)p - q\|
\]
exists for all \( t \in [0, 1] \). This completes the proof.

**Theorem 3.2.** Let \( E \) be a real uniformly convex Banach space such that its dual \( E^* \) has the Kadec-Klee property and \( K \) be a nonempty closed convex subset of \( E \) with \( K + K \subset K \). Let \( \{S_i\}_{i=1}^{N}, \{T_i\}_{i=1}^{N} : K \to K \) be \( 2N \) uniformly \( L \)-Lipschitzian asymptotically quasi-nonexpansive mappings with sequences \( \{k_i\}, \{h_i\} \subset [1, \infty) \) given in proposition 1.1 and \( F = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset \). Let \( \{x_i\} \) be the sequence defined by (7), where \( \{a_n^{(i)}\} \subset [a, 1 - a] \) for some \( a \in (0, 1) \) and all \( i \in I \) with the following restrictions:

(i) \( \sum_{n=1}^{\infty} (k_nh_n - 1) < \infty \);

(ii) \( \|x - T_iy\| \leq \|S_i(x - T_iy)\| \) for all \( x, y \in K \) and \( i \in I \).

If the mappings \( J - S_i \) and \( J - T_i \) for all \( i \in I \), where \( J \) denotes the identity mapping, are demiclosed at zero, then \( \{x_i\} \) converges weakly to a common fixed point of the mappings \( \{T_1, T_2, \ldots, T_N, S_1, S_2, \ldots, S_N\} \).

**Proof.** By Lemma 2.1, we know that \( \{x_n\} \) is bounded and since \( E \) is reflexive, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) which converges weakly to some \( p \in K \). By Lemma 2.4, we get that

\[
\lim_{j \to \infty} \|x_{n_j} - S_i x_{n_j}\| = 0 \quad \text{and} \quad \lim_{j \to \infty} \|x_{n_j} - T_i x_{n_j}\| = 0
\]

for all \( i \in I \). Since the mappings \( J - S_i \) and \( J - T_i \) for all \( i \in I \), where \( J \) denotes the identity mapping, are demiclosed at zero, therefore \( S_i p = p \) and \( T_i p = p \) for all \( i \in I \) which means \( p \in F \). Now, we show that \( \{x_n\} \) converges weakly to \( p \). Suppose \( \{x_n\} \) is another subsequence of \( \{x_n\} \) which converges weakly to some \( q \in K \). By the same method as above, we have \( q \in F \) and \( p, q \in w_{w}(x_n) \). By Lemma 3.1, the limit

\[
\lim_{n \to \infty} \|tx_n + (1 - t)p - q\|
\]
exists for all \( t \in [0, 1] \) and so \( p = q \) by Lemma 1.3. Thus, the sequence \( \{x_n\} \) converges weakly to \( p \in F \). This completes the proof.

**Remark 3.1.** Since a nonexpansive and an asymptotically nonexpansive mapping with \( F(T) \neq \emptyset \) are asymptotically quasi-nonexpansive mappings. Theorem 3.2 improves and generalizes Theorem 3.2 in [1] and Theorem 2 in [17].

**Example 3.1.** Let \( E = [-\pi, \pi] \) and let \( T \) be defined by

\[
Tx = x \cos x
\]
for each $x \in E$. Clearly $F(T) = \{0\}$. $T$ is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = |x| \cos \frac{x}{2} \leq |x| = |x - 0| = |x - z|,$$

and hence $T$ is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{x}{2}$ and $y = \pi$, then

$$|Tx - Ty| = \left| \frac{\pi}{2} \cos \frac{\pi}{2} - \pi \cos \frac{\pi}{2} \right| = \pi,$$

whereas

$$|x - y| = \left| \frac{\pi}{2} - \pi \right| = \frac{\pi}{2}.$$

**Example 3.2.** Let $E = \mathbb{R}$ and let $T$ be defined by

$$T(x) = \left\{ \begin{array}{ll} \frac{x}{2} \cos \frac{\pi}{2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{array} \right.$$  

Then $F(T) = \emptyset$ and $T$ is quasi nonexpansive. If $x \neq 0$, then $Tx \neq x$, because $Tx = x = \frac{x}{2} \cos \frac{\pi}{2}$ would give $2 = \cos \frac{\pi}{2}$ which is impossible. $T$ is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$|Tx - z| = |Tx - 0| = \left| \frac{x}{2} \cos \frac{\pi}{2} \right| \leq \frac{|x|}{2} < |x| = |x - 0| = |x - z|,$$

and hence $T$ is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2x}{\pi}$ and $y = \frac{1}{2}$, then

$$|Tx - Ty| = \left| \frac{\pi}{2} \cos \frac{\pi}{2} - \frac{1}{2\pi} \cos \frac{\pi}{2} \right| = \frac{1}{2\pi},$$

whereas

$$|x - y| = \left| \frac{2}{3\pi} - \frac{1}{\pi} \right| = \frac{1}{3\pi}.$$

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**References**


Convergence theorems for two finite families of asymptotically quasi-nonexpansive mappings...


FEKETE-SEZGŐ PROBLEMS FOR CERTAIN CLASS OF NON-BAZILEVIĆ FUNCTIONS INVOLVING THE DZIOK-SRIVASTAVA OPERATOR

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Abstract
In the present paper the sharp Fekete-Szegő’s inequalities for certain class of non-Bazilević functions involving the Dziok-Srivastava operator are obtained. Some of our results improve and generalize previously known results.

Keywords: analytic function, subordination, Hadamard product (or convolution), Fekete-Szegő’s inequality.

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1. INTRODUCTION

Let \( A \) denote the class of functions of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). If \( f(z) \) and \( g(z) \) are analytic in \( U \), we say that \( f(z) \) is subordinate to \( g(z) \), written as \( f < g \) in \( U \) or \( f(z) < g(z) \) (\( z \in U \)), if there exists a Schwarz function \( \omega(z) \), which (by definition) is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) (\( z \in U \)) such that \( f(z) = g(\omega(z)) \) (\( z \in U \)). Furthermore, if the function \( g(z) \) is univalent in \( U \), then we have the following equivalence holds (see \([10, 11]\)):

\[
f(z) < g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

For functions \( f, g \in A \), where \( f \) given by (1) and \( g \) is defined by

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k,
\]

then the Hadamard product (or convolution) \( f \ast g \) of the functions \( f \) and \( g \) is defined by

\[
(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g \ast f)(z).
\]
For positive real numbers $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s (\beta_j \not\in \mathbb{Z}_0 = \{0, -1, -2, \ldots\}; \ j = 1, \ldots, s)$, the generalized hypergeometric function $\,^qF_s$ is defined (see [17]) by the following infinite series:

$$\,^qF_s\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_q)_k \cdot z^k}{(\beta_1)_k \ldots (\beta_s)_k \cdot k!},$$

$$\left(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}\right),$$

where $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1 & (\nu = 0), \\ \theta(\theta + 1) \ldots (\theta + \nu - 1) & (\nu \in \mathbb{N}). \end{cases}$$

Corresponding a function $h\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right)$ defined by

$$h\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) = z \,^qF_s\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right),$$

(2)

Dziok and Srivastava [5] considered a linear operator $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \mathcal{A} \to \mathcal{A}$

defined by the following Hadamard product:

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = h\left(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z\right) * f(z),$$

(3)

$$\left(q \leq s + 1; q, s \in \mathbb{N}_0; z \in \mathbb{U}\right).$$

If $f \in \mathcal{A}$ is given by (1), then we have

$$H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z) = z + \sum_{k=2}^{\infty} \Gamma_k [\alpha_1] \ a_k z^k \ (z \in \mathbb{U}),$$

(4)

where

$$\Gamma_k [\alpha_1] = \frac{(\alpha_1)_{k-1} \ldots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \ldots (\beta_s)_{k-1}} \frac{1}{(k-1)!} \ (k \in \mathbb{N}).$$

(5)

To make the notation simple, we write

$$H_{q,s} [\alpha_1] f(z) = H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f(z).$$

In recent years, many interesting subclasses of analytic functions, associated with the Dziok–Srivastava operator $H_{q,s} [\alpha_1]$ and its many special cases, were investigated by (for example) Dziok and Srivastava [5, 6], Gangadharan et al. [7], Aouf and Seoudy [1] and others.
By making use of the linear operator $H_{q,s} [\alpha_1]$ and the above-mentioned principle of subordination between analytic functions, we now introduce the following subclass of non-Bazilević analytic functions.

**Definition 1.1.** Let $\phi(z)$ be an univalent starlike function with respect to 1 which maps the unit disk $\mathbb{U}$ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in A$ is said to be in the class $N_{\gamma,\delta}^{q,s}(\alpha_1; \phi)$ if it satisfies the following subordination condition:

$$
(1 + \gamma) \left( \frac{z}{H_{q,s} [\alpha_1] f(z)} \right)^\delta - \gamma \left( \frac{z}{H_{q,s} [\alpha_1] f(z)} \right)' \left( \frac{z}{H_{q,s} [\alpha_1] f(z)} \right)^{\delta+1} \prec \phi(z)
$$

($\gamma \in \mathbb{C}; 0 < \delta < 1; z \in \mathbb{U}$).

We note that:

(i) $N_{\gamma,\delta}^{q,s}(1, \beta_1; \beta_1; \phi) = N_{\gamma,\delta}^{q,s}(\phi)$, where $N_{\gamma,\delta}^{q,s}(\phi)$ is the class studied by Shanmugam et al. [16];

(ii) $N_{\gamma,\delta}^{q,s}(1, \beta_1; \beta_1; \frac{1 + A z}{1 + B z}) = N(\gamma, \delta; A, B)(-1 \leq B < A \leq 1)$, where $N(\gamma, \delta; A, B)$ is the class defined by Wang et al. [19];

(iii) $N_{\gamma,\delta}^{\rho,\delta}(1, \beta_1; \beta_1; \frac{1 + (1 - 2 \rho) z}{1 - z}) = N(\delta; \rho)(0 \leq \rho < 1)$, where $N(\delta; \rho)$ is the class of non-Bazilević functions of order $\rho$ which were considered by Tuneski and Daus [18];

(iv) $N_{\gamma,\delta}^{\rho,\delta}(1, \beta_1; \beta_1; \frac{1 + z}{1 - z}) = N(\delta)$, where $N(\delta)$ is the class of non-Bazilević functions which introduced by Obradovic [12].

We further, observe that, by the special choices for $\alpha_i (i = 1, ..., q)$ and $\beta_j (j = 1, ..., s)$, where $q, s \in \mathbb{N}_0$ our class $N_{q,s}^{\gamma,\delta}(\alpha_1; \phi)$ gives rise the following new subclasses involving different operators:

(i) $N_{\gamma,\delta}^{q,s}(a; b; c; \phi)$

$$
= \left\{ f \in A : (1 + \gamma) \left( \frac{z}{I_{a,b}^c f(z)} \right)^\delta - \gamma \left( \frac{z}{I_{a,b}^c f(z)} \right)' \left( \frac{z}{I_{a,b}^c f(z)} \right)^{\delta+1} \prec \phi(z) \right\},
$$

where the linear operator $I_{a,b}^c (a, b \in \mathbb{C}; c \notin \mathbb{Z}_0)$ was investigated by Hohlov [8];
The result is sharp for the functions given by
\[ p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}. \]
Lemma 1.2 ([9]). If \( p(z) = 1 + c_1 z + c_2 z^2 + \ldots \) is an analytic function with a positive real part in \( \mathbb{U} \), then

\[
|c_2 - vc_1^2| \leq \begin{cases} 
-4\nu + 2 & \text{if } \nu \leq 0, \\
2 & \text{if } 0 \leq \nu \leq 1, \\
4\nu - 2 & \text{if } \nu \geq 1,
\end{cases}
\]

when \( \nu < 0 \) or \( \nu > 1 \), the equality holds if and only if \( p(z) \) is \( (1 + z)/(1 - z) \) or one of its rotations. If \( 0 < \nu < 1 \), then the equality holds if and only if \( p(z) \) is \( (1 + z^2)/(1 - z^2) \) or one of its rotations. If \( \nu = 0 \), the equality holds if and only if

\[
p(z) = \left( \frac{1 + \alpha}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \alpha}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \alpha \leq 1)
\]

or one of its rotations. If \( \nu = 1 \), the equality holds if and only if \( p \) is the reciprocal of one of the functions such that equality holds in the case of \( \nu = 0 \).

Also the above upper bound is sharp, and it can be improved as follows when \( 0 < \nu < 1 \):

\[
|c_2 - vc_1^2| + \nu |c_1|^2 \leq 2 \quad \left( 0 \leq \nu \leq \frac{1}{2} \right)
\]

and

\[
|c_2 - vc_1^2| + (1 - \nu) |c_1|^2 \leq 2 \quad \left( \frac{1}{2} \leq \nu \leq 1 \right).
\]

In this paper, we obtain the Fekete-Szegö inequalities for \( N_{q,\delta}^{(\gamma)}(\alpha_1; \phi) \). The motivation of this paper is to improve and generalize the results obtained by Shanmugam et al. [16].

2. MAIN RESULTS

Unless otherwise mentioned, we assume throughout this paper that \( \Gamma_\delta [\alpha_1] \) is given by (5) and all powers are understood as principle values.

Theorem 2.1. Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f(z) \) given by (1) belongs to \( N_{q,\delta}^{(\gamma)}(\alpha_1; \phi) \) with \( \delta + \gamma \neq 0 \) and \( \delta + 2\gamma \neq 0 \), then

\[
|a_3 - \mu a_2^2| \leq \max \left\{ \left| \frac{B_1}{\delta + 2\gamma} \Gamma_\delta^3 [\alpha_1] \right|, \frac{(\delta + 1)(\delta + 2\gamma) B_1}{2(\delta + \gamma)^2} + \frac{\mu^{\delta + 2\gamma}}{(\delta + \gamma)^2} \Gamma_\delta^3 [\alpha_1] \right\}
\]

(7)

The result is sharp.

Proof. If \( f \in N_{q,\delta}^{(\gamma)}(\alpha_1; \phi) \), then there is a Schwarz function \( \omega \), analytic in \( \mathbb{U} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( \mathbb{U} \) such that

\[
(1 + \gamma) \left( \frac{\tilde{z}}{H_{q,\delta}^3 [\alpha_1]} \right)^\delta - \gamma \left( H_{q,\delta}^3 [\alpha_1] f(z) \right)^\delta = \phi(\omega(z)).
\]

(8)
Define the function \( p(z) \) by
\[
p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1z + c_2z^2 + \ldots.
\] (9)

Since \( \omega(z) \) is a Schwarz function, we see that \( \Re \{ p(z) \} > 0 \) and \( p(0) = 1 \). Therefore,
\[
\phi(\omega(z)) = \phi\left( \frac{p(z) - 1}{p(z) + 1} \right)
\]
\[
= \phi\left( \frac{1}{2} \left[ c_1z + \left( c_2 - \frac{c_1^2}{2} \right)z^2 + \left( c_3 - c_1c_2 + \frac{c_1^3}{4} \right)z^3 + \ldots \right] \right)
\]
\[
= 1 + \frac{1}{2}B_1c_1z + \left[ \frac{1}{2}B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2 \right]z^2 + \ldots.
\] (10)

Now by substituting (10) in (8), we have
\[
(1 + \gamma)\left( H_{q,s} \left[ \alpha_1 \right] f(z) \right)^\delta - \gamma \left( H_{q,s} \left[ \alpha_1 \right] f(z) \right)' \left( H_{q,s} \left[ \alpha_1 \right] f(z) \right)^{\delta+1}
\]
\[
= 1 + \frac{1}{2}B_1c_1z + \left[ \frac{1}{2}B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4}B_2c_1^2 \right]z^2 + \ldots.
\]

From this equation and (8), we obtain
\[
-(\delta + \gamma) \Gamma_2 \left[ \alpha_1 \right] a_2 = \frac{1}{2}B_1c_1,
\]
\[
-(\delta + 2\gamma) \Gamma_3 \left[ \alpha_1 \right] a_3 - \frac{\delta + 1}{2} \Gamma_2^2 \left[ \alpha_1 \right] a_3^2 = \frac{1}{2}B_1c_2 - \frac{1}{4}B_1c_1^2 + \frac{1}{4}B_2c_1^2,
\]

or, equivalently,
\[
a_2 = -\frac{B_1c_1}{2(\delta + \gamma) \Gamma_2 \left[ \alpha_1 \right]},
\]
\[
a_3 = -\frac{B_1}{2(\delta + 2\gamma) \Gamma_3 \left[ \alpha_1 \right] \left\{ c_2 - \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \frac{(\delta + 1)(\delta + 2\gamma)B_1}{2(\delta + \gamma)^2} \right]c_1^2 \right\}}.
\]

Therefore,
\[
a_3 - \mu a_2^2 = -\frac{B_1}{2(\delta + 2\gamma) \Gamma_3 \left[ \alpha_1 \right] \left\{ c_2 - \nu c_1^2 \right\}},
\] (11)

where
\[
\nu = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \frac{(\delta + 1)(\delta + 2\gamma)B_1}{2(\delta + \gamma)^2} - \frac{\mu (\delta + 2\gamma) \Gamma_3 \left[ \alpha_1 \right] B_1}{(\delta + \gamma)^2 \Gamma_2^2 \left[ \alpha_1 \right]} \right].
\] (12)
Our result now follows by an application of Lemma 1.1. The result is sharp for the functions
\[
(1 + \gamma) \left( \frac{z}{H_{\sigma,\beta}(z)} \right)^{\delta} - \gamma \left( H_{\sigma,\beta}(z) \right) \left( \frac{z}{H_{\sigma,\beta}(z)} \right)^{\delta+1} = \phi \left( z^{2} \right),
\]
and
\[
(1 + \gamma) \left( \frac{z}{H_{\sigma,\beta}(z)} \right)^{\delta} - \gamma \left( H_{\sigma,\beta}(z) \right) \left( \frac{z}{H_{\sigma,\beta}(z)} \right)^{\delta+1} = \phi \left( z \right).
\]
This completes the proof of Theorem 2.1.

Putting \( q = 2, s = 1, \alpha_1 = 1 \) and \( \alpha_2 = \beta_1 \) in Theorem 2.1, we obtain the following result.

**Corollary 2.1.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 \neq 0 \). If \( f(z) \) given by (1) belongs to \( N^{\gamma,\delta} \) (\( \phi \)) with \( \delta + \gamma \neq 0 \) and \( \delta + 2\gamma \neq 0 \), then
\[
|a_3 - \mu a_2^2| \leq \left| \frac{B_1}{\delta + 2\gamma} \right| \max \left\{ 1; \frac{B_2}{B_1} - \frac{(\delta + 1)(\delta + 2\gamma)B_1}{2(\delta + \gamma)^2} + \frac{\mu(\delta + 2\gamma)B_1}{(\delta + \gamma)^2} \right\}.
\]
The result is sharp.

**Remark 2.1.** For \( \gamma = -1, \phi(z) = \frac{1}{1 - \rho} \) \((0 \leq \rho < 1)\) in Corollary 2.1, we obtain the results of Tuneski and Darus [18, Theorem 1].

**Theorem 2.2.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let
\[
\sigma_1 = \frac{(\delta + 1)(\delta + 2\gamma)B_1^2 - 2(B_2 - B_1)(\delta + \gamma)^2}{2(\delta + \gamma)^2 \Gamma_2[\alpha_1]} B_1^2 \Gamma_2[\alpha_1],
\]
\[
\sigma_2 = \frac{(\delta + 1)(\delta + 2\gamma)B_1^2 - 2(B_2 + B_1)(\delta + \gamma)^2}{2(\delta + \gamma)^2 \Gamma_2[\alpha_1]} B_1^2 \Gamma_2[\alpha_1],
\]
\[
\sigma_3 = \frac{(\delta + 1)(\delta + 2\gamma)B_1^2 - 2B_2(\delta + \gamma)^2}{2(\delta + \gamma)^2 \Gamma_2[\alpha_1]} B_1^2 \Gamma_2[\alpha_1].
\]
If \( f(z) \) given by (1) belongs to \( N^{\gamma,\delta}(\alpha_1; \phi) \) with \( \delta + 2\gamma < 0 \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{B_1}{(\delta + 2\gamma) \Gamma_3[\alpha_1]} + \frac{(\delta + 1)B_1^2}{2(\delta + \gamma)^2 \Gamma_2[\alpha_1]} - \frac{\mu B_1^2}{(\delta + \gamma)^2 \Gamma_2[\alpha_1]} & \text{if } \mu \leq \sigma_1, \\
\frac{B_1}{(\delta + 2\gamma) \Gamma_3[\alpha_1]} + \frac{(\delta + 1)B_1^2}{2(\delta + \gamma)^2 \Gamma_2[\alpha_1]} - \frac{\mu B_1^2}{(\delta + \gamma)^2 \Gamma_2[\alpha_1]} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{B_1}{(\delta + 2\gamma) \Gamma_3[\alpha_1]} + \frac{(\delta + 1)B_1^2}{2(\delta + \gamma)^2 \Gamma_2[\alpha_1]} + \frac{\mu B_1^2}{(\delta + \gamma)^2 \Gamma_2[\alpha_1]} & \text{if } \mu \geq \sigma_2,
\end{cases}
\]
Further, if \( \sigma_1 \leq \mu \leq \sigma_3 \), then
\[
|a_3 - \mu a_2^2| \leq \frac{(\delta + 2\gamma)^2 \Gamma_2[\alpha_1]}{(\delta + \gamma)^2 \Gamma_3[\alpha_1] B_1^2} \left[ B_1^2 - B_2 + \frac{(\delta + 1)(\delta + 2\gamma)B_1^2}{2(\delta + \gamma)^2} - \frac{\mu(\delta + 2\gamma)^2 \Gamma_2[\alpha_1] B_1^2}{(\delta + \gamma)^2 \Gamma_2[\alpha_1]} \right] \mu a_2^2.
\]
The results of Theorem 2.2 follows by applying Lemma 1.2 to (11). To show

\[ f = \text{equality holds if and only if} \]

Shanmugam et al. [16, Theorem 2.1].

Proof. The results of Theorem 2.2 follows by applying Lemma 1.2 to (11). To show

that the bounds are sharp, we define the functions \( \mathcal{K}_m \) (\( n = 2, 3, \ldots \)) by

\[
(1 + \gamma) \left( \frac{z^2}{\mathcal{K}_m(z)} \right)^\delta - \gamma \left( \frac{z^2}{\mathcal{K}_m(z)} \right)^{\delta + 1} = \phi \left( \frac{z^{\alpha - 1}}{1 + \lambda z} \right),
\]

and the functions \( \mathcal{F}_\lambda \) and \( \mathcal{G}_\lambda \) (\( 0 \leq \lambda \leq 1 \)) by

\[
(1 + \gamma) \left( \frac{z^2}{\mathcal{F}_\lambda(z)} \right)^\delta - \gamma \left( \frac{z^2}{\mathcal{F}_\lambda(z)} \right)^{\delta + 1} = \phi \left( \frac{z^{\alpha - 1}}{1 + \lambda z} \right),
\]

\( \mathcal{F}_\lambda(0) = 0 = \mathcal{G}_\lambda(0) - 1 \)

and

\[
(1 + \gamma) \left( \frac{z^2}{\mathcal{G}_\lambda(z)} \right)^\delta - \gamma \left( \frac{z^2}{\mathcal{G}_\lambda(z)} \right)^{\delta + 1} = \phi \left( \frac{z^{\alpha - 1}}{1 + \lambda z} \right),
\]

\( \mathcal{G}_\lambda(0) = 0 = \mathcal{G}_\lambda(0) - 1 \).

Clearly, the functions \( \mathcal{K}_m, \mathcal{F}_\lambda \) and \( \mathcal{G}_\lambda \) \( \in \mathcal{N}_p^{\gamma, \lambda, \alpha} (\alpha; \phi) \). If \( \mu < \sigma_1 \) or \( \mu > \sigma_2 \), then

the equality holds if and only if \( f \) is \( \mathcal{K}_m \), or one of its rotations. When \( \sigma_1 < \mu < \sigma_2 \), the equality holds if and only if \( f \) is \( \mathcal{K}_m \), or one of its rotations. If \( \mu = \sigma_1 \), then

the equality holds if and only if \( f \) is \( \mathcal{F}_\lambda \), or one of its rotations. If \( \mu = \sigma_2 \), then the equality holds if and only if \( f \) is \( \mathcal{G}_\lambda \), or one of its rotations.

Taking \( q = 2, s = 1, \alpha_1 = 1 \) and \( \alpha_2 = \beta_1 \) in Theorem 2.2, we obtain the following result for the functions belonging to the class \( \mathcal{N}_p^{\gamma, \lambda, \alpha} (\phi) \) which improves the result of Shanmugam et al. [16, Theorem 2.1].
Corollary 2.2. Let \( \phi (z) = 1 + B_1 z + B_2 z^2 + \ldots \) with \( B_1 > 0 \) and \( B_2 \geq 0 \). Let

\[
\sigma_4 = \frac{(\delta + 1)(\delta + 2\gamma) B_1^2 - 2(B_2 - B_1)(\delta + \gamma)^2}{2(\delta + 2\gamma) B_1^2},
\]
\[
\sigma_5 = \frac{(\delta + 1)(\delta + 2\gamma) B_1^2 - 2(B_2 + B_1)(\delta + \gamma)^2}{2(\delta + 2\gamma) B_1^2},
\]
\[
\sigma_6 = \frac{(\delta + 1)(\delta + 2\gamma) B_1^2 - 2B_2(\delta + \gamma)^2}{2(\delta + 2\gamma) B_1^2}.
\]

If \( f(z) \) given by (1) belongs to \( N^{\gamma, \delta} (\phi) \) with \( \delta + 2\gamma < 0 \), then

\[
|a_3 - \mu a_2|^2 \leq \begin{cases} 
- \frac{B_2}{(\delta + 2\gamma) B_1} + \frac{(\delta + 1) B_1^2}{(\delta + 2\gamma)^2} - \frac{\mu B_1^2}{(\delta + 2\gamma)^2} & \text{if } \mu \leq \sigma_4, \\
- \frac{B_2}{(\delta + 2\gamma) B_1} - \frac{(\delta + 1) B_1^2}{(\delta + 2\gamma)^2} + \frac{\mu B_1^2}{(\delta + 2\gamma)^2} & \text{if } \sigma_4 \leq \mu \leq \sigma_5, \\
- \frac{B_2}{(\delta + 2\gamma) B_1} - \frac{(\delta + 1) B_1^2}{(\delta + 2\gamma)^2} + \frac{\mu B_1^2}{(\delta + 2\gamma)^2} & \text{if } \mu \geq \sigma_6,
\end{cases}
\]

Further, if \( \sigma_4 \leq \mu \leq \sigma_6 \), then

\[
|a_3 - \mu a_2|^2 \leq \begin{cases} 
- \frac{B_1}{(\delta + 2\gamma) \Gamma_3 [\alpha_1]} & \text{if } \mu \leq \sigma_4, \\
- \frac{B_1}{(\delta + 2\gamma) \Gamma_3 [\alpha_1]} & \text{if } \sigma_4 \leq \mu \leq \sigma_5, \\
- \frac{B_1}{(\delta + 2\gamma) \Gamma_3 [\alpha_1]} & \text{if } \mu \geq \sigma_6.
\end{cases}
\]

If \( \sigma_6 \leq \mu \leq \sigma_5 \), then

\[
|a_3 - \mu a_2|^2 \leq \frac{(\delta + \gamma)^2 \Gamma_2 [\alpha_1]}{(\delta + 2\gamma) \Gamma_3 [\alpha_1] B_1^2} \left[ B_1 + B_2 - \frac{(\delta + 1)(\delta + 2\gamma) B_1^2}{2(\delta + 2\gamma)^2} + \frac{\mu(\delta + 2\gamma) \Gamma_2 [\alpha_1] B_1^2}{(\delta + 2\gamma)^2 \Gamma_3 [\alpha_1]} \right] |a_2|^2
\]
\[
\leq - \frac{B_1}{(\delta + 2\gamma)}.
\]

The result is sharp.

Putting \( \phi (z) = 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \) in Theorem 2.2, we obtain the following corollary.
Corollary 2.3. Let
\[
\begin{align*}
\sigma_7 &= \frac{12(\delta + 1)(\delta + 2\gamma) + \pi^2(\delta + \gamma)^2}{24(\delta + 2\gamma)\Gamma_3[\alpha_1]}, \\
\sigma_8 &= \frac{12(\delta + 1)(\delta + 2\gamma) - 5\pi^2(\delta + \gamma)^2}{24(\delta + 2\gamma)\Gamma_3[\alpha_1]}, \\
\sigma_9 &= \frac{6(\delta + 1)(\delta + 2\gamma) - \pi^2(\delta + \gamma)^2}{12(\delta + 2\gamma)\Gamma_3[\alpha_1]}.
\end{align*}
\]

If \( f(z) \) given by (1) belongs to \( N_{q,\gamma,\phi}^{\alpha,\beta} \left( \alpha_1; 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{\tau}}{1 - \sqrt{\tau}} \right)^2 \right) \) with \( \delta + 2\gamma < 0 \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
-\frac{16}{3\pi^4(\delta + 2\gamma)^2\Gamma_3[\alpha_1]} + \frac{32(\delta + 1)}{\pi^4(\delta + \gamma)^2\Gamma_3[\alpha_1]} - \frac{64\mu}{\pi^4(\delta + \gamma)^2\Gamma_3^2[\alpha_1]} & \text{if } \mu \leq \sigma_7, \\
-\frac{\pi^2(\delta + 2\gamma)\Gamma_3[\alpha_1]}{16} + \frac{32(\delta + 1)}{3\pi^4(\delta + 2\gamma)^2\Gamma_3[\alpha_1]} + \frac{64\mu}{\pi^4(\delta + \gamma)^2\Gamma_3^2[\alpha_1]} & \text{if } \sigma_7 \leq \mu \leq \sigma_8, \\
\frac{8\mu}{\pi^2(\delta + 2\gamma)\Gamma_3[\alpha_1]} & \text{if } \mu \geq \sigma_8.
\end{cases}
\]
Further, if \( \sigma_7 \leq \mu \leq \sigma_9 \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{8}{\pi^2(\delta + 2\gamma)\Gamma_3[\alpha_1]} & \text{if } \sigma_7 \leq \mu \leq \sigma_9.
\end{cases}
\]

If \( \sigma_9 \leq \mu \leq \sigma_8 \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{8}{\pi^2(\delta + 2\gamma)\Gamma_3[\alpha_1]} & \text{if } \sigma_9 \leq \mu \leq \sigma_8.
\end{cases}
\]
The result is sharp.

Taking \( q = 2, s = 1, \alpha_1 = 1 \) and \( \alpha_2 = \beta_1 \) in Corollary 2.3, we obtain the following result for the functions of the class \( N_{q,\gamma,\phi}^{\alpha,\beta} (\phi) \) which improves the result of Shanmugam et al. [16, Corollary 2.2].

Corollary 2.4. Let
\[
\begin{align*}
\sigma_{10} &= \frac{12(\delta + 1)(\delta + 2\gamma) + \pi^2(\delta + \gamma)^2}{24(\delta + 2\gamma)}, \\
\sigma_{11} &= \frac{12(\delta + 1)(\delta + 2\gamma) - 5\pi^2(\delta + \gamma)^2}{24(\delta + 2\gamma)}, \\
\sigma_{12} &= \frac{6(\delta + 1)(\delta + 2\gamma) - \pi^2(\delta + \gamma)^2}{12(\delta + 2\gamma)}.
\end{align*}
\]
If $f(z)$ given by (1) belongs to $N^{\gamma, \delta} \left( 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2 \right)$ with $\delta + 2\gamma < 0$, then

$$|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
- \frac{16}{3\pi^2(\delta+2\gamma)} + \frac{32(\delta+1)}{\pi^4(\delta+2\gamma)^2} - \frac{64\mu}{\pi^4(\delta+2\gamma)^2} & \text{if } \mu \leq \sigma_{10}, \\
- \frac{8}{\pi^2(\delta+2\gamma)} & \text{if } \sigma_{10} \leq \mu \leq \sigma_{11}, \\
\frac{16}{3\pi^4(\delta+2\gamma)} - \frac{32(\delta+1)}{\pi^4(\delta+2\gamma)^2} + \frac{64\mu}{\pi^4(\delta+2\gamma)^2} & \text{if } \mu \geq \sigma_{11},
\end{array} \right.$$ 

Further, if $\sigma_{10} \leq \mu \leq \sigma_{12}$, then

$$|a_3 - \mu a_2^2| + \left( \frac{\delta+\gamma^2}{8(\delta+2\gamma)} \right) \left[ \frac{\pi^2}{3} + \frac{4(\delta+1)(\delta+2\gamma)}{2(\delta+2\gamma)^2} - \frac{8\mu(\delta+2\gamma)}{(\delta+2\gamma)^2} \right] |a_2|^2 \leq - \frac{8}{\pi^2(\delta+2\gamma)^2}.$$ 

If $\sigma_{12} \leq \mu \leq \sigma_{11}$, then

$$|a_3 - \mu a_2^2| + \left( \frac{\delta+\gamma^2}{8(\delta+2\gamma)} \right) \left[ \frac{\pi^2}{3} - \frac{4(\delta+1)(\delta+2\gamma)}{2(\delta+2\gamma)^2} + \frac{8\mu(\delta+2\gamma)}{(\delta+2\gamma)^2} \right] |a_2|^2 \leq - \frac{8}{\pi^2(\delta+2\gamma)^2}.$$ 

The result is sharp.

Remark 2.2. For special choices for $\alpha_i (i = 1, \ldots, q)$ and $\beta_j (j = 1, \ldots, s)$, where $q, s \in \mathbb{N}_0$ in the above results, we obtain corresponding results for the subclasses (i-v) involving the different operators mentioned in the introduction.

References


ON MULTIPLIERS OF SOME NEW ANALYTIC HARDY-TYPE $H^p_\vec{\alpha}$ AND BERGMAN-TYPE $H^p_\vec{\alpha}(\vec{\alpha})$
FUNCTION SPACES IN POLYDISC
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Abstract We provide several new results concerning coefficient multipliers of some new mixed norm analytic function spaces in polydisc $H^p_\vec{\alpha}$ and $H^p_\vec{\alpha}(\vec{\alpha})$. Our results extend previously well known classical assertions.

Keywords: analytic functions, multipliers, polydisc theorems.

1. INTRODUCTION

The goal of this paper is to continue the investigation of spaces of coefficient multipliers of some new homomorphic mixed norm spaces in the unit polydisc started in [1], [15], [16], [19]. These mixed norm spaces we study serve as a very natural extensions of the classical Hardy and Bergman spaces in the unit polydisc simultaneously. Also pointwise and coefficient multipliers of analytic spaces of Hardy and Bergman type in one and higher dimension were studied intensively by many authors during past several decades, see for example [3], [18], [19], and various references there. But the investigation in this new direction of coefficient multipliers of analytic mixed norm spaces in the unit polydisc was started in particular in recent papers of the author, see [15], [16], [17], [19]. Below we list standard notation and definitions which are needed and in the next section we formulate main results of this note.

Let $U = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc in $\mathbb{C}$, $\mathbb{T} = \partial U = \{ z \in \mathbb{C} : |z| = 1 \}$, $U^n$ is the unit polydisc in $\mathbb{C}^n$, $\mathbb{T}^n \subset \partial U^n$ is the distinguished boundary of $U^n$, $\mathbb{Z}_+ = \{ n \in \mathbb{Z} : n \geq 0 \}$, $\mathbb{Z}^n_+$ is the set of all multi indexes and $J = [0, 1)$.

We use the following notation: for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ we set $z^k = z_1^{k_1} \cdots z_n^{k_n}$ and for $z = (z_1, \ldots, z_n) \in U^n$ and $\gamma \in \mathbb{R}$ we set $(1 - |z|)^\gamma = (1 - |z_1|)^\gamma \cdots (1 - |z_n|)^\gamma$ and $(1 - z)^\gamma = (1 - z_1)^\gamma \cdots (1 - z_n)^\gamma$. Next, for $z \in \mathbb{R}^n$ and $w \in \mathbb{C}^n$ we set $w_z = (w_1z_1, \ldots, w_nz_n)$. Also, for $k \in \mathbb{Z}^n_+$ and $a \in \mathbb{R}$ we set $k + a = (k_1 + a, \ldots, k_n + a)$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$. For $k \in (0, +\infty)^n$ we set $\Gamma(k) = \Gamma(k_1) \cdots \Gamma(k_n)$. 

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The Lebesgue measure on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) is denoted by \( dV(z) \), normalized Lebesgue measure on \( \mathbb{T}^n \) is denoted by \( d\xi = d\xi_1 \ldots d\xi_n \) and \( dR = dR_1 \ldots dR_n \) is the Lebesgue measure on \([0, +\infty)^n\).

The space of all functions holomorphic in \( U^n \) is denoted by \( H(U^n) \). Every \( f \in H(U^n) \) admits an expansion \( f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k \). For \( \beta > -1 \) the operator of fractional differentiation is defined by (see [8])

\[
D^\beta f(z) = \sum_{k \in \mathbb{Z}^n} \frac{\Gamma(k + \beta + 1)}{\Gamma(\beta + 1)\Gamma(k + 1)} a_k z^k, \quad z \in U^n. \tag{1}
\]

Similarly we can easily define the operator of fractional derivative for \( \beta \)-vector, \( \beta = (\beta_1, \ldots, \beta_n) \), \( \beta_j > -1 \), \( j = 1, \ldots, n \).

\[
D^\beta f(z) = \sum_{k \in \mathbb{Z}^n} \frac{\Gamma(k_j + \beta_j + 1)}{\Gamma(\beta_j + 1)\Gamma(k_j + 1)} a_{k_1 \ldots k_n} z^{k_1 \ldots k_n}
\]

For \( f \in H(U^n) \), \( 0 < p < \infty \) and \( r \in \mathbb{I}^n \) we set

\[
M_p(f, r) = \left( \int_{T^n} |f(r\xi)|^p \, d\xi \right)^{1/p}, \quad d\xi = d\xi_1 \ldots d\xi_n \tag{2}
\]

with the usual modification to include the case \( p = \infty \). For \( 0 < p \leq \infty \) we define standard analytic Hardy classes in the polydisc:

\[
H^p(U^n) = \{ f \in H(U^n) : \| f \|_{H^p} = \sup_{r \in \mathbb{I}^n} M_p(f, r) < \infty \}. \tag{3}
\]

For \( n = 1 \) these spaces are well studied. The topic of multipliers of Hardy spaces in polydisc is relatively new, see for example [7], [3],[18],[15],[16]. These spaces are Banach spaces for \( p > 1 \) and complete metric spaces for all other positive values of \( p \). Also, for \( 0 < p \leq \infty \), \( 0 < q < \infty \) and \( \alpha > 0 \) we have mixed (quasi) norm spaces, defined below as follows

\[
A^{p,q}_\alpha(U^n) = \left\{ f \in H(U^n) : \| f \|_{A^{p,q}_\alpha} = \int_{T^n} M^q_p(f, R)(1 - R)^{\alpha q - 1} \, dR < \infty \right\}. \tag{4}
\]

If we replace the integration by \( I^n \) above by integration by unit interval \( I \) then we get other similar to these \( A^{p,q}_\alpha \) analytic spaces but on subframe and we denote them by \( B^{p,q}_\alpha \). These spaces are Banach spaces when both \( p \) and \( q \) are bigger than one, and complete metric spaces for all other values of \((p,q)\). We refer the reader for these classes in the unit ball and the unit disk to [3], [12], [13], [14] and references therein. Multipliers between \( A^{p,q}_\alpha \) spaces on the unit disc were studied in detail in [3],[7], [10], [11]. Multipliers of spaces on subframe are less studied.In forthcoming paper we will
study multipliers of related $B^{p,q}$ spaces. As we see from assertions below spaces of multipliers of classes on subframe depend also from dimension $n$.

As is customary, we denote positive constants by $c$ (or $C$), sometimes we indicate dependence of a constant on a parameter by using a subscript, for example $C_q$.

We define now one of the main objects of this paper.

For $0 < p, q < \infty$ and $\alpha > 0$ we consider Lizorkin-Triebel spaces $F^{p,q}_\alpha(U^n) = F^{p,q}_\alpha$ consisting of all $f \in H(U^n)$ such that

$$
\|f\|^{p,q}_{F^{p,q}_\alpha} = \left(\int_{U^n} \left(\int_{|I^n|} |f(R\xi)|^p(1 - R)^{q\alpha - 1}dR\right)^{q/p}d\xi\right)^{1/q} < \infty, \quad (5)
$$

It is not difficult to check that those spaces are complete metric spaces, if $\min(p, q) \geq 1$ they are Banach spaces. If we simply replace the integration by $I^n$ above by integration by unit interval $I$ then we get other similar to these $F^{p,q}_\alpha$ analytic spaces, but on subframe and we denote them by $T^{p,q}_\alpha$. They were studied in [20]. We refer the reader to this paper for various other properties on these type classes. We provide some new assertions on multipliers of this new $F^{p,q}_\alpha$ type spaces on subframe below. This paper probably is the second paper where multipliers of analytic classes on subframe are under attention. We note that this $F^{p,q}$ scale of spaces includes, for $p = q$, weighted Bergman spaces $A^p = F^{p,p}$ (see [8] for a detailed account of these spaces). On the other hand, for $q = 2$ these spaces coincide with Hardy-Sobolev spaces namely $H^p = F^{p,2}$, for this well known fact see [16], [8] and references therein.

Finally, for $\alpha \geq 0$ and $\beta \geq 0$ we set

$$
A^{\alpha,\infty}_{\alpha,\beta}(U^n) = \{f \in H(U^n) : \|f\|_{A^{\alpha,\infty}_{\alpha,\beta}} = \sup_{r \in I^n} (M_\infty(D^p f, r))(1 - r)^\beta < \infty\}. \quad (6)
$$

This is a Banach space and it is related with the well-known Bloch class in polydisc (see [3]). The Bloch class studied by many authors in various papers. (see, for example, [3], [6] for unit disk case and for polydisc case and also various references there). The Bloch space is a same

$$
A^{\alpha,\infty}_{\alpha,\beta}(U^n)
$$

space with $\alpha = \beta = 1$ in definition above. This space is also a Banach space. For all positive values of $p$ and $s$ we introduce the following two new spaces. We note first replacing $q$ by $\infty$ in a usual way we will arrive at some other spaces $F^{p,\infty}$ (the limit case of $F^{p,q}_\alpha$ classes.) The limit space case $F^{p,\infty}(U^n)$ is defined as a space of all analytic functions $f$ in the polydisc such that the function $\phi(\xi) = \sup_{r \in I^n} |f(r\xi)|(1 - r)^s$, $\xi \in \mathbb{T}^n$ is in $L^p(\mathbb{T}^n, d\xi)$, $s \geq 0$, $0 < p \leq \infty$.

Finally, the limit space case $A^{p,\infty}(U^n)$ is the space of all analytic functions $f$ in the polydisc such that (see [3])

$$
\sup_{r \in I^n} M_p(f, r)(1 - r)^s < \infty, s \geq 0, 0 < p \leq \infty
$$
Obviously the limit $F_{p, \infty, s}$ space is embedded in the last space we defined. These last two spaces are Banach spaces for all $p \geq 1$ and they are complete metric spaces for all other positive values of $p$.

The following definition of coefficient multipliers is well known in the unit disk. We provide a natural extension to the polydisc setup. This definition can be seen in papers [15], [16],[17],[19],[20] .

**Definition 1.** Let $X$ and $Y$ be quasi normed subspaces of $H(U^n)$. A sequence $c = \{c_k\}_{k \in \mathbb{Z}_n^+}$ is said to be a coefficient multiplier from $X$ to $Y$ if for any function $f(z) = \sum_{k \in \mathbb{Z}_n^+} a_k z^k$ in $X$ the function $h = M_c f$ defined by $h(z) = \sum_{k \in \mathbb{Z}_n^+} c_k a_k z^k$ is in $Y$. The set of all multipliers from $X$ to $Y$ is denoted by $M_T(X, Y)$.

The problem of characterizing the space of multipliers (pointwise multipliers and coefficient multipliers) between various spaces of analytic functions is a classical problem in complex function theory, there is vast literature on this subject, see [7], [8], [10], [11], and references therein.

In this paper we are looking for some extensions of already known classical theorems, namely we are interested in spaces of multipliers acting into analytic $H^\beta_\alpha$ and $H^\beta_\alpha(\alpha)$ Hardy and Bergman type spaces in the unit polydisc and from these spaces into certain well studied classes like mixed norm spaces, Bergman spaces and Hardy spaces. We note that the analogue of this problem of description of spaces of multipliers in $\mathbb{R}^n$ for various functional spaces in $\mathbb{R}^n$ was considered previously by various authors in recent decades (see [3]). The intention of this note is to study the spaces of coefficient multipliers of new analytic function spaces in polydisc. As we indicated this topic is well-known and various results by many authors were published in recent decades in this area starting from classical papers of Hardy- and Littlewood. (see , for example, [3] for many known results and various references there.) Nevertheless the study of coefficient multipliers of analytic function spaces in higher dimension namely in polydisc is a new area of research and only several sharp results are known till now see for example [3],[1],[15] – [18] and various references there. We complement these results and our results from [15],[16],[17] in this note providing new theorems on coefficient multipliers of some new analytic function spaces in polydisc.

We define also by $dm_2(\xi)$ the Lebesgue measure on the unit disk and replacing 2 by $2n$ the Lebesgue measure on the polydisc.

Let $L^p(\alpha)$ be the space of all measurable functions in $U^n$ so that

$$
\|f\|_{L^p(\alpha)} = \left( \int_U (1 - |\xi_n|)^{\alpha_n} \left( \int_U (1 - |\xi_{n-1}|)^{\alpha_{n-1}} \cdots \left( \int_U (1 - |\xi_1|)^{\alpha_1} |f(\xi_1, \ldots, \xi_n)|^{p_1} \times 
\times dm_2(\xi_1) \right)^{\frac{p_1}{\alpha_1}} \cdots dm_2(\xi_{n-1}) \right)^{\frac{p_n}{\alpha_n}} dm_2(\xi_n) \right)^{\frac{1}{p}} < +\infty.
$$
where all $p_j$ are positive and all $\alpha_j > -1$ for any $j = 1, \ldots, n$ Let $H^p(\partial^c) = L^p(\partial^c) \cap H(U^n)$. These spaces are one of the main objects of study of this paper (see [1], [2] for properties of these new classes).

We define now new mixed norm weighted Hardy analytic spaces in polydisc in the following natural way.

Let

$$H^p_{\alpha} = \sup_{r < 1} \left( \int_T \cdots \left( \int_T |f(r_1 \xi_1)|^{p_1} d\xi_1 \right)^{\frac{p_1}{p}} \cdots \int_T \right) \prod_{j=1}^n (1 - r_j)^{\alpha_j};$$

$p_j \in (0; \infty]$; $\alpha_j \geq 0$, $j = 1, \ldots, n$. Note in unit disk we get well-studied classical weighted Hardy spaces [3], [8], [9], [10], [19], $H^p_0 = H^p(U^n)$.

2. PRELIMINARY RESULTS

For formulation of all our main results of this note we need several lemmas, almost all of them are known and taken from recently published papers. The important ingredients of our work are estimates of Bergman kernel in polydisc in some new spaces with usual and unusual quasi-norms in polydisc. As usual we define positive constants by $C$ (or $c$) with indexes.

In particular we need the following lemma 1:

**Lemma 2.1** (see [1], [2]). Let $g(z_1, \ldots, z_n) = \prod_{j=1}^n \frac{1}{(1 - z_j)^{p_j}}$, $g_r(z) = g(rz), r \in I^n, z_j \in U, \beta_j > 0, j = 1, \ldots, n, g_r(z) = g(rz)$. Then for some large enough $\beta_0$ and all $\beta_j > \beta_0, j = 1, \ldots, n$ we have

$$\|g_r\|_{H^p(\partial^c)} \leq c \prod_{j=1}^n (1 - r_j)^{\alpha_j - \beta_j - 2};$$

for $\alpha_j > -1$ and all $0 < p_j < \infty, r_j \in I, j = 1, \ldots, n$; for $p_j = \infty$ we have $\beta_j > 0, j = 1, \ldots, n$ and

$$\|g_r\|_{H^\infty} \leq C \prod_{j=1}^n (1 - r_j)^{-\beta_j}, \quad r_j \in (0; 1).$$

for all $j = 1, \ldots, n$

As corollary of this lemma we will have theorems 1-2 which we will formulate below. One of the main ideas of this note is to study multipliers of some new analytic spaces with unusual norms and quasi-norms. For that reason first we find direct connections of these new analytic spaces with analytic function spaces with simpler norms and quasi-norms based on various simple embeddings and then use this connection and already known information about coefficient multipliers of analytic
classes with simpler quasi-norms to get information we search for. This shows the importance of embeddings connecting various analytic function spaces in polydisc. We in particular need these assertions.

**Lemma 2.2** (see [1], [2]). Let \( f \in H^p(\mathbb{C}^n), 0 < p_j \leq 1, j = 1, \ldots, n, \alpha_j > -1, \gamma_j = 2 - p_j, j = 1, \ldots, n. \) Then we have

\[
\int_0^1 (1 - r)^{\alpha_n + \gamma_n - 1} \ldots \int_0^1 (1 - r_2)^{\alpha_2 + \gamma_2 - 1} \int_0^1 (1 - r_1)^{\alpha_1 + \gamma_1 - 1} \times \\
M_1^{p_1}(r_1, \ldots, r_n, f) r_1 dr_1 \bigg( \int_0^1 r_2 dr_2 \bigg)^{\frac{p_2}{n}} \ldots r_n dr_n \leq c\|f\|_{H^p(\mathbb{C}^n)}^{\gamma_n};
\]

where

\[
M_1^{p}(r_1, \ldots, r_n, f) = \int_{\mathbb{T}^n} |f(r_1 \xi_1, \ldots, r_n \xi_n)|^p d\xi_1 \ldots d\xi_n;
\]

and for all \( 0 < p_j < \infty, \alpha_j > -1, j = 1, \ldots, n \)

\[
\sup_{z \in U^n} |f(z)| \prod_{j=1}^n (1 - |z_j|)^{\frac{2}{p_j} + \frac{\alpha_j}{\gamma_j}} \leq c\|f\|_{H^p(\mathbb{C}^n)}.\]

From lemma 2 we get

**Corollary 1.** Let \( f \in H^p(\mathbb{C}^n), 0 < p_j \leq 1, j = 1, \ldots, n. \) Then \( M_1(f, r)(1 - r)^r < \infty, r \in I, \) where \( \tau = \sum_{j=1}^n (\alpha_j - p_j + 2)\frac{1}{p_j}, \alpha_j > p_j - 2, j = 1, \ldots, n. \)

Similar result is true for \( M_1(f, r), r \in I. \) Here we just must replace \( M_1(f, r) \) by \( M_1(f, r_1, \ldots, r_n) \) and \( (1 - r)^r \) by \( \prod (1 - r_j)^{r_j} \), where \( r_j p_j = \alpha_j - p_j + 2, \) and where \( r_j \in I. \) We will use this corollary and this remark constantly later. We introduce now new spaces of analytic functions in polydisc and provide estimates of Bergman kernel in quasi-norms of these spaces in polydisk. Let for all positive values of indexes \( p, q, \gamma \)

\[
M_1^{p,q} = \{ f \in H(U^n) : \|f\|_{M_1^{p,q}} < \infty \}
\]

\[
\int_0^1 \left( \int_0^1 |f(r\zeta)|^q (1 - r)^{\gamma q - 1} dr \right) \frac{p}{n} d\zeta < \infty
\]

\[
0 < p, q < \infty, \gamma > 0.
\]

These spaces were studied recently in [20].

Note simply replacing \( T \) by \( T^n \) we get immediately known analytic \( F_{\alpha}^{p,q} \) spaces (see [19]). We define also \( \|f\|_{M_1^{p,\infty}} = \int_0^1 \left( \int_{\mathbb{T}^n} |f(r\zeta)((1 - r)^\alpha)|^p d\zeta \right)^{\frac{1}{p}} d\theta < \infty, 0 < p < \infty, \alpha \geq 0 \) and with usual modification for \( p = \infty, 0 < q < \infty \). We put \( A/p = 0 \) if \( p = \infty. \)
Lemma 2.3 (see [20]). Let
\[
g_R(z) = \frac{1}{(1 - Rz)^{\beta + 1}}, \quad R \in I, \ z \in U^n.
\]
Then
\[
\|g_R\|_{M_p^\gamma} \leq \frac{C_1}{(1 - R)^{n(\beta + 1) - \gamma n - \frac{1}{p}}}, \quad \beta > \frac{1}{np} - 1, \\
\|g_R\|_{M_p^\infty} \leq \frac{C_2}{(1 - R)^{n(\beta + 1) - \gamma n - \frac{1}{p}}}, \quad \beta > \frac{1}{np} - 1, \\
\|g_R\|_{M_\infty^\gamma} \leq \frac{C_3}{(1 - R)^{n(\beta + 1) - \gamma n - \frac{1}{p}}}, \quad \beta > \frac{1}{np} - 1.
\]

We note Lemma 3 in combination with standard arguments based on so-called closed graph theorem allows to show immediately the Theorem 4 below. Now we formulate an analogue of lemma 3 for $T_p^\gamma$ spaces in polydisc.

Lemma 2.4 (see [20]). Let
\[
g_R(z) = \frac{1}{(1 - Rz)^{\beta + 1}}, \quad R \in I, \ z \in U^n, \beta > 0.
\]
Then
\[
\|g_R\|_{T_p^\gamma} \leq \frac{C_1}{(1 - R)^{\beta + 1 - \frac{1}{p} - \frac{1}{p_j}}}, \quad \beta > \frac{\gamma}{n} + \frac{1}{np} - 1, \\
\|g_R\|_{T_\infty^\gamma} \leq \frac{C_2}{(1 - R)^{\beta + 1 - \frac{1}{p} - \frac{1}{p_j}}}, \quad \beta > \frac{\gamma}{n} - 1, \\
\|g_R\|_{T_p^\infty} \leq \frac{C_3}{(1 - R)^{\beta + 1 - \frac{1}{p} - \frac{1}{p_j}}}, \quad \beta > \frac{\gamma}{n} + \frac{1}{np} - 1.
\]

The proof of Lemma 5 is easy. We omit it leaving it to readers.

Lemma 2.5. For $g_R(z) = \prod_{j=1}^n \frac{1}{(1 - Rz_j)^{\beta_j + 1}}, \quad R \in I^n, \ z \in U^n$, we have
\[
\|g_R\|_{H_\alpha^p} \leq \frac{C}{\prod_{j=1}^n (1 - R_j)^{\beta_j + 1 - \frac{1}{p_j} - \alpha_j}}, \quad \beta_j > \alpha_j + \frac{1}{p_j} - 1, \quad j = 1, \ldots, n.
\]

The following is an analogue of lemma 3 for $A_\alpha^p, F_\alpha^p, H^\alpha$, namely we provide estimates for Bergman kernel in polydisc in these spaces.

Lemma 2.6 (see [19]). Let
\[
g_R(z) = \frac{1}{(1 - Rz)^{\beta + 1}}, \quad R \in I^n, \ z \in U^n, \beta > 0.
\]
Then
\[ \|g_R\|_{A_p^\gamma} \leq \frac{C_1}{(1 - R)^{\beta - \gamma - \frac{1}{p} + 1}}, \beta > \frac{1}{p} - 1, \]
\[ \|g_R\|_{F_p^\gamma} \leq \frac{C_2}{(1 - R)^{\beta - \gamma - \frac{1}{p} + 1}}, \beta > \frac{1}{p} - 1, \]
\[ \|g_R\|_{H_s} \leq \frac{C_3}{(1 - R)^{\beta - \gamma - \frac{1}{p} + 1}}, \beta > \frac{1}{s} - 1. \]

The following is an analogue of lemma 3 for \( A_p^\infty, A_\gamma^\infty, F_p^\infty, H^\infty, Bl \) namely we provide estimates for Bergman kernel in polydisc in these spaces.

**Lemma 2.7** (see [19]). Let
\[ g_R(z) = \frac{1}{(1 - Rz)^{\beta + 1}}, R \in F, z \in U, \beta > 0. \]

Then
\[ \|g_R\|_{A_p^\infty} \leq \frac{C_1}{(1 - R)^{\beta - \gamma - \frac{1}{p} + 1}}, \beta > \frac{1}{p} - 1, \]
\[ \|g_R\|_{A_\gamma^\infty} \leq \frac{C_2}{(1 - R)^{\beta - \gamma - 1}}, \beta > \gamma - 1, \]
\[ \|g_R\|_{F_p^\infty} \leq \frac{C_3}{(1 - R)^{\beta - \gamma - \frac{1}{p} + 1}}, \beta > \frac{1}{p} - 1, \]
\[ \|g_R\|_{Bl} \leq \frac{C_4}{(1 - R)^{\beta + 1}}, \beta > -1, \]
\[ \|g_R\|_{H^\infty} \leq \frac{C_5}{(1 - R)^{\beta + 1}}, \beta > -1. \]

Next lemma plays the same role as lemma 2 for \( H^\infty(\alpha) \).

**Lemma 2.8** (see [4]). Let \( 0 < p_j < 1, j = 1, \ldots, n \). Then
\[ \int_{U^n} |f(z_1, \ldots, z_n)\prod_{j=1}^n (1 - |z_j|)^{\frac{1}{p_j} - 2} dm_{2n}(z) \leq c \|f\|_{H^\infty} = \]
\[ = \sup_{0 < r_j \leq 1} \left\{ \int_T \left( \int_T |f(r_1 \xi_1, \ldots, r_n \xi_n)|^{p_1} d(\xi_1) \right)^{\frac{p_1}{p_2}} \ldots d(\xi_n) \right\}^{\frac{1}{p_2}}. \]

We get the following assertion from lemma 8 immediately using standard arguments.
Corollary 2. Let \( p_j \in (0, 1), j = 1, \ldots, n \), then

\[
\sup_{r \in I} M_1(f, r) \prod_{k=1}^{n} (1 - r_k)^{\frac{1}{p_k} - 1} \leq c\|f\|_{H^{\beta}}
\]

Weighted Hardy space case \( H^{\beta}_{\vec{a}} \) follows from corollary 2 if we apply it to \( f_R, R \in I \) function using then standard arguments.

This corollary 2 and it is weighted version we just mentioned can be used immediately instead of corollary 1 to get same theorems for \( H^{\beta}_{\vec{a}}(\vec{\alpha}) \) for those cases where \( M_f(X, H^{\beta}(\vec{a})) \) appear above in theorems (see theorems 3, 4, 5), we leave this procedure for readers.

Lemma 8 gives ways below to find easily simple necessary conditions for \( g \) function in terms of \( M_1(f, r) \) function to be a multiplier from \( X \) to \( H^{\beta}_{\vec{a}} \) for various \( X \) spaces we considered above also with the help of lemma’s 3-7 and the closed graph theorem.

The following lemma finally can give ways to provide necessary simple conditions on multipliers via growth conditions on \( M_p(f, r) \) only on diagonal of polydisc namely \( M_p(f, r, \ldots, r), 0 < p < \infty, r \in I \).

Lemma 2.9 (see [1], [2]). Let \( \alpha_j > -1, 0 < p_j < +\infty, j = 1, \ldots, n, D(f)(z) = f(z, \ldots, z), z \in U \). Then

\[
\int_U |f(z, \ldots, z)|^{p_0}(1 - |z|)^{\alpha_0} \prod_{j=1}^{n-1} ((1 - |z|)^{\alpha_j}/(1 - |z|)^2)^{p_j} dm_2(z) \leq c\|f\|_{H^{\beta}_{\vec{a}}}^{p_0}.
\]

3. MAIN RESULTS ON COEFFICIENT MULTIPLIERS IN SOME NEW SPACES OF ANALYTIC FUNCTIONS IN POLYDISC WITH MIXED NORMS

The intention of this section to provide main results of this paper. The base of all proofs is a standard argument based on closed graph theorem used before by many authors (see for example [3]) in one dimension and in higher dimension (see for example [15],[16],[17], [19] and references there). This in combination with various lemmas containing estimates of Bergman kernel in some new unusual analytic spaces we provided above leads directly to completion of proofs. Note that the same scheme in \( U^n \) was used by first author in his papers [15],[16],[17], [19]. The appropriate definitions of coefficient multipliers in polydisc and Bergman kernels in polydisc serve as important steps for solutions. Our first theorem and the second theorem provide new necessary conditions for multipliers into various known mixed norm spaces from \( H^{\beta}(\vec{a}) \) spaces which we study in this paper based on first lemmas. Note these two theorems are not new in case of unit disk (see [3]). So they can be viewed as extensions of these old results (see [3]). The following theorem can be seen also
as a corollary of lemma 1 above which provides estimates of Bergman kernel in mentioned mixed norm $H^p(\mathbb{C})$ spaces and the closed graph theorem.

Let $\beta_0$ be large enough positive number depending on parameters $p_j, \alpha_j, p, q, \gamma$, for example $\beta_0 = \beta_0(p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n)$.

**Theorem 3.1.** 1) Let $0 < p, q < \infty, g \in H(U^n)$, $g(z) = \sum_{|k| \geq 0} c_{k_1, \ldots, k_n} z_1^{k_1} \cdots z_n^{k_n}$, be a multiplier from $H^p(\mathbb{C})$ to $A^p_q(U^n)$ then

$$
\sup_{r \in I^n} M_p(D^g, r) \prod_{j=1}^n (1 - r_j)^{\gamma + \beta_j + 1 - 2/p - \alpha_j/p} < \infty,
$$

for all $\beta_j > \beta_0, j = 1, \ldots, n$ for some large enough $\beta_0$, and all $0 < p < \infty, \alpha_j > -1, \gamma > 0, j = 1, \ldots, n$.

2) Let $0 < p, q < \infty, g \in H(U^n)$, $g(z) = \sum_{k_1 \geq 0, \ldots, k_n \geq 0} c_{k_1, \ldots, k_n} z_1^{k_1} \cdots z_n^{k_n}$ be a multiplier from $H^p(\mathbb{C})$ to $F^p_q(U^n)$ then for $q \leq p$

$$
\sup_{r \in I^n} M_p(D^g, r) \prod_{j=1}^n (1 - r_j)^{\gamma + \beta_j + 1 - 2/p - \alpha_j/p} < \infty,
$$

for all $\beta_j > \beta_0$, for some large enough $\beta_0$, and all $0 < p < \infty, \alpha_j > -1, \gamma > 0, j = 1, \ldots, n$.

The following theorem also can be seen as a corollary of lemma 1 above which provides estimates of Bergman kernel in mentioned mixed norm $H^p(\mathbb{C})$ spaces and the closed graph theorem.

**Theorem 3.2.** Let $p, \gamma, p_j > 0$ and $\alpha_j > -1$ Let $g \in H(U^n)$, $g(z) = \sum_{|k| \geq 0} c_k z^k$

1) If $g \in M_{T}(H^p(\mathbb{C}); B_l)$ then

$$
\sup_{r \in I^n} M_{\infty}(D^g, r) \prod_{j=1}^n (1 - r_j)^{\beta_j + 2 - 2/p_j - \alpha_j/p_j} < \infty, \text{for all } \beta_j > \beta_0, 0 < p_j < \infty, j = 1, \ldots, n
$$

2) If $g \in M_{T}(H^p(\mathbb{C}); F^p_{q, \infty})$ then

$$
\sup_{r \in I^n} M_p(D^g, r) \prod_{j=1}^n (1 - r_j)^{\gamma + \beta_j + 1 - 2/p_j - \alpha_j/p_j} < \infty, \text{for all } \beta_j > \beta_0, j = 1, \ldots, n.
$$

3) If $g \in M_{T}(H^p(\mathbb{C}); A^p_{q, \infty})$ then

$$
\sup_{r \in I^n} M_p(D^g, r) \prod_{j=1}^n (1 - r_j)^{\gamma + \beta_j + 1 - 2/p_j - \alpha_j/p_j} < \infty, \text{for all } \beta_j > \beta_0, \alpha_j > -1, \gamma > 0, j = 1, \ldots, n
$$
Remark 3.1. Similarly conditions on $M_T(H^p(\vec{\alpha}), X)$ for $X = T^{p,q}_y$ can be found for $p < q$ or $p = q$. We have to use in addition that $T^{p,q}_y \subset B^{p,q}_y$, $p \leq q$ and the fact that

$$\sup_{r \in I} M_p(f, r)(1 - r)^\gamma \leq c\|f\|_{B^{p,q}_y}, \quad \gamma > 0, \quad 0 < p, q \leq \infty.$$ 

Proofs of theorems 1 and 2.

Here a sketch of proof is given for the main assertions contained in Theorem 1 and Theorem 2.

Assume $\{c_k\}_{k \in \mathbb{Z}_n} \in M_T(H^p(\vec{\alpha}), X)$ where $X = A^{p,q}_\alpha$ or $F^{p,q}_\alpha$, $0 < p, q \leq \infty$, $\alpha \geq 0$ or $X = B\ell, H^\infty$.

An application of the closed graph theorem gives

$$\|M_{c}f\|_X \leq c\|f\|_{H^p(\vec{\alpha})}.$$ 

Let $w \in U^n$ and set $f_w(z) = \frac{1}{1 - wz}$, $g_w = M_c f_w$. Then we have

$$D^\beta(g_w) = D^\beta M_c f_w = \frac{M_c}{1 - wz} g_w = c \left( M_c \frac{1}{(1 - wz)^{\beta + 1}} \right); \quad \beta > 0$$

which gives

$$\|D^\beta g_w\|_X \leq c\|f\|_{H^p(\vec{\alpha})} = c\left\| \frac{1}{(1 - wz)^{\beta + 1}} \right\|_{H^p(\vec{\alpha})}.$$ 

This give immediately by lemma 1 an estimate

$$\|D^\beta g_w\|_X \leq c \left[ \prod_{j=1}^n \frac{1}{(1 - r_j)^{-\frac{\alpha_j}{p_j}} - \frac{2}{p_j} + (\beta + 1)} \right].$$

We finish easily the proof using only one known embedding (depending on $X$ space).

For $X = A^{p,q}_\alpha$ we use embedding

$$\sup_{r \in I} (M_p(f, r)(1 - r)^\alpha \leq c\|f\|_{A^{p,q}_\alpha}, \quad f \in H(U^n), \quad 0 < p \leq \infty, \quad \alpha \geq 0.$$ 

For $X = F^{p,q}_\alpha$, $p \leq q$ we use the same embedding with inclusion

$$F^{p,q}_\alpha \subset A^{p,q}_\alpha, \quad p \leq q.$$ 

For $p = \infty$ or $q = \infty$ we have to replace above $p$ by $\infty$ or $q$ by $\infty$. The case if $X = B\ell$ or $X = H^\infty$ we have to modify the scheme we provided above.
For $X = T^{p,q}_\alpha$ we use same arguments and the embedding

$$\sup_{r \in \mathbb{R}} M_p(f, r)(1 - r)^\alpha \leq c\|f\|_{T^{p,q}_\alpha}; \quad 0 < p, q \leq \infty; \quad p \leq q; \quad \alpha > 0; \quad f \in H(U^n)$$

with obvious modification for $p = \infty$ or $q = \infty$. Note that the same embedding is true for $B^{p,q}_\alpha$. Similar arguments are valid if we replace $H^p(\alpha)$ by $H^{p}_\alpha$.

Indeed using lemma 5 instead of lemma 1 we can get complete analogues of all assertions of Theorem 1 and 2 we provided above but for $H^{p}_\alpha$ spaces. We leave this to readers.

As corollaries to Lemmas 1-9 we have Theorems 3 - 5 below. They provide various necessary conditions for a function to be a multiplier from mixed norm well-studied analytic Besov type and new Lizorkin-Triebel type spaces into new mixed norm spaces we defined and study in this paper namely $H^p(\alpha)$ and $H^{p}_\alpha$. Note some results are new even in case of unit disk, on the other hand others can be viewed as direct extensions of well-known one dimensional results to the higher dimensional case of polydisc and we also record them. Note we also here and above include in some cases "limit" spaces when some parameters involved are equal to infinity. We also leave some cases open for interested readers.

The following theorem can be seen as a corollary of lemma’s 5, 6, 7 and corollary 1 above which provides estimates of Bergman kernel in mentioned mixed norm spaces $A^{p,q}_\gamma, F^{p,q}_\gamma$ and the closed graph theorem.

**Theorem 3.3.** Let $p > 0, q > 0, \gamma > 0$. Let $\alpha_j > p_j - 2$. Let $0 < p_j \leq 1, j = 1, ..., n$ and let $g(z) = \sum_{k \in \mathbb{Z}_+^n} c_k z^k, g \in H(U^n)$, then

1) If $g \in M_T(A^{p,q}_\gamma, H^{p}_\alpha(\vec{\alpha}))$ then

$$\sup_{r \in \mathbb{R}} M_T(D^g, r_1, ..., r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty,$$

$$\tau_j = \frac{\alpha_j - p_j + 2}{p_j} + \beta - \gamma - 1, j = 1, ..., n.$$ $\beta > \max_j \left( \gamma + 1 - \frac{\alpha_j - p_j + 1}{p_j} \right)$. 

2) If $g \in M_T(F^{p,q}_\gamma, H^{p}_\alpha(\vec{\alpha}))$ then

$$\sup_{r \in \mathbb{R}} M_T(D^g, r_1, ..., r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty,$$

$$\tau_j = \frac{\alpha_j - p_j + 2}{p_j} + \beta - \gamma - 1, j = 1, ..., n.$$ $\beta > \max_j \left( \gamma + 1 - \frac{\alpha_j - p_j + 1}{p_j} \right)$.

3) If $g \in M_T(F^{p,\infty}_\gamma, H^{p}_\alpha(\vec{\alpha}))$ then

$$\sup_{r \in \mathbb{R}} M_T(D^g, r_1, ..., r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty,$$

$$\tau_j = \frac{\alpha_j - p_j + 2}{p_j} + \beta - \gamma - 1, j = 1, ..., n.$$ $\beta > \max_j \left( \gamma + 1 - \frac{\alpha_j - p_j + 1}{p_j} \right)$. 

Note some cases are open for interested readers.
\[ \tau_j = \frac{\alpha_j^{p_j+2}}{p_j} + \beta - \gamma - \frac{1}{p} + 1, \quad j = 1, \ldots, n, \beta > \max_j \left( \gamma + \frac{1}{p} - 1 - \frac{\alpha_j^{p_j+2}}{p_j} \right). \]

4) If \( g \in M_T(A_1^{p,\infty}, H^n(d)) \) then

\[ \sup_{r \in I} M_1(D^\alpha g, r_1, \ldots, r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty, \]

\[ \tau_j = \frac{\alpha_j^{p_j+2}}{p_j} + \beta - \gamma - \frac{1}{p} + 1, \quad j = 1, \ldots, n, \beta > \max_j \left( \gamma + \frac{1}{p} - 1 - \frac{\alpha_j^{p_j+2}}{p_j} \right). \]

5) If \( g \in M_T(A_1^{\infty,q}, H^n(d)) \) then

\[ \sup_{r \in I} M_1(D^\alpha g, r_1, \ldots, r_n) \prod_{j=1}^n (1 - r_j)^{\tau_j} < \infty, \]

\[ \tau_j = \frac{\alpha_j^{p_j+2}}{p_j} + \beta - \gamma - \frac{1}{p} + 1, \quad j = 1, \ldots, n, \beta > \max_j \left( \gamma + \frac{1}{p} - 1 - \frac{\alpha_j^{p_j+2}}{p_j} \right), \text{ with } p = \infty. \]

The following two theorems can be seen as a corollary of lemma’s 3 and 4 above which provides estimates of Bergman kernel in mentioned mixed norm spaces \( T_{\gamma}^{p,q} , M_{\gamma}^{p,q} \) including limit “infinity” cases of parameters and the closed graph theorem and lemma 8.

**Theorem 3.4.** Let \( p, q, \gamma > 0 \). Let \( 0 < p_j \leq 1, \quad j = 1, \ldots, n, g \in H(U^n); g(z) = \sum_{k \in \mathbb{Z}^n} c_k z^k, \alpha_j > p - 2, \quad \tau = \sum_{j=1}^n (\alpha_j - p_j + 2) \frac{1}{p_j}, \quad j = 1, \ldots, n. \) Then

1) If \( g \in (M_T(M_1^{p,q}, H^n(d))) \) then \( \sup_{r \in I} M_1(D^\alpha g, r)(1 - r)^{\frac{1 + \alpha + n - 1}{n}} < \infty; \)

for \( \beta > \gamma + \frac{1}{np} - \frac{\tau}{n} - 1, \)

2) If \( g \in (M_T(M_1^{p,\infty}, H^n(d))) \) then \( \sup_{r \in I} M_1(D^\alpha g, r)(1 - r)^{\frac{1 + \alpha + n - 1}{n}} < \infty; \)

for \( \beta > \gamma + \frac{1}{np} - \frac{\tau}{n} - 1, \)

3) If \( g \in (M_T(M_1^{\infty,q}, H^n(d))) \) then \( \sup_{r \in I} M_1(D^\alpha g, r)(1 - r)^{\frac{1 + \alpha + n - 1}{n}} < \infty \)

for \( \beta > \gamma - \frac{\tau}{n} - 1, \)

Replacing \( (M_T^{p,q}); (M_T^{p,\infty}); (M_T^{\infty,q}) \) by \( (T_{\gamma}^{p,q}); (T_{\gamma}^{p,\infty}); (T_{\gamma}^{\infty,q}) \) we have the following analogue of previous theorem for \( T \) type spaces of analytic function in polydisc based again on lemma’s 3, 4 and 8 we provided above. Note these results are new even in case of unit disk.
Let $p,q,\gamma > 0$ Let $0 < p_j \leq 1$, $j=1,...,n, g \in H(U^n), g(z) = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$.

Then

1) $f, g \in (M_T)(T^{\beta,\gamma}_T, H^\beta(\alpha))$ then $\sup_{r \in F^n} M_1(D^\beta g, r) \left\{ (1-r_j)^{\gamma+\beta+1-\frac{\gamma}{p_j}-\frac{1}{p_j}} < \infty, \right.$

$$\beta > \max_j \left( \frac{1}{p_j} + \frac{\gamma}{n} - \frac{\gamma}{p_j} - 1 \right), \tau_j = \frac{\alpha_j - p_j + 2}{p_j} > 0; \ j = 1, \ldots, n$$

2) $f, g \in (M_T)(T^{\beta,\infty}_T, H^\beta(\alpha))$ then $\sup_{r \in F^n} M_1(D^\beta g, r) \left\{ (1-r_j)^{\gamma+\beta+1-\frac{\gamma}{p_j}-\frac{1}{p_j}} < \infty, \right.$

$$\beta > \max_j \left( \frac{1}{p_j} + \frac{\gamma}{n} - \frac{\gamma}{p_j} - 1 \right), \tau_j = \frac{\alpha_j - p_j + 2}{p_j} > 0; \ j = 1, \ldots, n$$

3) $f, g \in (M_T)(T^{\beta,\gamma}_T, H^\beta(\alpha))$ then $\sup_{r \in F^n} M_1(D^\beta g, r) \left\{ (1-r_j)^{\gamma+\beta+1-\frac{\gamma}{p_j}-\frac{1}{p_j}} < \infty, \right.$

$$\beta > \max_j \left( \frac{1}{p_j} + \frac{\gamma}{n} - \frac{\gamma}{p_j} - 1 \right), \tau_j = \frac{\alpha_j - p_j + 2}{p_j} > 0, \ j = 1, \ldots, n.$$ 

with $p = \infty$.

Schemes of proofs of theorems 3, 4, 5.

Here is a short scheme of proof of various of type contained in theorem 3, 4, 5.

Assume $(c_k)_{k \in \mathbb{Z}_+^n} \in M_T(X, H^\beta(\alpha))$ (or $M_T(X, H^\beta_{\infty})$) then we have to use the closed graph theorem and corollary 1 for $H^\beta(\alpha)$ case or remark after corollary 2 for $H^\beta_{\infty}$ case.

We have the following estimates and equalities, first by closed graph theorem $\|M_{c,f}\|_{H^\beta(\alpha)} \leq \|f\|_X$. Let $w \in U^n$, and set $f_w(z) = \frac{1}{1-wz}, g_w = M_c(f_w)$. Then we have

$$D^\beta g_w = D^\beta(M_c(f_w)) = M_c(D^\beta(f_w)) = cM_{\frac{1}{(1-wz)^{\beta+1}}}, \beta > 0,$$

which gives

$$\|D^\beta g_w\|_{H^\beta(\alpha)} \leq c \frac{1}{(1-wz)^{\beta+1}}$$.

It remains to apply corollary 1 from left for $H^\beta(\alpha)$ spaces or remark after corollary 2 for $H^\beta_{\infty}$ spaces. From right for various $X$ spaces we have to use various estimates from lemma 3, 4, 6, 7 of Bergman kernel $g_\beta(z)$ in polydisc $U^n$ to get the desired result.

**Remark 3.2.** Some assertions in Theorems 3-5 seem to be sharpened, in the sense: the reverse implications given there may hold as well. We do not discuss these issues in our paper.
Remark 3.3. Note all assertions above on $H^p(\alpha)$ can be reformulated for $H^p_{\alpha}$ replacing lemma 1 by lemma 5 (we mean $M_f(H^p_{\alpha}, X)$ classes). Note also that using corollary 2 instead of corollary 1 all assertions of theorem 3,4,5 easily can be reformulated for weighted Hardy spaces.

4. FINAL REMARKS

We add in this section several useful remarks. Various similar new assertions on coefficient multipliers of analytic spaces which we study in this paper can be obtained from the following remark and descriptions of dual of $H^p(\alpha)$ spaces for $p_j > 1, j = 1, \ldots, n$ or $p_j < 1, j = 1, \ldots, n$ (see for this description [1], [2]). The idea is (see [3], [10], [11]) to use the following obvious fact if ($g \sim (c_k)$ is a multiplier from $X$ to $Y$ ($X, Y$ are quazynormed subspaces of $H(U^n)$) then it is a multiplier from dual spaces $Y^*$ to $X^*$ and then apply to this last pair a standard closed graph theorem. This procedure was used before by many authors (see for example [3]).

We provide first descriptions of dual spaces.

Proposition 3. Let $1 < p_j < \infty, \alpha_j > -1, j = 1, \ldots, n$. Then $(H^p(\alpha))^* = Z_1(U^n)$.

Let $p_j \leq 1, \alpha_j > -1, j = 1, \ldots, n$. Then $(H^p(\alpha))^* = Z_2(U^n)$, where $Z_1, Z_2$ are the following spaces of analytic functions in polydisc

$$Z_1(\beta, \alpha) = \{ f \in H(U^n) : ||D^{\alpha+1} f||_{H^p(\alpha)} < \infty, \frac{1}{p_j} + \frac{1}{q_j} = 1, j = 1, \ldots, n,$$

and

$$Z_2(\beta, \alpha) = \{ f \in H(U^n) : \sup |D^{\alpha+1} f| < \infty, (\alpha_j \geq \frac{2 + \alpha_j}{p_j}) \}.$$ 

As corollary of this and remark above, we can indicate $M_i, i = 1, \ldots, 3$ spaces and embeddings

1) $M_f(H^p, H^p_{\alpha}) \subset M_1(U^n)$;

2) $M_f(A^p_{\alpha}, H^p_{\alpha}) \subset M_2(U^n)$;

3) $M_f(A^p_{\alpha}, H^p_{\alpha}) \subset M_3(U^n)$; for $p \in (1; \infty)$, or $0 < p < 1$ and $\beta, \overline{\alpha} \in (1; \infty)$, $|\beta|, |\overline{\alpha}| \in (0; 1]$.

Note $(H^p)^*, (A^p_{\alpha})^*, (A^p_{\alpha})^*$ spaces are known (see for example [3]) to fix $M_f$ classes, $j = 1, \ldots, n$ following remarks above is easy now. We leave this to reader.

Obviously we can get similarly following the proof of lemma 1 estimates for Bergman kernel $g_\alpha(z)$ of some other mixed norm spaces in $U^n$ and then complete analogous of theorem 1 and 2 for them replacing $H^p(\alpha)$ in them.
We give examples. Let \( f \in H(U^n) \) and
\[
\|f\|_{X_i} = \left( \int_0^1 \left( \int_{|z|<r} \left( \int_{|w|<r} |f(w)|^{p_i} (1 - |w_1|)^{q_i} dm_2(w_1) \right)^{\frac{p_i}{n}} \right)^{\frac{2}{n}} \right)^{\frac{1}{2}} \times \\
\times \left( \int (1 - |w_2|)^{q_2} dm_2(w_2) \right)^{\frac{1}{2}},
\]
where \( 0 < p_j < \infty, \alpha_j > 1, j = 1, ..., n, q \in (0; \infty) \) or if we consider \( X_2 \) spaces with quasinorms
\[
\|f\|_{X_2} = \left( \int \left( \int \left( \int |f(w)|^{p_i} (1 - |w_1|)^{q_i} dm_2(w_1) \right)^{\frac{p_i}{n}} \right)^{\frac{2}{n}} \right)^{\frac{1}{2}},
\]
where \( f \in H(U^n), 0 < p_j < \infty, j = 1, ..., n, q \in (0; \infty) \), where as usual \( \Gamma_r(\zeta) = \{ z \in U, |1 - \zeta z| < r(1 - |z|), r > 1 \} \).

We also put as \( X \) below a fixed mixed norm space \( A^p_{\alpha} \) or \( F^p_{\alpha} \), 0 < \( p, q \leq \infty, \alpha > -1 \).

And similarly for cases when some \( p = \infty \) we consider
\[
\left( \int_0^1 \left( \sup_{|z|<r} \left( \sup_{|w|<r} |f(w)|^{p_i} (1 - |w_1|)^{\alpha_i} dm_2(w_1) \right)^{\frac{p_i}{n}} \right)^{\frac{2}{n}} \right)^{\frac{1}{2}}
\]
or
\[
\left( \int \left( \int_{\Gamma_r(\zeta)} \left( \sup_{z \in \Gamma_r(\zeta)} |f(z)|^{p_i} (1 - |z_1|)^{\alpha_i} dm_2(z_1) \right)^{\frac{p_i}{n}} \right)^{\frac{2}{n}} d\zeta \right)^{\frac{1}{2}}.
\]

We can use lemmas above and estimates (7), (8) below to find sizes of \( M_T(X_i, X), i = 1, 2 \). (complete analogues of theorem 1 and 2).

\[
\int_{\Gamma_r(\zeta)} (1 - |z|^\beta) \frac{dm_2(z)}{|1 - wz|^{\gamma}} \leq \frac{C_1}{|1 - wz|^{\gamma - \beta - 2}}, \zeta \in T, |w| < 1, \gamma > \beta + 2, \beta > -1. \quad (7)
\]

\[
\int_{|z|<r} (1 - |z|^\beta) \frac{dm_2(z)}{|1 - wz|^{\gamma}} \leq \frac{C_2}{|1 - rz|^{\gamma - \beta - 2}}, |w| < 1, 0 < r \leq 1, \gamma > \beta + 2, \beta > -1. \quad (8)
\]

These onedimensional estimates by simple iteration can be easily passed to polydisc case as we did above estimating Bergman kernel. So analogue of lemma 1 for \( X_i \)
spaces can be easily found. Similarly we can consider also spaces with quazinorms

$$\|f\|_{X_3} = \left[ \int_{\mathbb{D}} \left( \int_{\mathbb{D}} \ldots \left( \int_{\mathbb{D}} |f(w)|^{p_1} (1 - |w_1|)^{\alpha_1} dm_2(w_1) \right) \ldots \right)^{\frac{q}{p_1}} d\zeta \right]^{\frac{1}{q}},$$

$$0 < p_j < \infty, \alpha_j > -1, j = 1, \ldots, n, q \in (0; \infty)$$

and

$$\|f\|_{X_4} = \left[ \int_0^1 \left( \int_0^{\cdots} \left( \int_0^{\cdots} |f(w)|^{p_1} (1 - |w_1|)^{\alpha_1} dm_2(w_1) \right) \ldots \right)^{\frac{q}{p_1}} dr_1 \ldots dr_n \right]^{\frac{1}{q}},$$

$$0 < p_j < \infty, \alpha_j > -1, j = 1, \ldots, n, q \in (0; \infty)$$

and find similarly an information on classes $M_T(X_3, X)$, $M_T(X_i, X), i = 3, 4$ finding first estimates of Bergman kernel in $X_3$ and $X_4$

References


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PPF DEPENDENT FIXED POINTS IN A-CLOSED RAZUMIKHIN CLASSES

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Abstract
The PPF dependent fixed point result in algebraically closed Razumikhin classes due to Agarwal et al [Fixed Point Theory Appl., 2013, 2013:280] is identical with its constant class counterpart; and this, in turn, is reducible to a fixed point principle involving SVV type contractions (over the subsequent metric space of initial Banach structure), without any regularity conditions about the Razumikhin classes. The conclusion remains valid for all PPF dependent fixed point results founded on such global conditions.

Keywords: metric space, Picard operator, fixed point, SVV type contraction, Banach space, Razumikhin functional class, algebraical and topological closeness, PPF dependent fixed point, nonself contraction, iterative process.

2010 MSC: 47H10 (Primary), 54H25 (Secondary).

1. INTRODUCTION

Let \( X \) be a nonempty set. Call the subset \( Y \) of \( X \), almost singleton (in short: asingleton) provided \( y_1, y_2 \in Y \) implies \( y_1 = y_2 \); and singleton, if, in addition, \( Y \) is nonempty; note that, in this case, \( Y = \{ y \} \), for some \( y \in X \). Take a metric \( d : X \times X \to \mathbb{R}_+ := [0, \infty] \) over it; as well as a selfmap \( T \in \mathcal{F}(X) \). [Here, for each couple \( A, B \) of nonempty sets, \( \mathcal{F}(A, B) \) denotes the class of all functions from \( A \) to \( B \); when \( A = B \), we write \( \mathcal{F}(A) \) in place of \( \mathcal{F}(A, A) \). Denote \( \text{Fix}(T) = \{ x \in X; x = Tx \} \); each point of this set is referred to as fixed under \( T \). The determination of such elements is to be performed under the directions below, comparable with the ones in Rus [21, Ch 2, Sect 2.2] and Turinici [27]:

**Pic-1** We say that \( T \) is a Picard operator (modulo \( d \)) if, for each \( x \in X \), the iterative sequence \( (T^n x; n \geq 0) \) is \( d \)-convergent; and a globally Picard operator (modulo \( d \)) if, in addition, \( \text{Fix}(T) \) is an asingleton

**Pic-2** We say that \( T \) is a strong Picard operator (modulo \( d \)) if, for each \( x \in X \), the iterative sequence \( (T^n x; n \geq 0) \) is \( d \)-convergent with \( \lim_n (T^n x) \) belonging to \( \text{Fix}(T) \); and a globally strong Picard operator (modulo \( d \)) if, in addition, \( \text{Fix}(T) \) is an asingleton (hence, a singleton).

The basic result in this area is the 1922 one due to Banach [2]. Given \( k \geq 0 \), let us say that \( T \) is \( (d; k) \)-contractive, if:

\[(a01) \quad d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X.\]
Theorem 1.1. Assume that $T$ is $(d;k)$-contractive, for some $k \in [0,1]$. In addition, let $(X,d)$ be complete. Then, $T$ is globally strong Picard (modulo $d$).

This statement (referred to as: Banach’s fixed point principle) found some basic applications to different branches of operator equations theory. As a consequence, many extensions of it were proposed. From the perspective of this exposition, the following ones are of interest:

I) Contractive type extensions: the initial Banach contractive property is taken in a generalized way, as

\[(a02) \quad F(d(Tx,Ty),d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(Tx,y)) \leq 0,\]

for all $x,y \in X$;

where $F : R^6_+ \to R$ is a function. For the explicit case, some consistent lists of these may be found in the survey papers by Rhoades [20], Collaco and E Silva [6], Kincses and Totik [14], as well as the references therein. And, for the implicit case, certain particular aspects have been considered by Leader [16] and Turinici [25].

II) Structural extensions: the trivial relation $X \times X$ is replaced by a relation $\nabla$ over $X$ fulfilling or not certain regularity properties. For example, the case of $\nabla$ being reflexive, transitive (hence, a quasi-order) was considered in the 1986 paper by Turinici [26]. Two decades later, this result was re-discovered – in a (partially) ordered context – by Ran and Reurings [19]; see also Nieto and Rodriguez-Lopez [17]. On the other hand, the "amorphous" case ($\nabla$ has no regularity properties at all) was discussed (via graph techniques) in Jachymski [11]; and (from a general perspective) by Samet and Turinici [22]. Some other aspects involving additional convergence structures may be found in Kasahara [13].

III) Nonself extensions: $T$ is no longer a selfmap. In 1977, Bernfeld et al [3] introduced the concept of PPF (past-present-future) dependent fixed point for nonself-mappings (whose domain is distinct from their range). Furthermore, the quoted authors established – via iterative methods involving a certain Razumikhin class $R_c$ – some PPF dependent fixed point theorems for contractive mappings of this type. As precise there, the obtained results are useful tools in the study of existence and uniqueness questions for solutions of nonlinear functional differential/integral equations which may depend upon the past history, present data and future evolution. As a consequence, this theory attracted a lot of contributors in the area; see, for instance, Dhage [7, 8], Hussain et al [10], Kaewcharoen [12], Kutbi and Sintunavarat [15], as well as the references therein. However, as proved in a recent paper by Cho, Rassias, Salimi and Turinici [4], the starting conditions [imposed by the problem setting] relative to the ambient Razumikhin class $R_c$ may be converted into starting conditions relative to the constant class $K$; so, ultimately, we may arrange for these PPF dependent fixed point results holding over $K$. In this exposition we bring the discussion a step further, by establishing that

Fact-1) the algebraic closeness assumption [used in all these references] imposed upon the Razumikhin class $R_c$ yields, in a direct way, $R_c = K$. 

Theorem 1.1. Assume that $T$ is $(d;k)$-contractive, for some $k \in [0,1]$. In addition, let $(X,d)$ be complete. Then, $T$ is globally strong Picard (modulo $d$).
Fact-2) the PPF dependent fixed point problem attached to the constant class $\mathcal{K}$ is reducible to a (standard) fixed point problem in the (complete) metrical structure induced by our initial Banach one, under no regularity assumption about the ambient Razumikhin class.

Finally, as an application of these conclusions, we show that a recent PPF dependent fixed point result in Agarwal et al [1] is reducible to a fixed point problem involving a class of contractions over standard metric structures [taken as before] introduced under the lines proposed by Samet et al [23]. Further aspects will be discussed elsewhere.

2. RAZUMIKHIN CLASSES

Let $(E, ||.||)$ be a Banach space; and $d(., .)$ be the induced by norm metric on $E$: 

$$d(x, y) = ||x - y||,$$ 

hence, $(E, d)$ is a complete metric space. Further, let $I = [a, b]$ be a closed real interval, and $E_0 := C(I, E)$ stand for the class of all continuous functions $\varphi : I \to E$, endowed with the supremum norm

$$||\varphi||_0 = \sup\{||\varphi(t)||; t \in I\}, \varphi \in E_0.$$ 

As before, let $D(., .)$ stand for the induced by norm metric on $E_0$

$$D(\varphi, \xi) = ||\varphi - \xi||_0, \varphi, \xi \in E_0;$$

clearly, $(E_0, D)$ is a complete metric space.

Let $c \in I$ be fixed in the sequel. The Razumikhin class of functions in $E_0$ attached to $c$, is defined as

$$(b01) \quad \mathcal{R}_c = \{\varphi \in E_0; ||\varphi||_0 = ||\varphi(c)||\}.$$ 

Note that $\mathcal{R}_c$ is nonvoid; because any constant function belongs to it. To substantiate this assertion, a lot of preliminary facts are needed.

(A) For each $u \in E$, let $H[u]$ denote the constant function of $E_0$, defined as

$$(b02) \quad H[u](t) = u, t \in I.$$ 

Note that, by this definition,

$$||H[u]||_0 = ||u||, \quad H[u](c) = u;$$

whence $H[u] \in \mathcal{R}_c$. Denote, for simplicity $\mathcal{K} = \{H[u]; u \in E\}$; this will be referred to as the constant class of $E_0$. The following properties of this class are almost immediate; so, we do not give details.

**Proposition 2.1.** Under the above conventions,
The Razumikhin class

Suppose that\( \parallel u \parallel = \parallel H[u] \parallel \), \( \forall u \in E, \forall \lambda \in R \); hence, \( K \) is a linear subspace of \( E_0 \).

\[ \text{(cc2)} \quad \parallel u \parallel = \parallel H[u] \parallel \], \( \forall u \in E \)

\[ \text{(cc3)} \quad \text{the mapping } u \mapsto H[u] \text{ is an algebraic and topological isomorphism between } (E, \parallel \cdot \parallel) \text{ and } (K, \parallel \cdot \parallel) \]

\[ \text{(cc4)} \quad K \text{ is } D\text{-complete (hence, } D\text{-closed)} \text{ in } E_0. \]

\( \text{(B)} \) Returning to the general case, the following simple property holds.

**Lemma 2.1.** The Razumikhin class \( R_c \) is homogeneous, in the sense

\[ \lambda R_c = R_c, \quad \forall \lambda \in R \setminus \{0\}; \text{ whence, } R_c = -R_c. \] (1)

**Proof.** Given \([\lambda \in R, \varphi \in R_c]\), denote \( \xi = \lambda \varphi \). By definition,

\[ \parallel \xi \parallel_0 = |\lambda| \cdot \parallel \varphi \parallel_0, \quad \parallel \xi(c) \parallel = |\lambda| \cdot \parallel \varphi(c) \parallel. \]

This, along with the choice of \( \varphi \), gives \( \xi \in R_c \); and completes the argument.

\( \text{(C)} \) Let \( T : E_0 \to E \) be a (nonself) mapping. We say that \( \varphi \in E_0 \) is a PPF dependent fixed point \( \text{of } T \), when \( T\varphi = \varphi(c) \). The class of all these will be denoted as \( \text{PPF-Fix}(T; E_0) \).

Concerning existence and uniqueness properties involving such points, the first contribution to this theory is the 1977 statement in Bernfeld et al \[3\]; referred to as:

BLR theorem. The following concepts and constructions are necessary.

\( \text{I)} \) Given \( k \geq 0 \), call \( T \), \( k\text{-contractive} \), provided

\[ \text{(b03)} \quad d(T\varphi, T\xi) \leq kD(\varphi, \xi), \quad \text{for all } \varphi, \xi \in E_0. \]

\( \text{II)} \) Let us introduce a relation \( (\triangledown) \) over \( E_0 \), according to:

\[ \text{(b04)} \quad \varphi \triangledown \xi \text{ iff } T\varphi = \xi(c) \text{ and } \varphi - \xi \in R_c. \]

\( \text{III)} \) Finally, for the starting element \( \varphi_0 \in E_0 \), let us say that the sequence \( (\varphi_n; n \geq 0) \) in \( E_0 \) is \( (\varphi_0, \triangledown)\text{-iterative} \), in case \( \varphi_n \triangledown \varphi_{n+1}, \forall n. \)

**Theorem 2.1.** Suppose that \( T \) is \( k\text{-contractive} \), for some \( k \in [0, 1] \). Then,

\( \text{I)} \) Given the starting point \( \varphi_0 \in E_0 \), any \( (\varphi_0, \triangledown)\text{-iterative sequence } (\varphi_n; n \geq 0) \text{ in } E_0, D\text{-converges to an element of } \text{PPF-Fix}(T; E_0). \)

\( \text{ii)} \) Given the couple of starting points \( \varphi_0, \xi_0 \in E_0 \), and letting \( (\varphi_n; n \geq 0), (\xi_n; n \geq 0) \) be a \( (\varphi_0, \triangledown)\text{-iterative sequence} \) and \( (\xi_0, \triangledown)\text{-iterative sequence} \), respectively, we have the evaluation, for all \( n \geq 0, \)

\[ D(\varphi_n, \xi_n) \leq (1 - k)(D(\varphi_0, \varphi_1) + D(\xi_0, \xi_1)) + D(\varphi_0, \xi_0). \] (2)

In particular, if \( \varphi_0 = \xi_0 \), we have for all \( n \geq 0, \)

\[ D(\varphi_n, \xi_n) \leq (2 - k)D(\varphi_0, \varphi_1). \] (3)
iii) Let $(\varphi_n; n \geq 0), (\xi_n; n \geq 0)$ be as above. If $\varphi_n - \xi_n \in R_c$, for all $n \geq 0$, then, necessarily, $\lim_n \varphi_n = \lim_n \xi_n$.

iv) If $\varphi^*, \xi^*$ are in $PPF$-$\text{Fix}(T; E_0)$, and $\varphi^* - \xi^* \in R_c$, then $\varphi^* = \xi^*$.

[For completeness reasons, we shall provide a proof of the above result, at the end of this exposition.]

(D) Technically speaking, the BLR theorem is conditional in nature; because, for the starting $\varphi_0 \in R_c$ [hence, all the more, for the starting $\varphi_0 \in E_0$], the set of all $(\varphi_0, \nabla)$-iterative sequence $(\varphi_n; n \geq 0)$ in $E_0$ (taken as before) may be empty. To avoid this drawback, we have two possibilities:

Option-1) all considerations above are to be restricted to the constant class $K \subseteq R_c$; which, as a $D$-closed linear subspace of $E_0$, yields an appropriate setting for any algebraic and/or topological reasoning to be applied.

Option-2) the initial Razumikhin class $R_c$ remains as it stands; but, with the price of imposing further (strong) restrictions upon it.

A discussion of these is to be sketched under the lines below.

Part 1. Concerning the first option, we note that, the imposed $k$-contractive condition upon $T$ relates elements $\varphi, \xi \in E_0$ with elements $T \varphi, T \xi \in E$. On the other hand, at the level of constant class $K$, the underlying condition writes

\[ ||T \varphi - T \xi|| \leq k||\varphi(c) - \xi(c)||; \]

so that, it relates elements $\varphi(c), \xi(c) \in E$ with elements $T \varphi, T \xi \in E$. Hence, as long as we have a selfmap of $E$ that sends $\psi(c) \in E$ to $T \psi \in E$ (for all $\psi \in K$), the last condition is of selfmap type. The effectiveness of such a construction is illustrated by the considerations below. Let $T : E \rightarrow E$ be the selfmap of $E$ introduced as

\[ Tu = T(H[u]), \quad u \in E. \]

**Proposition 2.2.** Under these conventions, the following are valid:

i) If $x \in E$ is a fixed point of $T$, then $\xi := H[x] \in K$ is a PPF dependent fixed point of $T$.

ii) Conversely, if $\zeta = H[z] \in K$ is a PPF dependent fixed point of $T$, then $z \in E$ is a fixed point of $T$.

**Proof.** i) If $x \in E$ is a fixed point of $T$, we have $x = Tx = T(H[x])$; or, equivalently,

\[ x = T(\xi), \quad \text{where} \quad \xi := H[x] \in K. \]

This, by definition, gives $\xi(c) = T(\xi)$; which tells us that $\zeta \in K$ is a PPF dependent fixed point of $T$.

ii) Suppose that $\zeta = H[z] \in K$ is a PPF dependent fixed point of $T$; that is: $\xi(c) = T(\xi)$. This yields (by these notations)

\[ z = T(H[z]) = T\xi; \]
so that, \( z \in E \) is a fixed point of \( T \).

Having these precise, we may now proceed to the formulation of announced result. Two basic concepts appear.

I) Given \( k \ge 0 \), call \( T, k\)-contractive, provided

\[
(b07) \quad d(T\varphi, T\xi) \le kD(\varphi, \xi), \quad \text{for all } \varphi, \xi \in E.
\]

In particular, with \( \varphi = H[x], \xi = H[y] \) (where \( x, y \in E \)), this relation becomes

\[
(b08) \quad d(Tx, Ty) \le kD(H[x], H[y]) = kd(x, y), \quad \text{for all } x, y \in E;
\]
or, in other words: the associated selfmap \( T : E \to E \) is \((d; k)\)-contractive.

II) For the arbitrary fixed \( \varphi_0 = H[x_0] \in \mathcal{K} \), we say that the sequence \( \varphi_n \) is \((\varphi_0, T)\)-iterative, provided \( (T\varphi_n = \varphi_{n+1}(c), \forall n \ge 0) \). Note that, the family of such sequences is nonempty. In fact, for the starting \( x_0 \in E \), the \((x_0, T)\)-iterative sequence \( (x_n; n \ge 0) \) in \( E \) is well defined, according to the formula

\[
Tx_n = x_{n+1}; \quad n \ge 0.
\]

or, in other words: the sequence \( (\varphi_n := H[x_n]; n \ge 0) \) in \( \mathcal{K} \) is \((\varphi_0, T)\)-iterative. Conversely, if the sequence \( (\varphi_n := H[x_n]; n \ge 0) \) in \( \mathcal{K} \) is \((\varphi_0, T)\)-iterative, then (by the same formula), the sequence \( (x_n; n \ge 0) \) of \( E \) is \((x_0, T)\)-iterative; hence, for each \( \varphi_0 \in \mathcal{K} \), the family of all \((\varphi_0, T)\)-iterative sequences in \( \mathcal{K} \) is a singleton.

Putting these together, it follows, from the Banach fixed point principle we just exposed, the following coincidence point result involving our data (referred to as: Constant BLR theorem):

**Theorem 2.2.** Suppose that \( T : E \to E \) is \((d; k)\)-contractive, for some \( k \in [0, 1] \). Then,

i) \( T \) has (in \( \mathcal{K} \)) a unique PPF dependent fixed point \( \varphi^* \) in \( \mathcal{K} \).

ii) for the arbitrary fixed \( \varphi_0 \in \mathcal{K} \), the \((\varphi_0, T)\)-iterative sequence \( (\varphi_n; n \ge 0) \) in \( \mathcal{K} \), \( D\)-converges to \( \varphi^* \).

Note that – unlike the situation encountered at BLR theorem – the iterative sequences above are constructible in a precise way [by means of the associated selfmap \( T \)]; so, this result is an effective one.

**Part 2.** The second option above starts from the fact that, the construction in \( \mathcal{R}_c \) of iterative sequences given by BLR theorem requires the structural condition

\[
(b09) \quad \mathcal{R}_c \text{ is algebraically closed: } \varphi, \xi \in \mathcal{R}_c \implies \varphi - \xi \in \mathcal{R}_c
\]

(also referred to as: \( \mathcal{R}_c \) is \( a\)-closed); this assertion seems to have been formulated, for the first time, in Dhage [8, Observation I]. On the other hand, the existence property above is retainable, at the level of \( \mathcal{R}_c \), when

\[
(b10) \quad \mathcal{R}_c \text{ is topologically closed: } \mathcal{R}_c \text{ is a } D\text{-closed part of } E_0;
\]
cf. Dhage [8, Observation II]. Summing up, the following "existential" version of Theorem 2.1 (referred to as: Existential BLR theorem) enters into our discussion:

**Theorem 2.3.** Suppose that $T$ is $k$-contractive, for some $k \in [0, 1]$. In addition, let us assume that $R_c$ is algebraically and topologically closed. Then, $T$ has a unique PPF dependent fixed point in $R_c$.

Note that, this result is not present in the 1977 paper by Bernfeld et al [3]. The above formulation is a quite recent "by-product" of BLR theorem, under the lines imposed by the above remarks; cf. Kutbi and Sintunavarat [15]. Concerning the structural requirements above, we stress that, from a methodological perspective, the algebraically closed condition is a very strong one. Before explaining our assertion, let us give a useful characterization of this concept. [Since the verification is almost immediate, we omit the details].

**Lemma 2.2.** The following conditions are equivalent:

- $(b11)$ $R_c$ is algebraically closed
- $(b12)$ $R_c$ is additive: $\varphi, \psi \in R_c \Rightarrow \varphi + \psi \in R_c$
- $(b13)$ $R_c$ is a linear subspace of $E_0$.

Having these precise, we are in position to motivate our previous affirmation.

**Proposition 2.3.** Suppose that $R_c$ is algebraically closed; or, equivalently, additive. Then, necessarily, $R_c = K$; hence $R_c$ is topologically closed as well.

**Proof.** Suppose that $R_c \setminus K \neq \emptyset$, and take some function $\varphi$ in this set difference; hence, in particular, $\varphi(r) \neq \varphi(c)$, for at least one $r \in I$. (4) The function $\xi := H[\varphi(c)]$ belongs to the constant class $K$; hence, to the Razumikhin class $R_c$ as well. As $R_c$ is algebraically closed, the difference function $\delta := \varphi - \xi$ is an element of $R_c$; so that $||\delta||_0 = ||\delta(c)|| = 0$;

or, equivalently,

$\varphi(t) = \varphi(c)$, \hspace{1em} $\forall t \in I$;

in contradiction with the initial choice of $\varphi$. Hence, $R_c = K$, as claimed. The last affirmation is immediate, by the topological properties of $K$ (see above).

Summing up, the following conclusions are to be noted:

**Conc-1**) If the Razumikhin class $R_c$ is algebraically closed, we must have $R_c = K$; so that, Existential BLR theorem over $R_c$ is identical – in a trivial way – with Constant BLR theorem over $K$; which – as already shown – reduces to the Banach fixed point principle, without imposing any regularity condition upon $R_c$. This means that, the
algebraic (and/or topological) closeness condition upon $R_c$ has a null generalizing
effect, relative to constant BLR theorem.

**Conc-2** If the Razumikhin class $R_c$ is algebraically closed, we have [in view of
$R_c = K$], that any PPF dependent fixed point result on $R_c$ is identical – in a trivial way –
with the corresponding PPF dependent fixed point statement over $K$; which [by the
associated selfmap construction] is reducible to (standard) fixed point theorems over
the (complete) metric space $(E, d)$, without imposing any regularity condition upon
$R_c$. So, as before, the algebraic (and/or topological) closeness condition upon
$R_c$ has a null generalizing effect relative to the (variant of considered result over) constant
class $K$.

In particular, this latter conclusion tells us that all recent PPF dependent fixed point
results – based on such conditions upon $R_c$ – obtained in Cirić et al [5], Dhage [7, 8],
Harjani et al [9], Hussain et al [10], Kaewcharoen [12], Kutbi and Sintunavarat [15],
are in fact reducible to PPF dependent fixed point results over the constant class $K$;
and these, in turn, are deductible from corresponding fixed point theorems over the
supporting metrical structure $(E, d)$, without imposing any regularity condition to
$R_c$.

A verification of this assertion for the PPF dependent fixed point result in Agarwal et
al [1] is performed in the rest of our exposition. The remaining cases are to be treated
in a similar way; we do not give details.

### 3. SVV TYPE CONTRACTIONS

Let $X$ be a nonempty set. Take a metric $d : X \times X \to \mathbb{R}_+$ over it; as well as a
selfmap $T \in \mathcal{F}(X)$. The basic directions under which the question of determining the
fixed points of $T$ is to be solved were already sketched. As precise, a classical result
in this direction is the 1922 one due to Banach [2]. In the following, a conditional
version of the quoted statement is given, under the lines proposed in Samet et al [23].
Let the mapping $\alpha : X \times X \to \mathbb{R}_+$ be fixed in the sequel.

**I)** We say that $T$ is $\alpha$-admissible, if

\[(c01) \quad (\forall x, y \in X): \alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1.\]

**II)** Given $k \geq 0$, we say that $T$ is (SVV type) $(\alpha, k)$-contractive, provided

\[(c02) \quad \alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X.\]

**III)** Further, let us say that $\alpha$ is $X$-closed, provided

\[(c03) \quad \text{whenever the sequence } (x_n; n \geq 0) \text{ in } X \text{ and the element } x \in X \text{ fulfill }\]

\[\alpha(x_n, Tx_n) \geq 1, \forall n, \text{ and } x_n \xrightarrow{d} x, \text{ then } \alpha(x, Tx) \geq 1.\]

**IV)** Finally, let us say that $T$ is $X$-starting, if

\[(c04) \quad \text{there exists } x_0 \in X, \text{ such that } \alpha(x_0, Tx_0) \geq 1.\]

The following conditional fixed point result (referred to as: SVV theorem) involving
these data is then available:
Theorem 3.1. Suppose that \( T \) is (SVV type) \((\alpha, k)\)-contractive, for some \( k \in [0, 1] \). In addition, suppose that \( (X, d) \) is complete, \( T \) is \( \alpha \)-admissible, \( \alpha \) is \( X \)-closed, and \( T \) is \( X \)-starting. Then,

i) For the arbitrary fixed \( x_0 \in X \) with \( \alpha(x_0, Tx_0) \geq 1 \), the sequence \( (x_n; n \geq 0) \) in \( X \) defined as \( (x_{n+1} = Tx_n; n \geq 0) \) converges to a fixed point \( x^* \in X \) of \( T \), with \( \alpha(x^*, Tx^*) \geq 1 \).

ii) \( T \) has exactly one fixed point \( x^* \in X \), such that \( \alpha(x^*, Tx^*) \geq 1 \).

Proof. There are two assertions to be clarified.

Step 1. Let us firstly verify that \( T \) cannot have more than one fixed point \( x^* \in X \), such that \( \alpha(x^*, Tx^*) \geq 1 \). In fact, assume that, \( T \) would have another fixed point \( y^* \in X \) such that \( \alpha(y^*, Ty^*) \geq 1 \). From the (SVV type) contractive condition, we have

\[
\alpha(x^*, Tx^*)\alpha(y^*, Ty^*)d(Tx^*, Ty^*) \leq kd(x^*, y^*).
\]

This, along with the imposed properties, yields \( d(Tx^*, Ty^*) \leq kd(x^*, y^*) \); or, equivalently (as \( x^*, y^* \) are fixed points) \( d(x^*, y^*) \leq kd(x^*, y^*) \); wherefrom (as \( 0 \leq k < 1 \)), \( d(x^*, y^*) = 0 \); hence, \( x^* = y^* \).

Step 2. Let us now establish the existence part. Fix in the following some \( x_0 \in X \) with \( \alpha(x_0, Tx_0) \geq 1 \); and let \( (x_n := T^n x_0; n \geq 0) \) stand for the iterative sequence generated by it. From this very choice,

\[
\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1;
\]

whence, by the admissible property of \( T \),

\[
\alpha(x_1, Tx_1) = \alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1;
\]

and so on. By a finite induction procedure, one gets an evaluation like

\[
\alpha(x_n, Tx_n) = \alpha(x_n, x_{n+1}) \geq 1, \quad \forall n.
\]

(1)

This tells us that the (SVV type) contractive condition applies to \( (x_n, x_{n+1}) \), for all \( n \geq 0 \). An effective application of it gives

\[
\alpha(x_n, x_{n+1})\alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}), \quad \forall n;
\]

wherefrom

\[
d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}), \quad \forall n.
\]

This, again by a finite induction procedure, gives

\[
d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \quad \forall n;
\]

so that, as the series \( \sum_n k^n \) converges, \( (x_n; n \geq 0) \) is \( d \)-Cauchy. As \( (X, d) \) is complete, \( x_n \xrightarrow{d} x^* \) for some \( x^* \in X \); moreover, combining with (1) (and the closed property of
Denote, for simplicity, \( K = \phi, \xi \). As before, let 

\( \alpha(x, T x) a(x^*, T x^*) d(T x, T x^*) \leq k d(x, x^*), \forall n; \)

wherefrom

\[
d(x_{n+1}, T x^*) = d(T x, T x^*) \leq k d(x, x^*), \forall n. \tag{2}
\]

The sequence \((y_n := x_{n+1} ; n \geq 0)\) is a subsequence of \((x_n; n \geq 0)\); so that, \( y_n \stackrel{d}{\rightarrow} x^* \) as \( n \to \infty \). Passing to limit in (2), gives \( d(x^*, T x^*) = 0 \); wherefrom (as \( d \) is sufficient), \( x^* = T x^* \). The proof is thereby complete.

Note that, the obtained result is not very general in the area; but, it will suffice for our purposes. Various extensions of it may be found in Samet et al [23].

4. AKS THEOREM

Under these preliminaries, we may now pass to the announced result concerning PPF dependent fixed points.

Let \((E, \| \|)\) be a Banach space; and let \( d \) be the induced by norm metric on \( E \) \( d(x, y) = \| x - y \|, x, y \in E \); hence, \((E, d)\) is complete. Further, let \( I = [a, b] \) be a closed real interval, and \( E_0 := C(I, E) \) stand for the class of all continuous functions \( \varphi : I \to E \), endowed with the supremum norm \( (\| \varphi \|_0 = \sup\{\| \varphi(t) \|; t \in I\}, \varphi \in E_0) \).

As before, let \( D \) stand for the induced by norm metric on \( E_0 \) \( (d(\varphi, \xi) = \| \varphi - \xi \|_0, \varphi, \xi \in E_0) \); clearly, \((E_0, D)\) is complete.

Let \( c \in I \) be fixed in the sequel. The Razumikhin class of functions in \( E_0 \) attached to \( c \), is defined as

\[ \mathcal{R}_c = \{ \varphi \in E_0; \| \varphi \|_0 = \| \varphi(c) \| \}. \]

Note that \( \mathcal{R}_c \) is nonvoid; because any constant function belongs to it. Precisely, for each \( u \in E \), let \( H[u] \) stand for the constant function of \( E_0 \), defined as: \( H[u](t) = u, t \in I \). Note that, by this definition, \( \| H[u] \|_0 = \| u \|_0 \), \( H[u](c) = u \); whence \( H[u] \in \mathcal{R}_c \).

Denote, for simplicity, \( \mathcal{K} = \{ H[u]; u \in E \} \); this will be referred to as the constant class of \( E_0 \).

Now, let \( \mathcal{T} : E_0 \to E \) be a mapping. We say that \( \varphi \in E_0 \) is a PPF dependent fixed point of \( \mathcal{T} \), when \( \mathcal{T} \varphi = \varphi(c) \). As already noted, the natural way to determine such points is offered by the constant class \( \mathcal{K} \) of \( E_0 \). Then, the points in question appear as fixed points of the selfmap \( T : E \to E \), introduced as

\[ Tu = \mathcal{T}(H[u]), \quad u \in E. \]

Precisely (see above)

**fp-1** If \( z \in E \) is a fixed point of \( T \), then \( \zeta := H[z] \in \mathcal{K} \) is a PPF dependent fixed point of \( \mathcal{T} \).
Assume that the (nonself mapping) $T$ is a fixed point of $T$. The fixed point result to be applied is SVV theorem; so, we have to clarify whether the required conditions in terms of $T$ are obtainable via (nonself type) hypotheses in terms of $\mathcal{T}$. For an easy reference, we list the latter conditions. Let in the following $\alpha : E \times E \to R_+$ be a mapping.

I) We say that $\mathcal{T}$ is $\alpha$-admissible, if

$$(d01) \quad (\forall \varphi, \xi \in E_0): \alpha(\varphi(c), \xi(c)) \geq 1 \text{ implies } \alpha(T\varphi, T\xi) \geq 1.$$  

In particular, take $\varphi = H[x], \xi = H[y]$, where $x, y \in E$. Then, this condition yields

$$(\forall x, y \in E): \alpha(x, y) \geq 1 \text{ implies } \alpha(Tx, Ty) \geq 1;$$

or, in other words: the associated selfmap $T : E \to E$ is $\alpha$-admissible.

II) Given $k \geq 0$, we say that $\mathcal{T}$ is $(\alpha, k)$-contractive, provided

$$(d02) \quad \alpha(\varphi(c), T\varphi)\alpha(\xi(c), T\xi)d(T\varphi, T\xi) \leq kD(\varphi, \xi), \forall \varphi, \xi \in E_0.$$  

As before, take $\varphi = H[x], \xi = H[y]$, where $x, y \in E$. Then, by this condition,

$$\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in E;$$

i.e.: $T$ is (SVV type) $(\alpha, k)$-contractive.

III) Further, let us say that $\alpha$ is $E_0$-closed, provided

$$(d03) \quad \text{whenever the sequence } (\varphi_n; n \geq 0) \text{ in } E_0 \text{ and the element } \varphi \in E_0 \text{ fulfill}$$

$$[\alpha(\varphi_n(c), T\varphi_n) \geq 1, \forall n], \text{ and } \varphi_n \xrightarrow{D} \varphi, \text{ then } \alpha(\varphi(c), T\varphi) \geq 1.$$  

In particular, taking $(\varphi_n = H[x_n]; n \geq 0)$ and $\varphi = H[x]$, for some sequence $(x_n; n \geq 0)$ in $E$ and some point $x \in E$, this requirement becomes:

$$\text{whenever the sequence } (x_n; n \geq 0) \text{ in } E \text{ and the element } x \in E \text{ fulfill} [\alpha(x_n, Tx_n) \geq 1, \forall n], \text{ and } x_n \xrightarrow{d} x, \text{ then } \alpha(x, Tx) \geq 1;$$

or, in other words: $\alpha$ is $E$-closed.

IV) Finally, let us say that $\mathcal{T}$ is $E_0$-starting, if

$$(d04) \quad \text{there exists } \varphi_0 \in E_0, \text{ such that } \alpha(\varphi_0(c), T\varphi_0) \geq 1.$$  

Likewise, let us say that $\mathcal{T}$ is $E$-starting, if

$$(d05) \quad \text{there exists } \varphi_0 \in E, \text{ such that } \alpha(\varphi_0(c), T\varphi_0) \geq 1.$$  

Clearly, if $\mathcal{T}$ is $E$-starting, then $\mathcal{T}$ is $E_0$-starting as well. The reciprocal inclusion holds too, under an admissible property upon $T$.

**Proposition 4.1.** Assume that the (nonself mapping) $\mathcal{T}$ is $\alpha$-admissible and $E_0$-starting. Then, $\mathcal{T}$ is $E$-starting.
Proof. As \( \mathcal{T} \) is \( E_0 \)-starting, there exists \( \varphi_0 \in E_0 \) such that \( \alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1 \). On the other hand, for the point \( \mathcal{T}\varphi_0 \in E \), we may consider the function \( \xi_0 = H[\mathcal{T}\varphi_0] \) in the constant class \( \mathcal{K} \). This, by definition, means

\[ \xi_0(t) = \mathcal{T}\varphi_0, \ \forall t \in I; \text{ whence, } \xi_0(c) = \mathcal{T}\varphi_0. \]

The starting property of \( \mathcal{T} \) becomes:

\[ \alpha(\varphi_0(c), \xi_0(c)) \geq 1. \]

As \( \mathcal{T} \) is \( \alpha \)-admissible, this yields \( \alpha(\mathcal{T}\varphi_0, \mathcal{T}\xi_0) \geq 1 \). Combining with a preceding relation, we thus have

\[ \alpha(\xi_0(c), \mathcal{T}\xi_0) \geq 1; \]

which tells us that \( \mathcal{T} \) is \( \mathcal{K} \)-starting.

Now, as \( \mathcal{T} \) is \( \mathcal{K} \)-starting, there exists \( \varphi_0 = H[x_0] \) (where \( x_0 \in E \)) in \( \mathcal{K} \), such that \( \alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1 \). This, by definition, means \( \alpha(x_0, T x_0) \geq 1 \); and tells us that the associated selfmap \( T \) is \( E_0 \)-starting.

Putting these together, it results, via SVV theorem (and our preliminary facts), the following PPF dependent fixed point result (referred to as: AKS theorem), involving these data:

**Theorem 4.1.** Suppose that \( \mathcal{T} \) is \((\alpha, k)\)-contractive, for some \( k \in [0, 1] \). In addition, suppose that \( \mathcal{T} \) is \( \alpha \)-admissible, \( \alpha \) is \( E_0 \)-closed, and \( \mathcal{T} \) is \( E_0 \)-starting. Then

i) \( \mathcal{T} \) is \( \mathcal{K} \)-starting (see above)

ii) For the arbitrary fixed \( \varphi_0 \in \mathcal{K} \) with \( \alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1 \), the sequence \( (\varphi_n; n \geq 0) \) in \( \mathcal{K} \) defined as \( \varphi_{n+1} = \mathcal{T}\varphi_n; n \geq 0 \) converges to a PPF dependent fixed point \( \varphi^* \in \mathcal{K} \) of \( T \), with \( \alpha(\varphi^*(c), \mathcal{T}\varphi^*) \geq 1 \)

iii) \( \mathcal{T} \) has a exactly one PPF dependent fixed point \( \varphi^* \) in the constant class \( \mathcal{K} \), such that \( \alpha(\varphi^*(c), \mathcal{T}\varphi^*) \geq 1 \).

In particular, when the Razumikhin class \( \mathcal{R}_c \) is algebraically (as well as topologically) closed, and the starting condition is taken as

\[ (d06) \quad \mathcal{T} \text{ is } \mathcal{R}_c \text{-starting}; \text{ there exists } \varphi_0 \in \mathcal{R}_c, \text{ with } \alpha(\varphi_0(c), \mathcal{T}\varphi_0) \geq 1, \]

the obtained result is just the PPF dependent fixed point result in Agarwal et al [1]. However, as noted earlier, all these conditions have no generalizing effect upon our data; because, in view of \( \mathcal{R}_c = \mathcal{K} \), the obtained statement is again AKS theorem; which is deductible without any regularity condition upon \( \mathcal{R}_c \).

Finally, note that, by the same techniques, it follows that all PPF dependent fixed point results in Cirić et al [5], Dhage [7, 8], Harjani et al [9], Hussain et al [10], Kaewcharoen [12], Kutbi and Sintunavarat [15], are in fact reducible to (corresponding) fixed point statements over standard metric structures. Some other aspects will be delineated elsewhere.
5. PROOF OF BLR THEOREM

Let us now return to the BLR theorem. For completeness reasons, we shall provide a proof of this result which, in part, differs from the original one.

Let \((E, \|\|)\) be a Banach space; and let \(d\) be the induced by norm metric on \(E\) \((d(x, y) = \|x - y\|, \ x, y \in E)\); hence, \((E, d)\) is complete. Further, let \(I = [a, b]\) be a closed real interval, and \(E_0 := C(I, E)\) stand for the class of all continuous functions \(\varphi : I \rightarrow E\), endowed with the supremum norm \((\|\varphi\|_0 = \sup\{|\varphi(t)|; t \in I\}, \ \varphi \in E_0)\).

As before, let \(D\) stand for the induced metric \((D(\varphi, \xi) = \|\varphi - \xi\|_0, \ \varphi, \xi \in E_0)\); clearly, \((E_0, D)\) is complete.

Let \(c \in I\) be fixed in the sequel. The Razumikhin class of functions in \(E_0\) attached to \(c\), is defined as
\[
\mathcal{R}_c = \{\varphi \in E_0; \|\varphi\|_0 = \|\varphi(c)\|\}.
\]
Note that \(\mathcal{R}_c\) is nonvoid; because any constant function belongs to it.

Finally, let \(\mathcal{T} : E_0 \rightarrow E\) be a (nonself) mapping. We say that \(\varphi \in E_0\) is a PPF dependent fixed point of \(\mathcal{T}\), when \(\mathcal{T}\varphi = \varphi(c)\). The class of all these will be denoted as PPF-Fix(\(\mathcal{T}; E_0\)).

To establish the existence and uniqueness result in question, the following concepts and constructions are necessary.

I) Given \(k \geq 0\), call \(\mathcal{T}\), \(k\)-contractive, provided
\[
d(\mathcal{T}\varphi, \mathcal{T}\xi) \leq kD(\varphi, \xi), \ \text{for all} \ \varphi, \xi \in E_0.
\]
Note that, as a consequence of this, \(\mathcal{T}\) is \((D, d)\) continuous: \(\varphi_n \xrightarrow{D} \varphi\) implies \(\mathcal{T}\varphi_n \xrightarrow{d} \mathcal{T}\varphi\).

II) Let us introduce a relation \((\nabla)\) over \(E_0\), according to:
\[
\varphi \nabla \xi \text{ iff } \mathcal{T}\varphi = \xi(c) \text{ and } \varphi - \xi \in \mathcal{R}_c.
\]

III) Finally, given the starting element \(\varphi_0 \in E_0\), let us say that the sequence \((\varphi_n; n \geq 0)\) in \(E_0\) is \((\varphi_0, \nabla)\)-iterative, in case \(\varphi_n \nabla \varphi_{n+1}, \ \forall n\).

Having these precise, we may now proceed to the announced Proof. (BLR theorem) There are several steps to be considered.

Part 1. Take a starting point \(\varphi_0 \in E_0\), and let \((\varphi_n; n \geq 0)\) in \(E_0\) be some \((\varphi_0, \nabla)\)-iterative sequence. By definition, \(\varphi_0 \nabla \varphi_1\); which means
\[
\mathcal{T}\varphi_0 = \varphi_1(c), \ d(\varphi_0(c), \varphi_1(c)) = D(\varphi_0, \varphi_1).
\]
Further, \(\varphi_1 \nabla \varphi_2\); which means
\[
\mathcal{T}\varphi_1 = \varphi_2(c), \ d(\varphi_1(c), \varphi_2(c)) = D(\varphi_1, \varphi_2).
\]
Note that, as a combination of these, we have (by the contractive property of \(\mathcal{T}\))
\[
D(\varphi_1, \varphi_2) = d(\varphi_1(c), \varphi_2(c)) = d(\mathcal{T}\varphi_0, \mathcal{T}\varphi_1) \leq kD(\varphi_0, \varphi_1).
\]
This procedure may continue indefinitely; and gives the iterative relations

\[ D(\varphi_{n+1}, \varphi_{n+2}) \leq k D(\varphi_n, \varphi_{n+1}), \quad \forall n \geq 0. \quad (1) \]

By a finite induction procedure, one gets

\[ D(\varphi_n, \varphi_{n+1}) \leq k^n D(\varphi_0, \varphi_1), \quad \forall n \geq 0; \quad (2) \]

and since the series \( \sum_n k^n \) converges, the sequence \( (\varphi_n; n \geq 0) \) is \( D \)-convergent. As \((E_0, D)\) is complete, \( \varphi_n \xrightarrow[D]{} \varphi^* \) as \( n \to \infty \), for some \( \varphi^* \in E_0 \). Passing to limit as \( n \to \infty \) in the iterative relation that defines the sequence \( (\varphi_n; n \geq 0) \), one gets (as \( T \) is \( (D, d) \)-continuous) \( T \varphi^* = \varphi^*(c) \); i.e.; \( \varphi^* \) is an element of \( \text{PPF-Fix}(T; E_0) \).

**Part 2.** Take a couple of starting points \( \varphi_0, \xi_0 \in E_0 \), and let \( (\varphi_n; n \geq 0), (\xi_n; n \geq 0) \) be their corresponding \( (\varphi_0, \nabla) \)-iterative sequence and \( (\xi_0, \nabla) \)-iterative sequence, respectively. By the preceding part, \( (\varphi_n; n \geq 0) \) fulfills the iterative relations (1) and (2). Likewise, \( (\xi_n; n \geq 0) \) fulfills the iterative relations

\[ D(\xi_{n+1}, \xi_{n+2}) \leq k D(\xi_n, \xi_{n+1}), \quad \forall n \geq 0; \quad (3) \]

wherefrom (by a finite induction procedure)

\[ D(\xi_n, \xi_{n+1}) \leq k^n D(\xi_0, \xi_1), \quad \forall n \geq 0. \quad (4) \]

From the triangle inequality, one gets, for all \( n \geq 1, \)

\[ D(\varphi_n, \xi_n) \leq D(\varphi_{n-1}, \varphi_n) + D(\xi_{n-1}, \xi_n) + D(\varphi_{n-1}, \xi_{n-1}) \leq k^{n-1}[D(\varphi_0, \varphi_1) + D(\xi_0, \xi_1)] + D(\varphi_{n-1}, \xi_{n-1}). \]

By repeating the procedure, it follows that, after \( (n - 1) \) steps, the first part of conclusion **ii** follows. Moreover, if \( \varphi_0 = \xi_0 \), then \( T \varphi_0 = T \xi_0 \) (that is: \( \varphi_1(c) = \xi_1(c) \)); wherefrom (by the choice of our iterative sequences)

\[ D(\varphi_0, \varphi_1) = d(\varphi_0(c), \varphi_1(c)), \]
\[ D(\xi_0, \xi_1) = d(\xi_0(c), \xi_1(c)) = d(\varphi_0(c), \varphi_1(c)); \]

and, this yields the second part of conclusion **ii**.

**Part 3.** Let \( (\varphi_n; n \geq 0), (\xi_n; n \geq 0) \) be taken as in the premise above. By the very definition of these points and the working condition, we have, for all \( n \geq 1, \)

\[ D(\varphi_n, \xi_n) = d(\varphi_n(c), \xi_n(c)) = d(T \varphi_{n-1}, T \xi_{n-1}) \leq k D(\varphi_{n-1}, \xi_{n-1}). \]

This, by a finite induction procedure, gives

\[ D(\varphi_n, \xi_n) \leq k^n D(\varphi_0, \xi_0), \quad \forall n. \]

Passing to limit as \( n \to \infty \), and noting that both \( (\varphi_n; n \geq 0) \) and \( (\xi_n; n \geq 0) \) are \( D \)-convergent (see above), conclusion **iii** follows.
Part 4. Let $\varphi^*, \xi^*$ be two elements in $\text{PPF-Fix}(\mathcal{I}; E_0)$, taken as in the stated premise. By the working condition and the contractive property of $T$,

$$D(\varphi^*, \xi^*) = d(\varphi^*(c), \xi^*(c)) = d(\mathcal{I}\varphi^*, \mathcal{I}\xi^*) \leq kD(\varphi^*, \xi^*);$$

and, from this (as $D$ is a metric), $\varphi^* = \xi^*$. The proof is thereby complete.

As above said, this result – as well as its extensions due to Pathak [18] and Som [24] – is just a conditional one; because the iterative sequences appearing there are not effective. The correction of this drawback by taking $\Re_c$ as algebraically closed is, ultimately, without effect; for, as the preceding developments show, it brings the PPF dependent fixed point problem in question at the level of constant class, $\mathcal{K}$. From a theoretical viewpoint, this reduced problem may be of some avail in solving the (metrical) question we deal with. However, from a practical perspective, the resulting constant solutions for the nonlinear functional differential/integral equations based on these techniques – such as, the ones in Kutbi and Sintunavarat [15] – are not very promising. Further aspects will be discussed elsewhere.

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MATHEMATICAL MODELING OF THE PROBLEM ABOUT DYNAMIC STABILITY OF THE SHIELD IN A SUPERSONIC GAS FLOW
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Abstract
The stability of solutions of initial-boundary value problem for a coupled system of differential equations describing the dynamics of the elastic wall (shield) of a tank filled with liquid, the interaction with the walls of the fluid on the one hand, and with a supersonic flow of gas on the other. Definition of stability of the elastic body corresponds to the concept of dynamical systems by Lyapounov. The performed in this article stability investigations are based on the construction of Lyapunov type mixes functional. The paper also provides examples of numerical and analytical solutions of the problem, obtained by the Bubnov-Galerkin method.

Keywords: aerohydroelasticity, supersonic flow, differential equations, partial derivatives, Bubnov-Galerkin method, dynamic stability, Lyapunov functional.


1. INTRODUCTION
2. FORMULATION OF THE PROBLEM

Consider a planar problem of a dynamics of the elastic wall of the filled by a fluid reservoir $G^- = \{(x, y) \in \mathbb{R}^2 : 0 < x < l, -h < y < 0\}$, which is filled with liquid. Elastic wall is occupying the position of $y = 0$, $0 < x < l$ and modeled as an elastic plate. Other wall ($x = 0$, $x = l$ and $y = -h$) are considered undeformable (fig. 1). In the domain of $G^+ = \{(x, y) \in \mathbb{R}^2 : x \in (-\infty, +\infty), y \in (0, +\infty)\}$ supersonic flow of a gas flows along the axis $Ox$ at $V_0 > a_0$, where $a_0$ – velocity of sound. Assume that the Mach number is $M_0 = \frac{V_0}{a_0} > \sqrt{2}$.

Introduce the notation: $w(x, t)$ is the deformation function of a plate; $\varphi^-(x, y, t)$ – the velocity potential of a liquid in the domain $G^-$, $\varphi^+(x, y, t)$ – the velocity potential of a gas in the domain $G^+$.

The mathematical formulation of the problem:

\begin{align*}
\varphi^+_{tt} + 2V_0\varphi^+_{xt} + V_0^2\varphi^+_{xx} &= a_0^2(\varphi^+_{xx} + \varphi^+_{yy}), \quad (x, y) \in G^+; \\
\varphi^+_y(x, 0, t) &= \begin{cases} w_t + V_0 w_x, & x \in (0, l), \\ 0, & x \in (l, +\infty) \end{cases} \quad t \geq 0; \tag{1}
\end{align*}

\begin{align*}
\varphi^+_y(x, 0, t) &= \begin{cases} w_t + V_0 w_x, & x \in (0, l), \\ 0, & x \in (l, +\infty) \end{cases} \quad t \geq 0; \tag{2}
\end{align*}
Here the indices $x, y, t$ denote the derivatives on $x, y$ and $t$; $D$ and $m$ are the flexural stiffness and linear mass of the plate; $V_0, \rho^+, p^+$ – the gas velocity, the density and the pressure in the incoming homogeneous flow in the domain $G^+$; $\rho^-, p^-$ – the density and the fluid pressure in the domain $G^-$ at rest.

The equation (1) describes a flow of a gas in the domain $G^+$ in the model of an ideal compressible medium; (2), (6), (7) – the unflowing conditions; (3) – conditions of an absence of perturbations in front of the plate in the domain $G^+$; (4) – conditions of the absence of perturbations in initial time in the domain $G^+$; the equation (5) describes the fluid dynamics in the domain $G^-$ in the model of an ideal incompressible medium; the equation (8) describes the dynamics of the elastic wall of the fluid reservoir with the impact on it of a supersonic gas flow from above and from below the liquid; conditions (9) correspond to flexible coupling of the elastic plate of the fluid reservoir;
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(10) – the confirmed with (9) initial conditions; note that for other types of fixing the ends of the shield problem is solved similarly.

Equations and conditions (1) - (10) form of the initial-boundary value problem to determine three unknown functions \( \varphi^+(x,y,t), \varphi^-(x,y,t), w(x,t) \).

For the solving of this problem in the upper domain \( G^+ \) applies operational method. Let us turn the equation (1) and condition (2) in the dimensionless variables \( \varphi^+(x^*, y^*, t^*), w^*(x^*, t^*), x^*, y^*, t^* \):

\[
\varphi^*(x^*, y^*, t^*) = \frac{\varphi^+(x,y,t)}{V_0 t}, \quad w^*(x^*, t^*) = \frac{w(x,t)}{l}, \quad x^* = \frac{x}{l},
\]

\[
y^* = \frac{y}{l}, \quad t^* = \frac{V_0 t}{l}.
\]

Then the equation (1) and the condition (2) take the form:

\[
\varphi^*_{t^*} + 2\varphi^*_{x^*t^*} + \varphi^*_{x^*x^*} = M_0^{-2} (\varphi^*_{x^*x^*} + \varphi^*_{y^*y^*}),
\]

(12)

\[
\varphi^*_{y^*}(x^*, 0, t^*) = \begin{cases} w^*_0 + w^*_i, & x^* \in (0, 1), \quad t^* \geq 0, \\ 0, & x^* \in (1, +\infty) \quad t^* \geq 0. \end{cases}
\]

(13)

Applying the Laplace transform to the equation (12) by variables \( x^* \overrightarrow{A} t^* \), with conditions (3), (4), for double image at Laplace \( \tilde{\varphi}^*(p, y^*, q) \) obtain the ordinary differential equation

\[
[(p + q)^2 - M_0^{-2} p^2] \tilde{\varphi}^*(p, y^*, q) = M_0^{-2} \tilde{\varphi}^*_{y^*y^*}(p, y^*, q).
\]

(14)

Applying the Laplace transform on the variables \( x^* \overrightarrow{A} t^* \) to the unflowing boundary condition (13), one has

\[
\tilde{\varphi}^*_0(p, 0, q) = (q + p) \tilde{w}(p, q).
\]

(15)

General solution of the equation (14), satisfies the condition an attenuation at \( y^* \to \infty \) and the unflowing boundary condition (15), has the form

\[
\tilde{\varphi}^*(p, y^*, q) = \frac{(p + q) \tilde{w}(p, q)}{M_0 \sqrt{(p + q)^2 - M_0^{-2} p^2}} e^{-M_0 \sqrt{(p+q)^2 - M_0^{-2} p^2} y^*}.
\]

(16)

From expression (16) at \( y^* = 0 \) find the image of the term of right side of the equation (8)

\[
-p(\varphi^*_i(x, 0, t) + V_0 \varphi^*_e(x, 0, t)) = -\varphi^*_{x^*} M_0^2 (\varphi^*_r + \varphi^*_s),
\]

(17)

namely

\[
-\varphi^*_{x^*} M_0^2 (\varphi^*_r + \varphi^*_s)_{y^*=0} \leq
\]
Further solution to the problem consist in the finding of the original corresponding to the image
\[ \frac{\alpha M_0(p + q)^2 \tilde{w}(p, q)}{\sqrt{((p + q)^2 - M_0^2 p^2)}}, \] (18)

The approximate expression of the original, corresponding to an image (19), obtained on the basis of the quasi-static theory, where the formula for the pressure calculation produced by decomposition on reduced frequency of the exact expression for the pressure of two-dimensional unsteady flow [1,2], has in dimensional variables the form
\[ \frac{\rho^+ V_0}{\sqrt{M_0^2 - 1}} \left( V_0 w_x(x, t) + \frac{M_0^2 - 2}{M_0^2 - 1} w_t(x, t) \right) \] (20)

Where \( M_0 \to \infty \) we get the known expression of ”piston theory” of A.A. Ilyushin.

Then the dynamic equation of the elastic wall of the fluid reservoir (8) takes the form
\[ mw_{tt}(x, t) + Dw_{xxxx}(x, t) = (p^- - \rho^- \varphi^- (x, 0, t)) - p^+ - \]
\[ - \frac{\rho^+ V_0}{\sqrt{M_0^2 - 1}} \left( V_0 w_x(x, t) + \frac{M_0^2 - 2}{M_0^2 - 1} w_t(x, t) \right). \] (21)

The term
\[ -p^+ - \frac{\rho^+ V_0}{\sqrt{M_0^2 - 1}} \left( V_0 w_x(x, t) + \frac{M_0^2 - 2}{M_0^2 - 1} w_t(x, t) \right) \] (22)
in the equation (21) describes the effect on a supersonic gas flow on the plate.

Further a partial differential equation, which describes the dynamics of the elastic wall of the fluid reservoir with an aerohydrodynamic influence on it, containing only \( w(x, t) \) according to the posed problem (5) - (10).

Present the velocity potential \( \varphi^- (x, y, t) \) as
\[ \varphi^- (x, y, t) = \alpha(t) + \sum_{n=1}^{\infty} b_n(t) \cos(\lambda_n x)(e^{t \lambda_n y} + e^{-t \lambda_n y} e^{-2 t h}), \] (23)
where \( \alpha(t), b_n(t) \) – some arbitrary functions, and \( \lambda_n = \frac{n \pi}{l} \).
Equation (5) with conditions (7) and the first condition (6) carried out. The satisfying second condition (6), one has

\[ b_m(t) = \frac{2}{\lambda_m(1 - e^{-2\lambda_m h})} \int_0^l w_j(x,t) \cos(\lambda_m x) dx. \]  

(24)

Substitution (24) into (23), according to (8), gives the equation of the dynamics of the elastic plate

\[ mw_{tt}(x,t) + Dw_{xxxx}(x,t) = \\
= (p^- - p^+) - \rho^+ V_0 \sqrt{M_0^2 - 1} \left( V_0 w_x(x,t) + \frac{M_0^2 - 2}{M_0^2 - 1} w_j(x,t) \right) - \\
- \rho^- \left( \alpha_t(t) + \frac{2}{l} \sum_{n=1}^\infty \frac{\cos(\lambda_n x)(1 + e^{-2\lambda_n h})}{\lambda_n(1 - e^{-2\lambda_n h})} \int_0^l w_j(x,t) \cos(\lambda_n x) dx \right). \]  

(25)

The remaining arbitrary function \( \alpha_t(t) \) is determined satisfying in middle the equation (25) taking into account the incompressibility condition of the medium

\[ \alpha_t(t) = \left( -D \int_0^l w_{xxxx}(x,t) dx + (p^- - p^+) \right) \frac{1}{\rho^+}. \]  

(26)

3. STABILITY INVESTIGATION

Since the system (5) - (9) is linear, it is sufficient to study the stability of the zero solution of the corresponding homogeneous system. The homogeneous equation corresponding to equation (8), is the following:

\[ mw_{tt}(x,t) + Dw_{xxxx}(x,t) = \\
= -\rho^- \phi_t(x,0,t) - \rho^+ V_0 \sqrt{M_0^2 - 1} \left( V_0 w_x(x,t) + \frac{M_0^2 - 2}{M_0^2 - 1} w_j(x,t) \right). \]  

(27)

We are interested in the sufficient conditions for a stability of a zero solution of the boundary value problem (5) - (7), (9), (27) with respect perturbations of the initial conditions (10). Introduce the functional that the method of the construction of which is presented in [3]

\[ J(t) = \int_0^l \left( mw_t^2 + Dw_{xx}^2 + 2m\theta w_t + ak\theta w^2 + 2\rho^- \theta \phi(x,0,t) \right) dx + \]
The equation (5) with conditions (6) - (7), gives

\[ G \int \phi xx \phi x x + \phi yy \phi y y dx dy = l \int w_t \phi (x, 0, t) dx. \]  

(30)

Substituting (27) and (30) in (29), one has

\[ J_t(t) = 2 \int w_t (-Dw_xxx - \rho^- \phi_x x, 0, t) + \rho^- \theta w \phi_x (x, 0, t) - \alpha V_0 w_x - \alpha k w_t + \]

\[ + m \theta w_t^2 + \theta w (-Dw_xxx - \rho^- \phi_x x, 0, t) - \alpha V_0 w_x - \alpha k w_t + \]

\[ + \rho^- \theta w \phi_x (x, 0, t) + \rho^- \theta w \phi_x (x, 0, t) + \rho^- w_t \phi_x (x, 0, t)) dx. \]

(31)

The integration taking into account the conditions (9) gives

\[ \int w_{xxx} w_t dx = \int w_{xx} w_{tx} dx, \]

\[ \int w w_{xxx} dx = \int w_{xx}^2 dx, \int w w_x dx = 0. \]  

(32)

Multiplying the equation (5) on \( \phi(x, y, t) \) and integrating on the domain \( G^- \) one has

\[ G^- \int \phi (\phi_xx + \phi_yy) dx dy = l \int w_t \phi (x, 0, t) dx - G^- \int (\phi_x^2 + \phi_y^2) dx dy = 0. \]

(33)
From (33) it follows that
\[ \int_0^l \varphi(x,0,t)w_1 dx = \iint_G (\varphi_x^2 + \varphi_y^2) dxdy. \] (34)

Taking into account (32) - (34) according to (31)
\[ J(t) = -2 \int_0^l ((\alpha V_0 w_x + (\alpha k - m\theta) w_1^2 + \\
+ D\theta w_{xx}^2 - (\rho - \gamma + \gamma)w_t \varphi(x,0,t))dx - 2\gamma \iint_G (\varphi_x^2 + \varphi_y^2) dxdy, \] (35)

where \( \gamma > 0 \) is some positive parameter, introduced in the transformation process.

Making integrals estimates at the boundary conditions (7), (9), according to the Rayleigh’s inequality [4] one has
\[ \int_0^l w_{xx}^2 dx \geq \lambda_1 \int_0^l w_x^2 dx, \] (36)

where \( \lambda_1 = \frac{\pi^2}{l^2} \) is the smallest eigenvalue of the boundary value problem \( \psi_{xxxx} = -\lambda \psi_{xx}, \quad x \in [0,l] \) with the conditions corresponding to (9);
\[ \iint_G \varphi_x^2 dxdy \geq \eta_1 \iint_G \varphi^2 dxdy, \] (37)

where \( \eta_1 = \frac{\pi^2}{l^2} \) is the smallest eigenvalue of the boundary value problem \( \psi_{xx} = -\eta \psi, \quad x \in (0,l), \) with the conditions corresponding to (7). The considered boundary value problems are self-adjoint and completely defined.

According the Cauchy-Bunyakovsky inequality
\[ \left( \int_0^l \varphi dy \right)^2 \leq \int_0^y 1^2 dy \int_0^{\eta_1} \varphi^2 dy. \]

Consequently
\[ (\varphi(x,0,t) - \varphi(x,y,t))^2 \leq -y \int_0^y \varphi_x^2 dy. \] (38)
The integration (38) on the domain $G^-$ gives

$$
\iint_{G^-} (\varphi(x,0,t) - \varphi(x,y,t))^2 dxdy \leq \iint_{G^-} \left(-y \int_0^{-h} \varphi_y^2 dy \right) dxdy = \frac{h^2}{2} \iint_{G^-} \varphi_y^2 dxdy.
$$

(39)

Taking into account the inequalities (36), (37), (39) according to (35) we get for $J_i$ the estimate

$$
J_i(t) \leq -2 \iint_{G^-} \left( \frac{1}{h} \left( \alpha V_0 w_t w_x + (ak - m\theta)w_x^2 + \frac{D\theta\pi^2}{l^2} w_t^2 - \rho\varphi_x \varphi_x(x,0,t) + \gamma \left( \frac{\pi^2}{l^2} + \frac{2}{h^2} \right) \varphi^2(x,y,t) \right) \right) dxdy.
$$

(40)

Let us consider in (40) quadratic form with respect to $w_t$, $w_x$, $\varphi(x,0,t)$, $\varphi(x,y,t)$, the matrix of which has the form:

$$
\begin{pmatrix}
\frac{ak - m\theta}{h} & \frac{\alpha V_0}{2h} & -\frac{\rho\varphi_x}{2h} & 0 \\
\frac{\alpha V_0}{2h} & \frac{2h}{D\theta\pi^2} & \frac{\rho\varphi_x}{2h} & 0 \\
-\frac{\rho\varphi_x}{2h} & \frac{2\gamma}{h^2} & -\frac{\rho\varphi_x}{2h} & 0 \\
0 & 0 & \frac{\rho\varphi_x}{h^2} & \gamma \left( \frac{\pi^2}{l^2} + \frac{2}{h^2} \right)
\end{pmatrix}.
$$

(41)

On the Sylvester's criterion write out the non-negativity conditions of the quadratic form

$$
\Delta_1 > 0 \Rightarrow ak - m\theta > 0,
$$

(42)

$$
\Delta_2 > 0 \Rightarrow 4D\theta\pi^2(ak - m\theta) - \alpha^2 V_0^2 l^2 > 0,
$$

(43)

$$
\Delta_3 > 0 \Rightarrow 2\gamma(4D\theta\pi^2(ak - m\theta) - \alpha^2 V_0^2 l^2) - (\rho\varphi_x + \gamma)^2 D\theta\pi^2 h > 0,
$$

(44)

$$
\Delta_4 \geq 0 \Rightarrow \frac{\pi^2}{l^2} \left[ 2\gamma(4D\theta\pi^2(ak - m\theta) - \alpha^2 V_0^2 l^2) - (\rho\varphi_x + \gamma)^2 D\theta\pi^2 h \right] - \frac{2}{h^2} (\rho\varphi_x + \gamma)^2 D\theta\pi^2 h \geq 0,
$$

(45)
where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are principal minors of the matrix (41). Thus from (45) follows the fulfillment of the inequalities (42) - (44). Consider in (45) the case of equality and define the parameter $\gamma > 0$:

$$\left(\pi^2 h^2 + 2\ell^2\right) D \theta \gamma^2 - (8D\theta \rho^2 h(ak - m\theta) - 2\alpha^2 V_0^2 \ell^2 h - 2(\pi^2 h^2 + 2\ell^2) D \rho \theta^2) \gamma + 4(\pi^2 h^2 + 2\ell^2) D^2 (\rho^{-1})^2 \theta^3 = 0.$$  \hspace{1cm} (46)

Let the root of the quadratic equation (46) be $\gamma > 0$. Then the following inequalities (47), (48) must be carried out

$$8D\theta \pi^2 h(ak - m\theta) - 2\alpha^2 V_0^2 \ell^2 h - 2(\pi^2 h^2 + 2\ell^2) D \rho \theta^2 > 0,$$

$$D = (8D\theta \pi^2 h(ak - m\theta) - 2\alpha^2 V_0^2 \ell^2 h - 2(\pi^2 h^2 + 2\ell^2) D \rho \theta^2)^2 - 4(\pi^2 h^2 + 2\ell^2) D^2 \rho^2 \theta^3 \geq 0,$$

where $D$ is the discriminant of the equation (46). The parameter $\gamma$ is defined by the following equality

$$\gamma = \frac{8D\theta \pi^2 h(ak - m\theta) - 2\alpha^2 V_0^2 \ell^2 h - 2(\pi^2 h^2 + 2\ell^2) D \rho \theta^2 + \sqrt{D}}{(\pi^2 h^2 + 2\ell^2) 2D \theta}.$$  \hspace{1cm} (49)

Thus, at the conditions (47), (48) realization from (40) we get

$$J(t) \leq 0 \Rightarrow J(t) \leq J(0)$$  \hspace{1cm} (50)

Consider the boundary value problem $\psi_{xxxx} = \eta \psi, \; x \in [0, l]$ with boundary conditions (9). This problem is self-adjoint and completely defined. According to Rayleigh’s inequality [4], one has

$$\int_0^l w_x^2 \, dx \geq \eta_1 \int_0^l w^2 \, dx,$$  \hspace{1cm} (51)

where $\eta_1 = \frac{\pi^4}{l^4}$ is the smallest eigenvalue of the boundary value problem.

Estimate $J(t)$, using the inequality

$$J(t) = \int_0^l \left( m w_t^2 + D(1-\chi)w_{xx}^2 + D\chi w_{xx}^2 + 2m\theta \psi_t w_t + akw^2 + +2\rho^{-1}\theta w_\psi(x,0,t)\right) \, dx + \rho^{-1} \int_G (\psi_x^2 + \psi_y^2) \, dx \, dy \geq$$
\[
\begin{align*}
&\geq \int_G \left( \frac{1}{\hbar} \left( mw_i^2 + \left( D\pi^4 + ak\theta \right) w^2 + 2m\theta w_i + 2\rho^- \theta w \varphi(x,0,t) \right) + \\
&\quad + \rho^- \left( \frac{\pi^2}{l^2} + \frac{2}{h^2} \right) \varphi^2(x,y,t) - \frac{4\rho^-}{h^2} \varphi(x,0,t) \varphi(x,y,t) + \\
&\quad + \frac{2\rho^-}{h^2} \varphi^2(x,0,t) dxdy \right) + \int_0^l \frac{\pi^2}{l^2} D(1-\chi) w_x^2 dx,
\end{align*}
\]  

(52)

where \( \chi \) is the artificially introduced parameter, \( \chi \in (0,1) \).

In (52) we have a quadratic form with respect to \( w_i(x,t), w(x,t), \varphi(x,0,t), \varphi(x,y,t) \) with the matrix following:

\[
\begin{pmatrix}
\frac{m}{\hbar} & \frac{m\theta}{h} & 0 & 0 \\
\frac{m\theta}{h} & D\pi^4 + \frac{ak\theta}{h} & \rho^- \theta & 0 \\
0 & \frac{\rho^- \theta}{h} & 0 & 2\rho^- \\
0 & 0 & 2\rho^- & \frac{\rho^- \pi^2}{l^2} + \frac{2\rho^-}{h^2}
\end{pmatrix}.
\]

(53)

According to Sylvester’s criterion, write out the non-negativity quadratic form conditions

\[
\Delta_1 > 0 \Rightarrow \frac{m}{\hbar} > 0,
\]

(54)

\[
\Delta_2 > 0 \Rightarrow \frac{D\pi^4}{h^4} + ak\theta - m\theta^2 > 0,
\]

(55)

\[
\Delta_3 > 0 \Rightarrow 2\left( \frac{D\pi^4}{h^4} + ak\theta - m\theta^2 \right) - \rho^- h\theta^2 > 0,
\]

(56)

\[
\Delta_4 > 0 \Rightarrow \pi^2 h(2\frac{D\pi^4}{h^4} + ak\theta - m\theta^2) - \rho^- h\theta^2 - 2\rho^- \theta^2 l^2 \geq 0,
\]

(57)

where \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) are principal minors of the matrix (53). Here from (57) following the inequalities (54) - (56). Consider in (57) the case of equality and define the parameter \( \chi \):

\[
\chi = \frac{(2\rho^- \theta^2 l^2 + \rho^- h^2 \theta^2 \pi^2 - 2ak\theta + 2m\theta^2)l^4}{D\pi^4} \in (0,1).
\]

(58)

Thus, at the condition (58) fulfillment from (52) follows

\[
J(t) \geq \int_0^l \frac{\pi^2}{l^2} D(1-\chi) w_x^2 dx.
\]

(59)
According to the Cauchy-Bunjakovsky inequality, one has

$$\int_0^l w^2 dx \geq \frac{1}{l} \int_0^l w(x,t)^2 dx . \tag{60}$$

Finally

$$\frac{\pi^2}{l^3} D(1 - \chi) w^2(x,t) \leq J(0) = \int_0^l (mw_t^2(x,0) + Dw_{xx}^2(x,0) +$$

$$+ 2m\theta w(x,0) w_t(x,0) + \alpha \theta w^2(x,0) + 2p^- \theta w(x,0) \varphi(x,0,0)) dx +$$

$$+ \rho^- G \int_0^l (\varphi^2_x(x,y,0) + \varphi^2_y(x,y,0)) dxdy , \tag{61}$$

whence follow the statement

**Theorem 3.1.** *Let the conditions (47), (48) and (58) are satisfied. Then the solution \( w(x,t) \) of (5) - (8) is stable with respect to perturbations of the initial values of \( w(x,0), w_t(x,0), \varphi(x,0,0), \varphi_x(x,y,0), \varphi_y(x,y,0) \), if \( w(x,t) \) satisfies the boundary conditions (9).*

Conditions (47), (48) and (58) impose restrictions on a stream velocity \( V_0 \), a flexural rigidity of a plate \( D \) and other parameters of a mechanical system.

4. **NUMERICAL SOLUTION AND RESULTS**

For solving the boundary value problem (25), (9), (10) is applied the Galerkin method. The equation (25) is reduced to the form

$$w_t(x,t) = \frac{1}{m} (-Dw_{xxxx}(x,t) + p - \theta V_0 w_x - \theta \gamma w_t) -$$

$$- \frac{1}{m} \rho \left\{ a_t(t) + \frac{2}{l} \sum_{n=1}^{\infty} \cos(\lambda_n x) K_n \int_0^l w_t(x,t) \cos(\lambda_n x) dx \right\} , \tag{62}$$

where \( p = (p^- - p^+) , \theta = \frac{\rho^+ V_0}{\sqrt{M_0^2 - 1}} , K_n = \frac{(1 + e^{-2\lambda_n h})}{\lambda_n (1 - e^{-2\lambda_n h})} , \gamma = \frac{M_0^2 - 2}{M_0^2 - 1} . \)

According to Galerkin the test solution \( w(x,t) \) will be sought in the form

$$w(x,t) = \sum_{k=1}^{N} w_k(t) \sin(\lambda_k x) , \tag{63}$$
where \( \{ \sin(\lambda_k x) \}_{k=1}^\infty \) is the complete system of basis functions on the interval \([0, l]\), chosen so as to satisfy the given boundary conditions (9).

Orthogonality conditions for the discrepancies of the equation (62) with regard to (63) allow to write out the system of equations for \( w_k(x, t) \)

\[
\ddot{w}_k = \frac{2}{lm} \left( -D l^2 \lambda_k^4 w_k - \theta V_0 \sum_{k=1}^{N} \lambda_k w_k H_{k,m}^1 - \theta \gamma \frac{l}{2} \ddot{w}_k + H_{m}^2 (p - \rho \alpha(t)) \right) - \frac{4}{l^2 m} \left( \sum_{n=1}^{\infty} K_n \sum_{k=1}^{N} w_k H_{k,m}^3 H_{m,n}^4 \right),
\]

(64)

where \( H_{k,m}^1 = \int_0^l \cos(\lambda_k x) \sin(\lambda_m x) dx, \)

\( H_{k,m}^3 = \int_0^l \sin(\lambda_k x) \cos(\lambda_m x), \)

\( H_{m,n}^4 = \int_0^l \sin(\lambda_m x) \cos(\lambda_n x). \)

The initial conditions for the \( w_k(t) \) we obtain according to (10) with regard to (63)

\[
w_k(0) = \frac{2}{7} l \int_0^l f_1(x) \sin(\lambda_k x) dx, \quad \dot{w}_k(0) = \frac{2}{7} l \int_0^l f_2(x) \sin(\lambda_k x) dx.
\]

(65)

Thus, we have the Cauchy problem for a system of ordinary differential equations (64) with initial conditions (65).

We give examples of numerical calculation for the problem (64), (65). The following graphs will show deformation of an elastic wall of the fluid reservoir at supersonic gas flow with appropriate parameters of a mechanical system.

We assume that an elastic element is made from aluminum, i.e. \( E = 7 \times 10^{10} \) (Pa) – elastic modulus, \( \rho_{pl} = 2699 \) (kg/m³) – density. The element is flowed by supersonic air flow with density \( \rho^+ = 1.3 \) (kg/m³), the fluid reservoir is filled with water with density \( \rho^- = 998.2 \) (kg/m³). Other parameters of a mechanical system are \( l = 100 \) (m); \( h = 100 \) (m); \( h_{pl} = 0.5 \) (m) – plate thickness; \( m = 269.9 \) (kg/m) – mass per unit length; \( \nu = 0.34 \) – Poisson’s ratio; \( D = \frac{E h_{pl}^3}{12(1 - \nu^2)} = 6.5958 \cdot 10^8 \) (N*m) – flexural stiffness. All values are given in the SI system. The initial conditions (in meters) \( w(x, 0) = -0.0015 \sin \left( \frac{2\pi x}{l} \right) \dot{w}(x, 0) = 0. \)

With the software MATLAB [20] solved the problem of Cauchy (64), (65) (the order of approximation \( N = 15 \)) and received plots of \( w(x, t) \) at the point \( \frac{l}{4} \) for different values of the flow velocity \( V \).
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1. $V = 600$

![Fig. 2.: The deformation of the elastic wall of the fluid reservoir at a point $x^* = l/4$](image1)

According to the graphs in Fig. 2 and Fig. 3, we can conclude that the solution of (64), (65) at $V = 600$ is stable.

2. $V = 1070$

![Fig. 3.: Deflection of the elastic wall of the fluid reservoir at points: $t = 1, t = 80$](image2)

![Fig. 4.: The deformation of the elastic wall of the fluid reservoir at a point $x^* = l/4$](image3)

According to the graphs in Fig. 4 and Fig. 5, we can conclude that the solution of (64), (65) at $V = 1070$ is unstable.

5. CONCLUSIONS

The mathematical model of the tank filled with liquid and containing an elastic wall is offered. The wall interacts with this liquid on the one hand, and with a supersonic flow of gas with another. The model represents an initial-boundary value problem for system of the integro-differential equations with partial derivatives for
determination of three unknown functions - deformation (deflection) of a wall, velocity potential of liquid in internal area of the tank and velocity potential of gas in external area. The solution of aerohydrodynamic part of a problem in the external area, based on operational method is given. The initial-boundary value problem for determination of two unknown functions - deformations of a wall and velocity potential of liquid in the tank is as a result received. On the basis of construction of the Lyapunov type mixed functional the stability conditions of solutions of this problem are received. Definition of stability of an elastic body corresponds to the concept of stability of dynamic systems by Lyapunov. The solution of aerohydrodynamic part of a task in internal area of the tank, based on Fourier’s method is also given. The initial-boundary value problem for determination only one unknown function - wall deformation is as a result received. Examples of the numerical and analytical solutions of this problem, obtained by Bubnov-Galerkin’s method are given.

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References


ASYMPTOTIC EQUATIONS OF THE NONLINEAR TRANSONIC GAS FLOWS AND THEIR SOLUTIONS
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Abstract
In the article there is received an equation for the transonic gas flows which considers transversals disturbance surpassing the main flow disturbance. Some exact individual solutions of this equation are shown as well as their application to the solutions of a number of tasks.

Keywords: partial differential equations, asymptotic decomposition, transonic gas flows, partial solutions.

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1. DEDUCTION OF THE ASYMPTOTIC EQUATION

Irrotational isentropic gas flows in the cylindrical dimensionless coordinates \( x, r, \theta \) are being described with the help of the equation:

\[
\Phi_{tt} + 2\Phi_t \Phi_{xt} + 2\Phi_t \Phi_{rt} + \frac{2}{r^2} \Phi_\theta \Phi_{t\theta} + 2\Phi_x \Phi_r \Phi_{rx} + \frac{2}{r^2} \Phi_x \Phi_\theta \Phi_{r\theta} + \\
+ \frac{2}{r^2} \Phi_\theta \Phi_r \Phi_{r\theta} + \Phi_x^2 \Phi_{xx} + \Phi_r^2 \Phi_{rr} + \frac{1}{r^2} \Phi_\theta^2 \Phi_{\theta\theta} - \\
a^2 \left( \Phi_{rr} + \frac{1}{r} \Phi_r + \frac{1}{r^2} \Phi_{\theta\theta} \right) = 0, \\
a^2 = \rho^{\chi-1} = \frac{\rho^{\chi-1}}{p} = \frac{x}{2} + \frac{1}{2} - \frac{\chi - 1}{2} \left( 2\Phi_r + \Phi_x^2 + \Phi_r^2 + \frac{1}{2r^2} \Phi_\theta^2 \right). 
\]

In (1) \( \Phi(x, r, \theta, t) \) means the potential of speed, \( t \) is the time, \( a \) is the sound speed, \( \rho \) is the density, \( p \) is the pressure. Let us put into an asymptotic decomposition for \( \Phi(x, r, \theta, t) \)

\[
\Phi = x + \epsilon \psi(r, \theta, t^0) + \epsilon^3 \varphi(x^0, r, \theta, t^0) + ..., \ x = \epsilon x^0, \ t = \frac{1}{\epsilon} t^0, 
\]

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where $\epsilon$ is a small parameter. If we substitute (2) in (1) and leave the part of the main order aside, we will get a transonic equation for the function $\varphi(x^0, r, \theta, t^0)$:

$$2\varphi,_{\varphi,\rho} + (\chi + 1)\varphi,_{\varphi,\varphi} + 2\psi,_{\varphi,\varphi} + \frac{2}{r^2}\psi,_{\varphi,\theta} + \frac{\chi - 1}{2} \left(2\psi,_{\rho} + \psi,_{\theta} + \frac{1}{r^2}\psi,_{\theta}^2\right) \varphi,_{\varphi,\varphi} - \Delta \varphi = L(\psi).$$

(3)

In (3) are put into the following designations

$$\Delta \varphi \equiv \varphi,_{rr} + \frac{1}{r}\varphi,_{r} + \frac{1}{r^2}\varphi,_{\theta\theta},$$

$$-L(\psi) \equiv \psi,_{\rho,\rho} + 2\psi,_{\rho,\rho} + \frac{2}{r^2}\psi,_{\rho,\theta} + \psi,_{\rho,\theta} + \frac{1}{r^2}\psi,_{\theta}^2\psi,_{\theta\theta} + \frac{2}{r^2}\psi,_{\rho,\rho} + \psi,_{\theta} - \frac{1}{r^2}\psi,_{\theta}^2.$$

The function $\psi(r, \theta, t^0)$ satisfies the Laplace’s equation $\Delta \psi = 0$. In case $\psi \equiv 0$ we get a classical transonic Lin-Raissner-Tzyan’s equation

$$2\varphi,_{\varphi,\rho} + (\chi + 1)\varphi,_{\varphi,\varphi} - \varphi,_{rr} - \frac{1}{r}\varphi,_{r} - \frac{1}{r^2}\varphi,_{\theta\theta} = 0,$$

which proceeds to the Karman-Facovitch’s mixed type equation in permanent case:

$$(\chi + 1)\varphi,_{\varphi,\varphi} - \varphi,_{rr} - \frac{1}{r}\varphi,_{r} - \frac{1}{r^2}\varphi,_{\theta\theta} = 0.$$

The equation (3) describes transonic gas flows, which appear because of the influence on the streamline solid the side (with respect to the main stream course which coincides with the axis $x$) disturbance of the main transonic flow (for the disturbing cross flow $\Phi_x, \Phi_z = \epsilon$, for the main flow $\Phi_y, \Phi_z \sim \epsilon^3$). The disturbance for the external streamlines of the aircrafts is, for instance, the side wind, which changes its intensity depending on time $\psi = V_{\infty}(t) r \cos(\theta + \alpha(t))$. For the internal streamline, for example, the disturbance can be a flow twist ($\psi = \Gamma(t) \theta$).

We can get the conditions at the front a shock wave $x^0 = x^0(r, \theta, t^0)$ from the Renckin-Gyugonio conditions, substituting the decomposition (2) in them and leaving the main order parts aside:

$$2\frac{\partial x^0}{\partial \rho} + \left(\frac{\partial x^0}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial x^0}{\partial \theta}\right)^2 + 2\psi,_{\rho} + \frac{2}{r^2}\psi,_{\theta} = \chi - 1$$

$$= \chi - 1 \left(2\psi,_{\rho} + \psi,_{\theta} + \frac{1}{r^2}\psi,_{\theta}^2\right) + \chi + \frac{1}{2} \left(\varphi,_{\varphi,\rho} + \varphi,_{\varphi,\varphi}\right), \varphi = \varphi^*. \quad (4)$$

If we put $\varphi \equiv \varphi^*$ in (4), we will get a defining equation for (3).

Let us write down the conditions on the streamline surface, which differs from the cylindrical surface just a little bit, giving it as

$$r = r_0(\theta, t^0) + r_2(x^0, \theta, t^0) \epsilon^4 + .... \quad (5)$$
Substituting (2) and (5) into the exact non-passing condition

\[ -\Phi_x x + \Phi_r - r^{-2} r_0 \Phi_\theta = r_t \]

and leaving the main order parts aside, then we get:

\[
\psi_r - \frac{1}{r_0^2} \frac{\partial r_0}{\partial \theta} \psi_\theta = \frac{\partial r_0}{\partial \theta}, \quad \varphi_r - \frac{1}{r_0^2} \frac{\partial r_0}{\partial \theta} \varphi_\theta = \frac{\partial r_2}{\partial \chi^0}.
\]

(6)

The values of \( \varphi_r, \varphi_\theta, \psi_r, \psi_\theta \) in (6) is being calculated when \( r = r_0(\theta, t^0) \).

The sound surface equation \( (V^2 = a^2) \) in transonic approximation takes the form

\[
N \equiv \chi + \frac{1}{2} \left( \psi_r^2 + \frac{1}{r^2} \psi_\theta^2 + (\chi - 1)\psi_\rho + (\chi + 1)\varphi_\rho \right) = 0.
\]

(7)

For steady flow the equation (3) has the mixed type.

Substituting (2) into the expression for the pressure, conducting the decomposition into the Taylor’s series and leaving the main order parts aside, we get an asymptotic formula for the pressure definition

\[
P = 1 - \chi e^2 \left( \psi_\rho + \varphi_\rho + \frac{1}{2} \psi_r^2 + \frac{1}{2r^2} \psi_\theta^2 \right).
\]

(8)

2. SOME OF THE SOLUTIONS OF THE ASYMPTOTIC EQUATION (3)

Let us indicate some of the partial solutions of the equation (3).

Let us mark the automodel class of the solutions (indices zero of variables \( x, t \) is left out here and further):

\[
\psi = \tilde{r}^{\beta} \psi(\zeta, \eta), \quad \varphi = \tilde{r}^{2\beta - 1} \varphi(\xi, \zeta, \eta), \quad \xi = \frac{x}{\tilde{r}^\beta}, \quad \zeta = \frac{r}{\tilde{r}^{\beta+1}/2}, \quad \eta = \theta + \alpha \ln t,
\]

(9)
where \( \alpha, \beta \) are arbitrary numbers. After substitution (9) into the equation (3), we will get a equation for function \( \varphi = \varphi(\xi, \zeta, \eta) \)

\[
2 \left( (\beta - 1) \varphi_{\xi} - \beta \varphi_{\xi\xi} \xi - \frac{\beta + 1}{2} \varphi_{\xi\xi} + \alpha \varphi_{\xi\eta} \right) + (\chi + 1) \varphi_{\xi\zeta} + 2 \varphi_{\zeta} \varphi_{\xi\zeta} + \frac{\xi}{\zeta} \varphi_{\eta} \varphi_{\xi\eta} - \Delta \varphi + \chi \left( 2 \beta \varphi - (\beta + 1) \varphi \varphi_{\zeta} + 2 \alpha \varphi_{\eta} + \varphi_{\xi}^2 + \frac{1}{\zeta^2} \varphi_{\eta}^2 \right) \varphi_{\xi\xi} = L(\varphi),
\]

\[
\Delta \varphi = \frac{1}{\xi} \varphi_{\xi\xi} + \frac{1}{\zeta} \varphi_{\zeta\zeta} + \frac{1}{\eta} \varphi_{\eta\eta}.
\]

Function \( \varphi \) satisfies the Laplace’s equation: \( \varphi_{\xi\xi} + \frac{1}{\xi} \varphi_{\xi} + \frac{1}{\zeta} \varphi_{\zeta} + \frac{1}{\eta} \varphi_{\eta} = 0 \).

Equation (3) permits polynomial type of solution

\[
\varphi = \sum_{k=0}^{3} \varphi_k(r, \theta, t) x^k. \tag{10}
\]

After substitution (10) into the equation (3) we will get a system of equations for functions \( \varphi_0(r, \theta, t), \varphi_1(r, \theta, t), \varphi_2(r, \theta, t), \varphi_3(r, \theta, t) \):

\[
\begin{cases}
18 \varphi_3^2 (\chi + 1) - \Delta \varphi_3 = 0, \\
6 \varphi_3 + 18 (\chi + 1) \varphi_2 \varphi_3 + 6 \varphi_3 + \frac{6}{r^2} \varphi_{\rho} \varphi_{\phi} = 0, \\
4 \varphi_2 + (\chi + 1) (6 \varphi_1 \varphi_3 + 4 \varphi_3^2) + 4 \varphi_3 \varphi_{\phi} + \frac{4}{r^2} \varphi_{\rho} \varphi_{\phi \phi} = 0, \\
+ 6 (\chi + 1) \varphi_3 \left( \varphi_3 + \frac{1}{2} \varphi_{\rho}^2 + \frac{1}{2} \varphi_{\phi}^2 \right) = 0, \\
2 \varphi_1 + 2 (\chi + 1) \varphi_1 \varphi_2 + 2 \varphi_1 \varphi_{\phi} + \frac{2}{r^2} \varphi_{\rho} \varphi_{\phi \phi} = 0, \\
+ 2 (\chi + 1) \varphi_2 \left( \varphi_1 + \frac{1}{2} \varphi_{\rho}^2 + \frac{1}{2} \varphi_{\phi}^2 \right) = -L(\varphi).
\end{cases}
\]

In the class of solutions (10) there is a solution in case of the steady flows, which describes the gas flow in the Lavales’s nozzles with the permanent acceleration
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Fig. 1.: The streamlined surface of the form (12) at \( r_0 = 3, r_0 = 5 \).

(\( \varphi_{xx} = \text{const} \)) and considers the flow twist (\( \psi = \Gamma \theta, \Gamma = \text{const} \)):

\[
\varphi = a x^2 + (\chi + 1) a^2 r^2 x + \left[ \frac{\chi - 1}{2} \Gamma^2 a \ln r + \frac{1}{8} (\chi + 1)^2 a^3 r^4 \right]. \tag{11}
\]

If \( r \to 0 \), the speed constituent \( V_r = \epsilon^3 \varphi_r \) has a peculiarity \( \ln r \), which is conditioned by the primordial peculiarity of the constituent assignment \( V_\theta(V_0 = \epsilon \Gamma / r) \). The sound surface equation for (11), according to (7), takes the form

\[
x = -\frac{\chi + 1}{2} a + \frac{\Gamma^2}{4ar^2}.
\]

If \( \chi \to 1 \), the influence of the flow twist on the speed distribution gets smaller and the peculiarity for \( \varphi_r \) disappears if \( r \to 0 \). Conditions (6) take a form:

\[
\frac{\partial r_0}{\partial \theta} = 0, \quad \frac{\partial r_2}{\partial x} = \varphi_r, \quad (\varphi_r \text{ is being calculated at } r = r_0(\theta)).
\]

Defining functions \( r_0(\theta) \) and \( r_2(x, \theta) \), we will get a streamline surface equation:

\[
r = \epsilon^4 \left[ r_0 a^2 (\chi + 1) x^2 + \left( (\chi - 1) \Gamma^2 a \ln r_0 - \frac{1}{2} (\chi + 1)^2 a^3 r_0^3 \right) x + C(\theta) \right] + r_0, \quad r_0 = \text{const}.
\tag{12}
\]

The solution (11) can be used for describing the flows in the annular channel, the equations of the internal and external sides can be gotten from (12) at \( r_0 = r_0^{(1)} \), \( r_0 = r_0^{(2)} \) (see fig.1).

In case \( \Gamma = 0 \) in (11) we will get a known solution that describes the flow in the center of the Lavales’s nozzle. Let us define the solution for cases \( \psi = \Gamma \theta \) that has features:

\[
V_x, V_r, V_\theta \to 0 \text{ at } r \to \infty \quad \psi = f(x, \theta) r^{-2} + g(r, \theta), \quad \Delta g = 0.
\tag{13}
\]
Substituting the solution (13) and \( \psi \) into the equation (3), we will get an equation for the function \( f(x, \theta) \):

\[
(\chi + 1)f_xf_{xx} + 2\Gamma f_x\theta + (\chi - 1)\Gamma^2 f_{xx} - 4f - f_{\theta\theta} = 0.
\]

The solution (13) permits the generalization: \( \varphi = f(\xi, \eta)\theta^{-2} + g(r, \theta), \xi = x + \alpha \ln r, \eta = \theta + \beta \ln r, \alpha, \beta \) are arbitrary numbers.

Let us examine the case, corresponding the gas flow between revolving flatnesses \( \theta = \theta_1(t), \theta = \theta_2(t) \). In this case

\[
\psi = r^2(a(t) \cos 2\theta + b(t) \sin 2\theta) = r^2 f(\theta, t),
\]

at the same time in (3)

\[
L(\psi) = r^2 (8(a' + b' b) + 8(a \cos 2\theta + b \sin 2\theta)(2ab \sin 4\theta + +a^2 \cos 4\theta - b^2 \cos 4\theta) + a'' \cos 2\theta + b'' \sin 2\theta) = r^2 G(\theta, t).
\]

Functions \( a(t), b(t) \) are being defined from the conditions of non-passing (6):

\[
f_\theta(\theta_k(t), t) = \frac{\partial \theta_k(t)}{\partial t}, \quad k = 1, 2,
\]

and are equal

\[
a(t) = \frac{1}{A} (\theta'_2 \cos 2\theta_2 - \theta'_1 \cos 2\theta_1), \quad b(t) = \frac{1}{A} (\theta'_2 \sin 2\theta_1 - \theta'_1 \sin 2\theta_2),
\]

where \( A = 2 \sin 2(\theta_1 - \theta_2) \). Substituting (14) into (3), we will get the equation for \( \varphi(x, r, \theta, t) \):

\[
2\varphi_{xl} + (\chi + 1)\varphi_x \varphi_{xx} + 4r(a \cos 2\theta + b \sin 2\theta)\varphi_{x\theta} + 4(b \cos 2\theta - -a \sin 2\theta)\varphi_{x\theta} - \Delta \varphi + (\chi - 1)r^2(a' \cos 2\theta + b' \sin 2\theta + 2a^2 + +2b^2)\varphi_{xx} = -r^2 G(\theta, t).
\]

The equation (15) permits the solution that takes a form of (10). Then we will get a system of four equations for the functions \( \varphi_0(\theta, \theta, t), \varphi_1(\theta, \theta, t), \varphi_2(\theta, \theta, t), \varphi_3(\theta, \theta, t) \):

\[
\begin{cases}
18\varphi_3(\chi + 1) - \Delta \varphi_3 = 0, \\
6\varphi_3 + 18(\chi + 1)\varphi_2 \varphi_3 + 12(b \cos 2\theta - a \sin 2\theta)\varphi_{3\theta} + +12(r(a \cos 2\theta + b \sin 2\theta)\varphi_{3\theta} - \Delta \varphi_2 = 0, \\
4\varphi_2 + (\chi + 1)(6\varphi_1 \varphi_3 + 4\varphi_2^3) + 8r(a \cos 2\theta + b \sin 2\theta)\varphi_2, - -\Delta \varphi_1 + 8(b \cos 2\theta - a \sin 2\theta)\varphi_{2\theta} + 6(\chi - 1)r^2 \varphi_3(a' \cos 2\theta + +b' \sin 2\theta + 2a^2 + 2b^2) = 0, \\
2\varphi_1 + 2(\chi + 1)\varphi_1 \varphi_2 + 4r(a \cos 2\theta + b \sin 2\theta)\varphi_{1\theta} + 4(b \cos 2\theta - -a \sin 2\theta)\varphi_{1\theta} - \Delta \varphi_0 + 2(\chi - 1)r^2(a' \cos 2\theta + b' \sin 2\theta + +2a^2 + 2b^2)\varphi_2 = r^2 G(\theta, t).
\end{cases}
\]
At the same time we can put \( \varphi_3 = g(\theta, t)r^{-2} \), where \( g(\theta, t) \) is defined from the equation: \( 18(\chi + 1)g^2 - 4g - g_{\theta \theta} = 0 \).

In a particular case, let us consider the solution for the steady flows

\[
\varphi(x, r, \theta) = \varphi_2(\theta) x^2 + \varphi_1(\theta) r^2 x + \varphi_0(\theta) r^4 .
\]  

(17)

Substituting (17) into the system (16), we will get a system of three equations for the functions \( \varphi_2(\theta), \varphi_1(\theta), \varphi_0(\theta) \):

\[
\begin{align*}
\varphi''_2 &= 0, \\
4(\chi + 1)\varphi'_2 + 8(b \cos 2\theta - a \sin 2\theta)\varphi'_2 - 4\varphi_1 - \varphi''_1 &= 0, \\
2(\chi + 1)\varphi_1 + 4(b \cos 2\theta - a \sin 2\theta)\varphi_2' - 16\varphi_0 - \varphi''_0 + 8(a \cos 2\theta + \\
+ b \sin 2\theta)\varphi'_1 + 4(\chi - 1)(a^2 + b^2)\varphi_2 &= G(\theta).
\end{align*}
\]

Considering the non-passing conditions \( \frac{\partial \varphi_k}{\partial \theta}(\theta = \theta_1, \theta_2) = 0 \), \( k = 0, 1, 2 \), we will get \( \theta_2 - \theta_1 = \frac{\pi n}{2}, n \in N \), \( \varphi_2 = \text{const} = C_1 \). Then \( \varphi_1 = C_1^2(1 + \chi) + C_2 \cos 2\theta \), and the function \( \varphi_0 \) is defined from the equation:

\[
\begin{align*}
\varphi''_0 + 16\varphi_0 &= 2(\chi + 1)C_1(C_1^2(1 + \chi) + C_2 \cos 2\theta) - 8C_2 \sin 2\theta(b \cos 2\theta - \\
- a \sin 2\theta) + 8(a \cos 2\theta + b \sin 2\theta)(C_1^2(1 + \chi) + C_2 \cos 2\theta) + \\
+ 4(\chi - 1)(a^2 + b^2)C_1 - 8(a \cos 2\theta + b \sin 2\theta)(2ab \sin 4\theta + a^2 \cos 4\theta - \\
- b^2 \cos 4\theta) = H(\theta).
\end{align*}
\]

Let us consider the streamline surface, which differs from the cylinder just a little bit \( (r_0 = R) \). In this case, assuming the cross-flow of the surface is unseparated, we can suggest \( \psi = V_\infty \cos \theta(r + R^2r^{-1}) \). Then the equation (3) takes the form

\[
\begin{align*}
2\varphi_{x\theta} + (\chi + 1)\varphi_x \varphi_{x\theta} + 2V_\infty \cos \theta \left( 1 - \frac{R^2}{r^2} \right) \varphi_{x\theta} - \Delta \varphi - \\
- 2V_\infty \sin \theta \left( 1 + \frac{R^2}{r^2} \right) \varphi_{x\theta} + \frac{1 - \chi}{2} V_\infty^2 \varphi_{xx} \left( 1 - \frac{R^2}{r^2} \right) \cos 2\theta + \\
\left( \frac{R^4}{r^4} \right) = -2 \left( \frac{R^2}{r^3} \right) V_\infty^3 \cos \theta \left( 1 - \frac{2R^2}{r^2} + \frac{R^4}{r^4} - 4 \sin^2 \theta \right).
\end{align*}
\]

(18)

We can free ourselves from the right side \( (a(r, \theta)) \) of the equation (18) with the help of the new function \( \tilde{\varphi} = \varphi + g(r, \theta) \), \( \Delta g = \alpha(r, \theta) \). Then in a permanent case we will get an equation for the function \( \tilde{\varphi} \):

\[
\begin{align*}
(\chi + 1)\tilde{\varphi}_{x\theta} \tilde{\varphi}_{x\theta} + 2V_\infty \cos \theta \left( 1 - \frac{R^2}{r^2} \right) \tilde{\varphi}_{x\theta} - \Delta \tilde{\varphi} - 2V_\infty \sin \theta \left( 1 + \frac{R^2}{r^2} \right) \tilde{\varphi}_{x\theta} + \\
\frac{1 - \chi}{2} V_\infty^2 \tilde{\varphi}_{xx} \left( 1 - \frac{2R^2}{r^2} \cos 2\theta + \frac{R^4}{r^4} \right) = 0.
\end{align*}
\]

(19)
The equation (19) has a solution with the type (10) with $\varphi_k$ depended on $r$, $\theta$, at the same time we can assume $\varphi_3 = g(\theta)r^{-2}$ (in particular, $g = 0$ or $g = 1/(3(\chi+1)\cos^2 \theta)$). For the distant from the solid flow $r \to 0$, the solution for the equation (19) can be being found in the form

$$\varphi = r^\lambda f(\xi, \theta) + \ldots, \quad \xi = xr^{-n}.$$ 

We can assume $\lambda = 3n - 2$, because at $V_\infty = 0$ we must get an asymptotic corresponded to the classical equation $(\chi + 1)\varphi_{xx} - \Delta \varphi = 0$. In the case of the formal transition to (18) at $r \to \infty$ we will get a limit equation

$$M(\varphi) = 2\varphi_{xt} + (\chi + 1)\varphi_x\varphi_{xx} + 2V_\infty \cos \theta \varphi_{xt} - (2/r)V_\infty \sin \theta \varphi_{tt} - \Delta \varphi + ((\chi - 1)/2)V_\infty^2 \varphi_{xx} = -(2R^2/r^3)V_\infty^3 \cos \theta(1 - 4 \sin^2 \theta),$$

(20)

which permits an exact solution in a permanent case

$$\varphi = rf(\xi, \theta) + g(\theta)r^{-1}, \quad \xi = x/r, \quad g(\theta) = -R^2V_\infty^3 \sin^2 \theta \cos \theta.$$ 

In case $R \ll 1$, an approximate solution of the equation (18) in a permanent case can be being found in the form

$$\varphi = \sum_{n=0}^{\infty} \varphi_n(x, r, \theta)R^{2n}.$$ 

Then for $\varphi_0$ we will get the equation (20) in a permanent case and without the right side ($M(\varphi_0) = 0$), which has a solution (10), where $\varphi_0$ doesn’t depend on $t$, a solution $\varphi_0 = rf(\xi, \eta) + g(r, \theta)$, $\xi = x/r$, $\eta = \theta + \alpha \ln r$, $\Delta g = 0$. In case $V_\infty \ll 1$ the solution (18) can be being found in the form

$$\varphi = \sum_{n=0}^{\infty} \varphi_n(x, r, \theta)V_n^{m}.$$ 

In this case we will can a known transonic equation for $\varphi_0$

$$(\chi + 1)\varphi_0_{xx} - \Delta \varphi_0 = 0.$$ 

In case of the separation streamline and assuming that two vortical rectilinear shrouds of infinite length with permanent and opposite in sign intensions $\Gamma$ mare
coming off the surface of the cylinder, the expression for \( \psi(r, \theta) \) can be written as:

\[
\psi(r, \theta) = V_\infty \cos \theta \left( r + \frac{R^2}{r} \right) + \frac{\Gamma}{2\pi} \int_0^\infty \left( \arctg \frac{r \sin \theta - R}{r \cos \theta - S} - \arctg \frac{r \sin \theta + R}{r \cos \theta - S} \right) dS + \\
\frac{\Gamma}{2\pi} \int_0^\infty \left( \arctg \frac{r \sin \theta(S^2 + R^2) + R^3}{r \cos \theta(S^2 + R^2) - S R^2} \right) - \arctg \frac{r \sin \theta(S^2 + R^2) - R^3}{r \cos \theta(S^2 + R^2) - S R^2} \right) dS
\]

The come-off points of the whirlwinds are points \( r = R, \theta = \pm \frac{\pi}{2} \). According to (8), it is necessary to know \( \psi_\theta(R, \theta) (\psi_r(R, \theta) = 0) \) in non-passing condition (6), which is satisfied by the function (21) for the definition of the power influence on the streamline solid. Defining \( \psi_\theta \) from (21) with the help of the \( \theta \) differentiation and making \( S \) integration, we get

\[
\psi_\theta(R, \theta) = -(\Gamma + 2V_\infty)R \sin \theta + \frac{\Gamma R}{2\pi} \left( \cos \theta \left( \frac{1 - \sin \theta}{1 + \sin \theta} \right) - 2 \sin \theta \arctg \left( \frac{\cos \theta}{1 - \sin \theta} \right) + \arctg \left( \frac{\cos \theta}{1 + \sin \theta} \right) \right).
\]

The investment into the expression for \( C_0 = -\int_0^{2\pi} P \cos \theta d\theta \), corresponding (21) is defined according to (8), (22) by the formula (\( V_\infty = R = 1 \)):

\[
\int_0^{2\pi} \psi_\theta^2 \cos \theta d\theta = 2\Gamma^2 + 4\Gamma, \Gamma < -2.
\]

3. CONCLUSIONS

The article is devoted to the development of the mathematical theory of gas flow with a speed close to the speed of sound, namely transonic gas flows, i.e. flows that contain both subsonic and supersonic area. The main problems arising in the study of such flows should be classified as non-linearity and mixed type equations describing transonic flow. Transonic gas flows taking into account the transverse perturbations are studied on the basis of nonlinear equation obtained in this paper. Some exact particular solutions of this equation are constructed and their application to solving a number of transonic aerodynamics problems are shown. In particular, a solution of polynomial form describing axisymmetric gas flow in Laval nozzles with constant
acceleration and flow swirling is obtained. The unsteady flows in the channels between the rotating planes are researched. The asymptotic equation describing the flows arising at the unseparated and separated flow past the body that is practically identical to the cylindrical one is derived.

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