THE THEORETICAL APPROACH OF FACE SEALS PRESSURE FOR HYDRODYNAMIC OPERATING MODEL

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Abstract
The power series approach is used to solve the Reynolds equation for hydrodynamic face seal lubrication. The angular misalignment of the stator and rotor is considered. Obtaining another set of partial differential equations, the Fourier series approaches the pressure components. Solving the particulars non-homogenous Euler-Cauchy equations, the analytical expression of hydrodynamic film pressure between the seal rings is computed.

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1. INTRODUCTION

The face seals are used within the mechanical systems where efficient operation is required. This is the reason for that physicists, mathematicians and engineers are interested by the face seals study.

The face seal operation is conditioned by minimum friction losses, that means the full separation of the ring surfaces. By the other way, a great interspace supposes great seal leakage. The pressure distribution within the fluid supposes a separation field model [1]. The simplest model is considered the model with flat, smooth and parallel surfaces, with uniform film and linear pressure distribution. The nature of the film was the subject of many researches, most of authors considering hydrodynamic behaviour [1, 2]. The thin film hydrodynamic approach was introduced by O. Reynolds and this is the fundamental basis for the theoretical studies [4].

To solve Reynolds equation the numerical methods are widely used, as shown in [3], but the analytical solution is important from seal optimisation point of view. In [5] the pressure was approached by Fourier series and was obtained a set of ordinary differential equations, easy to solve. Pressure, load capacity and leakage rate are plotted depending by the tilt parameter. To solve analytically the Reynolds equation in cylindrical coordinates, two approaches are considered: power series and Fourier series. The theory behind the power series method to solve differential equations is rather simple but the algebraic procedures involved could be quite complex, partic-
2. THE REYNOLDS EQUATION FOR FACE SEAL

Most of the studies considers that the thickness between the seal rings centers is constant. For the viscous fluid movement between the surfaces of a mechanical face seal, the following hypothesis are assumed:
- the fluid is Newtonian and the flow is laminar;
- the exterior gravity forces and inertia forces are neglected;
- the film thickness is very small related to other dimensions of seal interface;
- the fluid velocity along the film height is very small related to the velocity components along the other direction;
- the velocity ratio is the same with the dimensions ratio;
- the height roughness are small related to film thickness.

On the $S_1$ and $S_2$ surfaces (with $z = H_1$ and $z = H_2$), the following conditions are considered:

$$
\begin{align*}
    v^r &= v_1^r; v^\theta = v_1^\theta, v^z = v_1^z, \text{ for } z = H_1 \ (r, \theta, t) \\
    v^r &= v_2^r; v^\theta = v_2^\theta, v^z = v_2^z, \text{ for } z = H_2 \ (r, \theta, t)
\end{align*}
$$

(1)

where: $r$, $\theta$, $z$ are the cylindrical co-ordinates; $v^r$, $v^z$, $v^\theta$ are the velocity components in cylindrical co-ordinates; $p$ is the pressure, $\rho$ is the specific density and $\mu$ is the fluid viscosity.

The fluid pressure depends by seal ring radius and angle, $p = p(r, \theta, t)$ and the Reynolds equation in cylindrical coordinates is given below [4]:

$$
\begin{align*}
    \frac{\partial}{\partial r} \left[ \frac{1}{\mu} (H_2 - H_1)^3 \frac{\partial p}{\partial r} \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{\mu} (H_2 - H_1)^3 \frac{\partial p}{\partial \theta} \right] &= \\
    = 6r (H_2 - H_1) \frac{\partial (v_1^r + v_2^r)}{\partial r} + 6r (v_1^r - v_2^r) \frac{\partial (H_2 - H_1)}{\partial r} + \\
    + 6 (H_2 - H_1) \frac{\partial (v_1^\theta + v_2^\theta)}{\partial \theta} + 6 (v_1^\theta - v_2^\theta) \frac{\partial (H_2 - H_1)}{\partial \theta} + 12r (v_1^r - v_2^r).
\end{align*}
$$

(2)

We suppose that $S_1$ and $S_2$ are circular and coaxial rings, where $S_1$ is stationary and $S_2$ mobile with $\omega$ as angular velocity around OZ axis, as shown in fig.1.

The height $H_1 = H(r, \theta)$ isn’t time dependent because defines the stationary ring position and to find it the point $N_1$ is projected over an horizontal reference surface and for $\theta = 0$ as shown in fig.1 and fig.2, it obtains:

$$
\begin{align*}
    z(r) &= NN_1 = OO_1 \pm r \tan(N_1O_1P) = OO_1 \pm r \tan \chi_1
\end{align*}
$$

Considering $M_1$ another point of the $r$ radius circle and $\theta$ the angle between the $N_1$ and $M_1$ points, using the fig.1 the following relation can be deduced:

$$
\begin{align*}
    z(r, \theta) &= NN_1 = OO_1 \pm r \tan \chi_1 \sin \theta
\end{align*}
$$
The theoretical approach of face seals pressure for hydrodynamic operating model

Fig. 1.: Face seal model.

Fig. 2.: Seal rings, planar section.
Because parameter $\chi_1$ is small, we approach $\tan \chi_1 \approx \sin \chi_1 \approx \chi_1$, so:

$$H_1(r, \theta) = OO_1 \pm r\chi_1 \sin \theta$$  \hspace{1cm} (3)

Similarly, for $H_2$ (but replacing $\theta$ with $\theta - \omega t$ due to $S_2$ motion) we obtain:

$$H_2(r, \theta, t) = OO_1 + h_0 \pm r\chi_2 \sin(\theta - \omega t)$$

Because the fluid is viscous, it adheres to the surfaces, so:

$v_1^h = 0, v_1^0 = 0, v_1^2 = 0, v_2^0 = r\omega, v_2^2 = 0$.

Considering $h_0$ constant we suppose that $v_2^0 = 0$. The equation (2) becomes:

$$\frac{\partial}{\partial r} \left( \frac{r}{\mu} (H_2 - H_1)^3 \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{1}{\mu} (H_2 - H_1)^3 \frac{\partial p}{\partial \theta} \right) = -6v_2^0 \frac{\partial (H_2 - H_1)}{\partial \theta}$$  \hspace{1cm} (4)

We denote $h = H_2 - H_1$ and considering $\mu$ constant the equation (4) becomes:

$$\frac{\partial}{\partial r} \left( rh^3 \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{h^3}{r} \frac{\partial p}{\partial \theta} \right) = -6\mu \omega r^2 [\chi_2 \cos(\theta - \omega t) + \chi_1 \cos \theta]$$  \hspace{1cm} (5)

Following non-dimensional parameters are defined: thickness $\tilde{h} = \frac{h}{h_0}$; radiuses $\tilde{r} = \frac{r}{R_e}$ and $\tilde{r}_l = \frac{R_e}{R_l}$; relative tilt $\tilde{X} = \frac{X_1}{X_2}$; film tilt-thickness parameter $\tilde{\chi}_2 = \frac{\chi_2}{\chi_0}$ and pressure $\tilde{p} = \frac{k_0}{\mu \kappa_0} \omega p$.

Then, considering $r \in [R_1, R_e], \bar{r} \in [\bar{r}_l, 1]$ and $\frac{\partial}{\partial \bar{r}} = \frac{\partial}{\partial \tilde{r}} \frac{\partial}{\partial \bar{r}}$, so $\frac{\partial}{\partial \tilde{r}} = \frac{1}{\kappa_0} \frac{\partial}{\partial \tilde{r}}$ and the same sign for the last members of the $H1$ and $H2$, the thickness film equation is presented below:

$$h = H_2 - H_1 = h_0 + r\chi_2 \left[ \sin(\theta - \omega t) - \frac{\chi_1}{\chi_2} \sin \theta \right]$$

Dividing the equation above by $h_0$ gives:

$$\tilde{h} = 1 + \tilde{r}e_2 [\sin(\theta - \Omega) - \chi \sin \theta]$$  \hspace{1cm} (6)

Using the non-dimensional parameters, the equation (5) can be written as,

$$\frac{\partial}{\partial \bar{r}} \left( \tilde{r}^3 \frac{\partial \tilde{p}}{\partial \bar{r}} \right) + \frac{\partial}{\partial \bar{\theta}} \left( \tilde{h}^3 \frac{\partial \tilde{p}}{\partial \bar{\theta}} \right) = -e_2 \tilde{r}^2 [\cos(\theta - \Omega) - \chi \cos \theta]$$  \hspace{1cm} (7)

### 3. Reynolds Equation Approach

To simplify the notations we use instead of $\tilde{h}, \tilde{r}, \tilde{p}$, the symbols $h, r, p$. Because $e_2 < 1$, (from $R_e(\chi_1 + \chi_2) < h_0$ so $e_2(1 + \chi) < 1$) we develop $p$ as powers series after $e_2$.

$$h = 1 + e_2 h_1(r, \theta), \text{ where } h_1(r, \theta) = r [\sin(\theta - \Omega) - \chi \sin \theta]$$

$$p = p_0(r, \theta) + e_2 p_1(r, \theta) + e_2^2 p_2(r, \theta) + \cdots$$  \hspace{1cm} (8)
Replacing the expression of pressure given by the relation (8), the equation (7) becomes:

\[
\frac{\partial}{\partial r} \left[ r(1 + e_2 h_1)^3 \frac{\partial}{\partial r} \left( \sum_{k \geq 0} e_k^2 p_k \right) \right] + \frac{\partial}{\partial \theta} \left[ \frac{1}{r} (1 + e_2 h_1)^3 \frac{\partial}{\partial \theta} \left( \sum_{k \geq 0} e_k^2 p_k \right) \right] = 0
\]

\[
= -e_2 r^2 [cos(\theta - \Omega) - \chi \cos \theta]
\]

Applying the partial derivation, the following relation is obtained:

\[
\left[ (1 + e_2 h_1)^3 + 3 e_2 r(1 + e_2 h_1)^2 \frac{\partial h_1}{\partial r} \sum_{k \geq 0} e_k^2 \frac{\partial p_k}{\partial r} + r(1 + e_2 h_1)^3 \sum_{k \geq 0} e_k^2 \frac{\partial^2 p_k}{\partial r^2} + \right.

\[
+ \frac{3 e_2}{r} (1 + e_2 h_1)^2 \frac{\partial h_1}{\partial \theta} \sum_{k \geq 0} e_k^2 \frac{\partial p_k}{\partial \theta} + \frac{1}{r} (1 + e_2 h_1)^3 \sum_{k \geq 0} e_k^2 \frac{\partial^2 p_k}{\partial \theta^2} = 0
\]

\[
= -e_2 r^2 [cos(\theta - \Omega) - \chi \cos \theta]
\]

Using the identity of \( e_k^2 \) coefficients gives:

- for \( e_0^2 \):

\[
\frac{\partial p_0}{\partial r} + r \frac{\partial^2 p_0}{\partial r^2} + \frac{1}{r} \frac{\partial^2 p_0}{\partial \theta^2} = 0
\]

The limit conditions \( p_0(r, \theta) = p_0(1, \theta) = 0 \) and \( p_0(r, \theta) = p_0(r, \theta + 2\pi) = 0 \) will give \( p_0 = 0 \).

- for \( e_1^2 \):

\[
\frac{\partial p_1}{\partial r} + r \frac{\partial^2 p_1}{\partial r^2} + \frac{1}{r} \frac{\partial^2 p_1}{\partial \theta^2} = -r^2 [cos(\theta - \Omega) - \chi \cos \theta].
\]

- for \( e_2^2 \):

\[
\frac{\partial p_2}{\partial r} + \left( 3h_1 + 3r \frac{\partial h_1}{\partial r} \right) \frac{\partial p_2}{\partial r} + \frac{3 r h_1}{r} \frac{\partial^2 p_2}{\partial r^2} + \frac{3 h_1 \frac{\partial^2 p_2}{\partial r^2}}{r} = 0.
\]

- for \( e_3^2 \):

\[
\frac{\partial p_3}{\partial r} + \left( 3h_1 + 3r \frac{\partial h_1}{\partial r} \right) \frac{\partial p_3}{\partial r} + \left( 3h_1^2 + 6r h_1 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_3}{\partial r} + r \frac{\partial^2 p_3}{\partial r^2} + \frac{3 r h_1}{r^2} \frac{\partial^2 p_3}{\partial r^2} + \frac{3 h_1}{r} \frac{\partial^2 p_3}{\partial r^2} + \frac{3 h_1}{r} \frac{\partial^2 p_3}{\partial r^2} = 0.
\]

For \( e_m^m \) member where \( m \geq 4 \):

\[
\frac{\partial p_m}{\partial r} + \left( 3h_1 + 3r \frac{\partial h_1}{\partial r} \right) \frac{\partial p_m}{\partial r} + \left( 3h_1^2 + 6r h_1 \frac{\partial h_1}{\partial r} \right) \frac{\partial p_m}{\partial r} + \frac{3 h_1}{r^2} \frac{\partial^2 p_m}{\partial r^2} + \frac{3 h_1^2}{r} \frac{\partial^2 p_m}{\partial r^2} + \frac{3 h_1}{r} \frac{\partial^2 p_m}{\partial r^2} + \frac{3 h_1}{r} \frac{\partial^2 p_m}{\partial r^2} = 0.
\]
The equations of the coefficients \( e_1^1 \) and \( e_2^2 \) are presented below and will be analysed:

\[
\frac{r^2}{2} \frac{\partial^2 p_1}{\partial r^2} + \frac{1}{r} \frac{\partial^2 p_1}{\partial r \partial \theta} + \frac{\partial p_1}{\partial r} = -r^2 \left( \cos(\theta - \Omega) - \chi \cos \theta \right) \tag{9}
\]

\[
\left( \frac{r^2}{2} \frac{\partial^2 p_2}{\partial r^2} + \frac{1}{r} \frac{\partial^2 p_2}{\partial r \partial \theta} + \frac{\partial p_2}{\partial r} \right) = - \left( 3h_1 + 3r \frac{\partial h_1}{\partial r} \right) \frac{\partial p_1}{\partial r} - 3 \rho h_1 \frac{\partial^2 p_1}{\partial r^2} - \frac{3h_1}{r} \frac{\partial^2 p_1}{\partial \theta^2}. \tag{10}
\]

A. Solving the equation (9)

Because \( p(r, \theta) = p(r, \theta + 2\pi) \) then \( p_1(r, \theta) = p_1(r, \theta + 2\pi) \). To solve the differential equation (9), the Fourier series is used to develop pressure \( p_1(r, \theta) \):

\[
p_1(r, \theta) = p_1^0(r) + \sum_{k \geq 1} \left[ p_1^{k1}(r) \cos k \theta + p_1^{sk}(r) \sin k \theta \right] \tag{11}
\]

Substitution of the form (11) into equation (9) gives:

\[
r \left[ \frac{r^2}{2} \frac{\partial^2 p_1^0}{\partial r^2} + \sum_{k \geq 1} \left( \frac{r^2}{2} \frac{\partial^2 p_1^{k1}}{\partial r^2} \cos k \theta + \frac{r^2}{2} \frac{\partial^2 p_1^{sk}}{\partial r^2} \sin k \theta \right) \right] - \frac{1}{r} \left[ \sum_{k \geq 1} \left( \frac{\partial p_1^{k1}}{\partial r} \cos k \theta + \frac{\partial p_1^{sk}}{\partial r} \sin k \theta \right) \right] + \frac{\partial p_1^0}{\partial r} + \sum_{k \geq 1} \left( \frac{d p_1^{k1}}{d r} \cos k \theta + \frac{d p_1^{sk}}{d r} \sin k \theta \right) = -r^2 \left[ (\cos \Omega - \chi) \cos \theta + \sin \Omega \sin \theta \right]
\]

Identification of the coefficients of equal trigonometric functions gives the following set of differential equations:

- for the free terms: \( r \frac{d p_0^0}{d r} + \frac{d p_0^0}{d r} = 0 \),
- for \( \cos \theta \): \( r \frac{d^2 p_1^{k1}}{d r^2} - \frac{1}{r} p_1^{k1} + \frac{d p_1^{k1}}{d r} = -r^2 (\cos \Omega - \chi) \)
- for \( \sin \theta \): \( r \frac{d^2 p_1^{sk}}{d r^2} - \frac{1}{r} p_1^{sk} + \frac{d p_1^{sk}}{d r} = -r^2 (\sin \Omega) \)
- for \( \cos k \theta, k > 2 \): \( r \frac{d^2 p_1^{k1}}{d r^2} - \frac{k^2}{r} p_1^{k1} + \frac{d p_1^{k1}}{d r} = 0 \)
- for \( \sin k \theta, k > 2 \): \( r \frac{d^2 p_1^{sk}}{d r^2} - \frac{k^2}{r} p_1^{sk} + \frac{d p_1^{sk}}{d r} = 0 \)

Because \( p_r = p_1 = 0 \), we write the limit conditions for \( r = r_1 \) and \( r = 1 \).

The equations (12)_1 and (12)_4,5 have trivial solutions, so:

\[
p_1^0 = 0, p_1^{k1} = 0, p_1^{sk} = 0, \text{ for } k > 2 \tag{13}
\]

The equations (12)_2,3, which differ only in polynomial terms, are particular cases of non-homogenous Euler-Cauchy equation [7]:

\[
r^2 \frac{d^2 x}{d t^2} + \frac{d x}{d t} - x = at^3 \tag{14}
\]
To solve the equation (14), setting \( t = e^s (s = \ln t) \) and \( x(t) = y(s) \) we have,
\[
\frac{ds}{dt} = \frac{dy}{ds} \left( \frac{1}{\frac{dy}{ds} \frac{1}{t}} \right),
\]
\[
\frac{d^2 x}{dt^2} = \frac{d}{dt} \left( \frac{dy}{ds} \frac{1}{t} \right) = \frac{d}{ds} \frac{dy}{ds} \frac{1}{t} - \frac{dy}{ds} \frac{1}{t^2} = \frac{d^2 y}{ds^2} \frac{1}{t} - \frac{dy}{ds} \frac{1}{t^2}
\]
We return to relation (14) and the following equation is obtained:
\[
\frac{d^2 y}{ds^2} - y = ae^{3s}
\] (15)
The differential equation presented above has a particular solution as \( y_0(s) = be^{3s} \).
Substitution of particular solution into equation (15) gives:
\[
9be^{3s} - be^{3s} = ae^{3s}
\]
where \( b = \frac{a}{8} \) and \( y_0(s) = \frac{a}{8} e^{3s} \).
The homogenous equation attached to the equation (15) is given below:
\[
\frac{d^2 \overline{y}}{ds^2} - \overline{y} = 0
\]
The characteristic equation is: \( \lambda^2 - 1 = 0 \) with \( \lambda_{1,2} = \pm 1 \). So, the solution of homogenous equation is:
\[
\overline{y}(s) = C_1 e^{s} + C_2 e^{-s}
\]
The solution of the differential equation (15) is given as:
\[
y(s) = y_0(s) + \overline{y}(s) = \frac{a}{8} e^{3s} + C_1 e^{s} + C_2 e^{-s}
\]
and the solution of the equation (14) is:
\[
x(t) = \frac{a}{8} t^3 + C_1 t + C_2 \frac{1}{t}
\]
The general solution of equation (12) is:
\[
p^{(1)}_1(r) = -\frac{1}{8} (\cos \Omega - \chi) r^3 + C_1 r + C_2 \frac{1}{r}
\]
Using the initial conditions \( p^{(1)}_1(r_i) = 0, p^{(1)}_1(1) = 0 \) we obtain a linear equation system with \( C_1 \) and \( C_2 \) as unknowns.
Solving the linear equation system, the coefficients \( C_1 \) and \( C_2 \) are computed as,
\[
C_1 = \frac{1}{8} (\cos \Omega - \chi) (r_i^2 + 1); \quad C_2 = -\frac{1}{8} (\cos \Omega - \chi) r_i^2
\]
then:
\[ p_1^1(r) = -\frac{1}{8} (\cos \Omega - \chi) \left[ r^3 - (r_i^2 + 1)r + \frac{r_j^3}{r} \right] \]  

(16)

Replacing \((\cos \Omega - \chi)\) with \(\sin \Omega\) into relation (16), the solution of the equation (12) is,

\[ p^1_1(r) = \frac{1}{8} \sin \Omega \left[ r^3 - (r_i^2 + 1)r + \frac{r_j^3}{r} \right] \]  

(17)

From relations (11), (13), (16) and (17) is deduced pressure \(p_1\) as,

\[ p_1(r, \theta) = \frac{1}{8} \sin \Omega \left[ r^3 - (r_i^2 + 1)r + \frac{r_j^3}{r} \right] \cdot \left[ (\cos \Omega - \chi) \cos \theta + \sin \Omega \sin \theta \right] \]  

(18)

**B. Solving the equation (10)**

Because \(p(r, \theta) = p(r, \theta + 2\pi)\) then \(p_2(r, \theta) = p_2(r, \theta + 2\pi)\). To solve the differential equation (10), the Fourier series is used to develop pressure \(p_2(r, \theta)\):

\[ p_2(r, \theta) = p_2^0(r) + \sum_{k \geq 1} \left[ p_2^{ck}(r) \cos k\theta + p_2^{sk}(r) \sin k\theta \right] \]  

(19)

Substitution of the relation (19) into the equation (10) gives:

\[ r \left[ \frac{d^2 p_2^0}{dr^2} + \sum_{k \geq 1} \left( \frac{d^2 p_2^{ck}}{dr^2} \cos k\theta + \frac{d^2 p_2^{sk}}{dr^2} \sin k\theta \right) \right] - \frac{1}{r} \left[ \sum_{k \geq 1} k^2 \left( p_2^{ck} \cos k\theta + p_2^{sk} \sin k\theta \right) \right] + \frac{dp_2^0}{dr} + \sum_{k \geq 1} \left( \frac{dp_2^{ck}}{dr} \cos k\theta + \frac{dp_2^{sk}}{dr} \sin k\theta \right) = \frac{3}{4} \left( 5r^3 - \frac{L^2}{r} \right) \cdot E_1 \]  

(20)

where \(E_1\) is denoted as the following expression:

\[ E_1 = \frac{1}{2} \sin 2\theta \cdot (\cos \Omega - \chi)^2 - \sin^2 \Omega - \cos 2\theta \cdot (\cos \Omega - \chi) \cdot \sin \Omega \]  

(21)
The theoretical approach of face seals pressure for hydrodynamic operating model

The coefficients identity for equal trigonometric functions from relation (20) gives the following set of differential equations:

- for the free terms: \( r \frac{d^2 p_0}{dr^2} + \frac{dp_0}{dr} = 0 \)
- for \( \cos \theta \): \( r \frac{d^2 p_1}{dr^2} - \frac{1}{r} p_2^1 + \frac{dp_1}{dr} = 0 \)
- for \( \sin \theta \): \( r \frac{d^2 p_1}{dr^2} - \frac{1}{r} p_2^1 + \frac{dp_1}{dr} = 0 \)
- for \( \cos 2\theta \): \( r \frac{d^2 p_2}{dr^2} - \frac{4}{r} p_2^2 + \frac{dp_2}{dr} = \frac{3}{4} \left( 5r^3 - \frac{r_i^3}{r} \right) \cdot \sin \Omega (\cos \Omega - \chi) \) (22)
- for \( \sin 2\theta \): \( r \frac{d^2 p_2}{dr^2} - \frac{4}{r} p_2^2 + \frac{dp_2}{dr} = \frac{3}{8} \left( 5r^3 - \frac{r_i^3}{r} \right) \cdot (\cos \Omega - \chi)^2 - \sin^2 \Omega \)
- for \( \cos k\theta, k > 3 \): \( r \frac{d^2 p_k}{dr^2} - \frac{k^2}{r} p_2^k + \frac{dp_k}{dr} = 0; \)
- for \( \sin k\theta, k > 3 \): \( r \frac{d^2 p_k}{dr^2} - \frac{k^2}{r} p_2^k + \frac{dp_k}{dr} = 0; \)

Considering the limit conditions for the unknown functions \( r = r_i \) and \( r = 1 \), the following results are obtained:

\[ p_0^0 = 0, \ p_2^1 = 0, \ p_2^2 = p_2^k = 0, \ k > 3 \] (23)

The equations (22)\(_{4,5}\) are particular cases of non-homogenous Euler-Cauchy equation [7]:

\[ t^2 \frac{d^2 x}{dt^2} + t \frac{dx}{dt} - 4x = a_0 + a_4 t^4 \] (24)

Setting \( t = e^s (s = \ln t) \) and \( x(t) = y(s) \) we have,

\[ \frac{dx}{dt} = \frac{dy}{ds} \frac{1}{t} \text{ and } \frac{d^2 x}{dt^2} = \frac{d^2 y}{ds^2} \frac{1}{t^2} - \frac{dy}{ds} \frac{1}{t^2} \]

The equation (24) becomes:

\[ \frac{d^2 y}{ds^2} - 4y = a_0 + a_4 e^{4s} \] (25)

For this kind of equation it searches a particular solution as below:

\[ y_0(s) = b_0 + b_4 e^{4s} \]

Substitution of the particular solution into the equation (25) gives:

\[ b_0 = -\frac{a_0}{a_4} \text{ and } b_4 = \frac{a_0}{a_4} \]

The homogenous equation attached to the equation (25) is given below:

\[ \frac{d^2 y}{ds^2} - 4y = 0 \]
This equation has \( \lambda^2 - 4 = 0 \) as characteristic equation, with \( \lambda_{1,2} = \pm 2 \) as solutions, so the solution of the homogenous equation is:

\[
\overline{y}(s) = C_0 e^{2s} + C_1 e^{-2s}
\]

The general solution of equation (25) is presented below:

\[
y(s) = y_0(s) + \overline{y}(s) = -\frac{a_0}{4} + \frac{a_4}{12} e^{2s} + C_0 e^{2s} + C_1 e^{-2s}
\]

Replacing the function \( t = e^s \), is obtained the general solution for the equation (25):

\[
x(t) = C_0 t^2 + C_1 \frac{1}{t^2} - \frac{a_0}{4} + \frac{a_4}{12} t^4 \quad (26)
\]

The integration constants \( C_0 \) and \( C_1 \) are determined using the boundary conditions: \( p_2'^2(r_1) = 0 \) and \( p_2'^2(1) = 0 \).

The integration constants are given below:

\[
C_0 = \frac{1}{1 + r_i^2} \left[ \frac{a_0}{4} - \frac{a_4}{12} (r_i^4 + r_i^2 + 1) \right] = \frac{15r_i^4 + 27r_i^2 + 15}{48(r_i^2 + 1)} \cdot \sin \Omega (\cos \Omega - \chi)
\]

\[
C_1 = \frac{r_i^2}{1 + r_i^2} \left[ \frac{a_0}{4} + \frac{a_4}{12} r_i^2 \right] = -\frac{1}{8} \frac{r_i^4}{1 + r_i^2} \cdot \sin \Omega (\cos \Omega - \chi)
\]

So,

\[
p_2'^2(r) = \left( \frac{1}{48} \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 - \frac{1}{8} \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} - \frac{3r_i^2}{16} - \frac{15r_i^4}{48} r_i^4 \right) \cdot \sin \Omega (\cos \Omega - \chi) \quad (27)
\]

Replacing \( \sin \Omega (\cos \Omega - \chi) \) with \( \frac{1}{2} \left[ (\cos \Omega - \chi)^2 - \sin^2 \Omega \right] \) into relation (27) the solution of the equation (22) is computed,

\[
p_2'^2(r) = \frac{1}{2} \left( -\frac{1}{48} \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 + \frac{1}{8} \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} + \frac{3r_i^2}{16} + \frac{15r_i^4}{48} r_i^4 \right) \cdot \left[ (\cos \Omega - \chi)^2 - \sin^2 \Omega \right] \quad (28)
\]

The pressure \( p_2(r, \theta) \) is deduced by substitution of the relations (27) and (28) into (19):

\[
p_2(r, \theta) = p_2'^2 + p_2'^2 = \left( -\frac{1}{48} \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 + \frac{1}{8} \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} + \frac{3r_i^2}{16} + \frac{15r_i^4}{48} r_i^4 \right) \cdot \left[ \frac{1}{2} \sin 2\theta (\cos \Omega - \chi)^2 - \sin^2 \Omega \right] \quad (29)
\]

It returns to initial notations as \( \overline{p}, \overline{\theta}, \overline{\tau}, \overline{r} \) and non-dimensional pressure becomes:

\[
\overline{p} = e_2 \overline{p}_1(r, \theta) + e_2^2 \overline{p}_2(r, \theta) = -\frac{e_2}{8} \left[ \overline{r}^3 - (\overline{r}_i^2 + 1) \overline{r} + \overline{r}_i^2 \right] \cdot \left[ (\cos \Omega - \chi) \cos \theta - \sin \Omega \sin \theta \right] + \frac{e_2^2}{8} \left( -\frac{1}{6} \frac{15r_i^4 + 27r_i^2 + 15}{r_i^2 + 1} r_i^2 + \frac{r_i^4}{1 + r_i^2} \frac{1}{r_i^2} + \frac{3r_i^2}{2} + \frac{15r_i^4}{6} r_i^4 \right) \cdot \left[ \frac{1}{2} \sin 2\theta (\cos \Omega - \chi)^2 - \sin^2 \Omega \right] \quad (30)
\]
4. CONCLUSION

The proposed method using power series and Fourier series gives analytical solution for the pressure distribution within face seal interface. In this way, the influence of the constructive parameters on the interface pressure is shown.

References


