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FIRST PASSAGE TO A SEMI-INFINITE LINE FOR A TWO-DIMENSIONAL WIENER PROCESS

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Abstract Assume that $X(t)$ and $Y(t)$ are independent Wiener processes with drift -1 and 0, respectively, and diffusion coefficient equal to 2 (in both cases). Let $I(x, y)$ be the indicator function of the event $\{\tau(x, y) < \infty\}$, where $\tau(x, y) = \inf\{t > 0 : Y(t) = 0, X(t) \geq 0 \mid X(0) = x, Y(0) = y\}$, in which $y \neq 0$ or $x < 0$. We obtain an explicit expression for $\phi(x, y) = E\left[e^{-aX[\tau(x,y)]}I(x, y)\right]$. An application to an optimal control problem is also presented.

Keywords: Brownian motion, first exit time, Kolmogorov backward equation, optimal stochastic control.

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1. INTRODUCTION

We consider the two-dimensional Wiener process $(X(t), Y(t))$ defined by the system of stochastic differential equations

$$\begin{aligned} dX(t) &= -dt + \sqrt{2}\, dB_1(t), \\ dY(t) &= \sqrt{2}\, dB_2(t), \end{aligned}$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions. Let

$$\tau(x, y) = \inf\{t > 0 : Y(t) = 0, X(t) \geq 0 \mid X(0) = x, Y(0) = y\},$$

where $y \neq 0$ or $x < 0$. We define

$$I(x, y) = \begin{cases} 1 & \text{if } \tau(x, y) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $I(x, y)$ is the indicator function of the event $\{\tau(x, y) < \infty\}$.

The function

$$\phi(x, y) := E\left[e^{-aX[\tau(x,y)]}I(x, y)\right], \tag{1}$$

where $a > 0$, satisfies the Kolmogorov equation

$$\phi_{yy} + \phi_{xx} - \phi_x = 0,$$

and is subject to the conditions $\phi(x, 0) = e^{-ax}$ if $x \geq 0$, and $\phi(x, y) \to 0$ if $x^2+y^2 \to \infty$.

Since the stochastic processes $X(t)$ and $Y(t)$ are independent, if we replace the first-passage time $\tau(x, y)$ by

$$\tau_0(x, y) = \inf\{t > 0 : Y(t) = 0 \mid X(0) = x, Y(0) = y\},$$

where $y \neq 0$ and $x \in \mathbb{R}$, then the function $\phi_0(x, y)$ that corresponds to $\phi(x, y)$ is easy to obtain. Indeed, first we can state that $\tau_0(x, y)$ actually does not depend on the variable x. Moreover, it is well known that $P[\tau_0(y) < \infty] = 1$. Therefore, we can write that

$$\phi_0(x, y) = E\left[e^{-aX[\tau_0(y)]} \,\middle|\, X(0) = x, Y(0) = y\right].$$

Next, making use of the fact that $X(t)$ has a Gaussian distribution with mean $x - t$ and variance $2t$, and of the following formula for the probability density function of the random variable $\tau_0(y)$ (see Lefebvre [3], for instance):

$$f_{\tau_0(y)}(t) = \frac{|y|}{\sqrt{4\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} \quad \text{for } t > 0, \tag{2}$$

we can derive an explicit (and exact) expression for $\phi_0(x, y)$ by conditioning on the random variable $\tau_0(y)$. That is, we write that

$$\begin{aligned}
\phi_0(x, y) &= E\left[e^{-aX[\tau_0(y)]} \,\middle|\, X(0) = x, Y(0) = y\right] \\
&= E\left[E\left[e^{-aX[\tau_0(y)]} \,\middle|\, \tau_0(y), X(0) = x, Y(0) = y\right]\right] \\
&= \int_0^\infty E\left[e^{-aX[\tau_0(y)]} \,\middle|\, \tau_0(y) = t, X(0) = x, Y(0) = y\right] f_{\tau_0(y)}(t) dt \\
&= \int_0^\infty \int_{-\infty}^\infty \frac{e^{-au}}{\sqrt{4\pi t}} \exp\left\{-\frac{(u - x + t)^2}{4t}\right\} \frac{|y|}{\sqrt{4\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} du dt \\
&= \frac{|y|}{4\pi} \int_0^\infty \int_{-\infty}^\infty \frac{e^{-au}}{t^2} \exp\left\{-\frac{(u - x + t)^2 + y^2}{4t}\right\} du dt.
\end{aligned}$$

The main difficulty in computing the function $\phi(x, y)$ stems from the fact that it is discontinuous on the boundary $y = 0$. A related problem for which the function is discontinuous on the boundary has been considered by the author and Whittle (see Lefebvre and Whittle [4]) in an optimization context. They defined the two-dimensional diffusion process $(X(t), Y(t))$ by

$$dX(t) = Y(t)\, dt, \tag{3}$$

$$dY(t) = bu(t)dt + \sigma\, dB(t), \tag{4}$$

where $b \neq 0$ is a constant, $u(t)$ is the control variable and $B(t)$ is a standard Brownian motion. Hence, $X(t)$ is a controlled integrated Brownian motion. They looked for the control u^* that minimizes the expected value of the cost function

$$J(x) = \int_0^{T_d(x)} \left(\frac{1}{2} q u^2(t) - \lambda \right) dt,$$

in which

$$T_d(x) = \inf\{t > 0 : |X(t)| = d \mid X(0) = x\},$$

with $-d < x < d$, and q and λ are positive constants. By appealing to a theorem proved in Whittle [5], the authors were able to express the value of u^* in terms of the following mathematical expectation for the uncontrolled process $(\xi(t), \eta(t))$ that corresponds to $(X(t), Y(t))$:

$$\phi_1(x) := E\left[e^{\lambda \tau_d(x)/\alpha} \,\middle|\, \xi(0) = x \right],$$

where

$$\alpha = \frac{\sigma^2 q}{b^2}$$

and $\tau_d(x)$ is the same as $T_d(x)$, but for the process $(\xi(t), \eta(t))$ obtained by setting $u(t)$ equal to 0 in (4).

The function $\phi_1(x)$ is also discontinuous on the boundaries $x = d$ and $x = -d$, because the process $X(t)$ cannot hit the boundary $x = d$ for the first time with $y < 0$ or, equivalently, the boundary $x = -d$ with $y > 0$.

The authors were not able to derive an exact expression for $\phi_1(x)$. Instead, they used a technique that enabled them to obtain an approximate solution for the optimal control.

Actually, a few years later, Lachal [2] considered, in particular, the problem of computing the probability density function of the random variable

$$\tau_b(x, y) := \inf\{t > 0 : X(t) = b \mid X(0) = x, Y(0) = y\}$$

for the two-dimensional diffusion process $(X(t), Y(t))$ defined by

$$\begin{aligned} dX(t) &= Y(t)\, dt, \\ dY(t) &= dB(t). \end{aligned}$$

That is, $X(t)$ is the integral of the standard Brownian motion $Y(t)$. He derived the following exact expression:

$$\begin{aligned} f_{\tau_b(x,y)}(t) &= \epsilon \left[\sqrt{\frac{3}{2\pi}} \left(\frac{3}{2} \frac{b - x}{t^{5/2}} - \frac{1}{2} \frac{y}{t^{3/2}} \right) \exp\left\{ -\frac{3(b - x - ty)^2}{2t^3} \right\} \right. \\ &\quad \left. + \int_0^\infty z dz \int_0^t f_{\tau_0(0,-z)}(s) q(x, y; b, z; t - s) ds \right], \end{aligned}$$

where

$$f_{\tau_0(0,-z)}(s) = \int_0^\infty \frac{3\mu}{\sqrt{2}\pi s^2} \exp\left\{-\frac{2}{s}(z^2 - \mu z + \mu^2)\right\} d\mu \int_0^{4\mu z/s} e^{-3\theta/2} \frac{d\theta}{\sqrt{\pi\theta}},$$

in which ϵ is the sign of $(b - x)$, $z > 0$ and

$$q(x, y; u, v; t) = p(x, y; u, v; t) - p(x, y; u, -v; t),$$

the function $p(x, y; u, v; t)$ being the joint density function of the random vector $(X(t), Y(t))$, which is known to be

$$\begin{aligned} p(x, y; u, v; t) \quad = \quad & \frac{\sqrt{3}}{\pi t^2} \exp\left\{-\frac{6}{t^3}(u - x - ty)^2 + \frac{6}{t^2}(u - x - ty)(v - y)\right. \\ & \left. -\frac{2}{t}(v - y)^2\right\}. \end{aligned}$$

We see that the exact solution to such a one-boundary problem is quite complicated, and we can expect the solution in the case of a two-boundary problem to be even more complicated. In the context of an optimization problem, such as in Lefebvre and Whittle [4], this exact solution would not have been very useful, at any rate, because one must be able to give an expression for the optimal control that the optimizer can actually implement.

In Section 2, by making use of the Wiener-Hopf technique, we will calculate the Fourier transform of $\phi(x, y)$. We will invert this transform in the case when $y = 0$. Finally, with the help of probabilistic arguments, we will obtain an explicit expression for $\phi(x, y)$.

In Section 3, an application to an optimal control problem will be presented, and we will conclude this work with a few remarks in Section 4.

2. COMPUTATION OF THE FUNCTION $\phi(X, Y)$

To obtain an exact expression for the function $\phi(x, y)$, we will first compute its Fourier transform, with the help of the Wiener-Hopf technique. Let

$$\Phi(\omega, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x, y) e^{i\omega x} dx.$$

We find that $\Phi(\omega, y)$ satisfies the ordinary differential equation

$$\frac{d^2\Phi(\omega, y)}{dy^2} - (\omega^2 - i\omega)\Phi(\omega, y) = 0. \tag{5}$$

The Wiener-Hopf technique consists in assuming that $\phi(x, 0)$ is known for all $x \in \mathbb{R}$, and not only for $x \geq 0$. We write that

$$\phi(x, 0) = \begin{cases} e^{-ax} & \text{if } x \geq 0, \\ u(x) & \text{if } x < 0, \end{cases}$$

where $u(x)$ is a function that will need to be determined later.

Next, the solution of Eq. (5) that tends to 0 as $|y|$ increases to ∞ is

$$\Phi(\omega, y) = \left[U(\omega) + \frac{1}{\sqrt{2\pi}}\frac{1}{1 - i\omega}\right]\exp\left(-|y|\sqrt{\omega^2 - i\omega}\right),$$

where

$$U(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} u(x)e^{i\omega x}dx$$

is the Fourier transform of $u(x)$.

It can be shown (see Zwillinger [6], pp. 383-386) that, when $a = 1$,

$$U(\omega) = -\frac{1}{\sqrt{\omega - i}}\left[\frac{\sqrt{\omega - i} - \sqrt{-2i}}{\sqrt{2\pi}(1 - i\omega)}\right],$$

from which we deduce that

$$\Phi(\omega, y) = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{-2i}}{\sqrt{\omega - i}(1 - i\omega)}\exp\left(-|y|\sqrt{\omega^2 - i\omega}\right). \tag{6}$$

Remark 2.1. *There are a few misprints in Zwillinger's book. In particular, in Eq. (104.4), p. 384, it should be ϕ_x instead of ϕ_y. Moreover, the formula for the function $U(\omega)$ should be as above, rather than as in Eq. (104.17). That is, it is $(1 - i\omega)$ in the denominator, instead of $\sqrt{1 - i\omega}$.*

Next, the formula for $\Phi(\omega, y)$ in the case when $a > 0$ can be found in Davies [1], p. 281:

$$\Phi(\omega, y) = \frac{1}{\sqrt{2\pi}}\frac{e^{-i\pi/4}\sqrt{1 + a}}{\sqrt{\omega - i}(a - i\omega)}\exp\left(-|y|\sqrt{\omega^2 - i\omega}\right). \tag{7}$$

Remark 2.2. *In Davies [1], the Fourier transform of $f(x)$ is defined as follows:*

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx.$$

Therefore, we must multiply the formula on p. 281 by $1/\sqrt{2\pi}$. It is easy to check that if we set a equal to 1 in (7), then we indeed retrieve Eq. (6).

In order to obtain the function $\phi(x, y)$ that we are looking for, we must invert the Fourier transform $\Phi(\omega, y)$. However, it turns out to be a very difficult task in the general case when $y \in \mathbb{R}$. We can, however, invert this transform when $y = 0$. Indeed, making use of the mathematical software *Maple*, we find that

$$\phi(x, 0) = \begin{cases} e^{-ax} & \text{if } x \geq 0, \\ e^{-ax}\left[1 - \operatorname{erf}(\sqrt{-x}\sqrt{1 + a})\right] & \text{if } x < 0, \end{cases} \tag{8}$$

in which *erf* is the *error function*.

Remark 2.3. *In Maple, the Fourier transform of $f(x)$ is defined as follows:*

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

Now, with the help of Eq. (8) and probabilistic arguments, we can obtain an explicit expression for $\phi(x, y)$ for any real y. First, we define (as in the Introduction)

$$\tau_0(y) = \inf\{t > 0 : Y(t) = 0 \mid Y(0) = y\},$$

where $y \neq 0$. That is, $\tau_0(y)$ is the first-passage time to 0 for the process $Y(t)$, independently of the value of $X[\tau_0(y)]$.

Next, we condition on $\tau_0(y)$ and $X[\tau_0(y)]$:

$$\begin{aligned} \phi(x, y) &= \int_0^{\infty} \int_{-\infty}^{\infty} E\left[e^{-aX[\tau(x,y)]} I(x, y) \mid X[\tau_0(y)] = x_1, \tau_0(y) = t\right] \\ &\quad \times f_{X(\tau_0)|\tau_0}(x_1 \mid t) f_{\tau_0}(t) dx_1 dt. \end{aligned}$$

We can write that

$$\begin{aligned} \phi(x, y) &= \int_0^{\infty} \int_{-\infty}^{0} \phi(x_1, 0) f_{X(\tau_0)|\tau_0}(x_1 \mid t) f_{\tau_0}(t) dx_1 dt \\ &\quad + \int_0^{\infty} \int_0^{\infty} e^{-ax_1} f_{X(\tau_0)|\tau_0}(x_1 \mid t) f_{\tau_0}(t) dx_1 dt. \end{aligned}$$

Finally, we mentioned in the Introduction that $X(\tau_0) \mid \{\tau_0 = t\} \sim \mathrm{N}(x-t, 2t)$ and the probability density function of the random variable $\tau_0(y)$ is given in Eq. (2). Hence, we can now state the following proposition.

Proposition 2.1. *The function $\phi(x, y)$ defined in (1) is given by*

$$\begin{aligned} \phi(x, y) &= \int_0^{\infty} \int_{-\infty}^{0} e^{-ax_1} \left[1 - \operatorname{erf}(\sqrt{-x_1}\sqrt{1+a})\right] \\ &\quad \times \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{1}{4t}(x_1 - x + t)^2\right\} \frac{|y|}{2\sqrt{\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} dx_1 dt \\ &\quad + \int_0^{\infty} \int_0^{\infty} e^{-ax_1} \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{1}{4t}(x_1 - x + t)^2\right\} \\ &\quad \times \frac{|y|}{2\sqrt{\pi t^3}} \exp\left\{-\frac{y^2}{4t}\right\} dx_1 dt. \end{aligned}$$

In the next section, we will briefly mention a possible application of the previous proposition in stochastic optimal control.

3. AN OPTIMAL CONTROL APPLICATION

In Lefebvre and Whittle [4], the authors used the process defined by (3), (4) as a rudimentary model for an airplane. The process $X(t)$ denoted the height of the airplane, the value $x = -d$ represented ground level and $x = d$ was a height at which the airplane was likely to be detected by a radar. The aim of the optimizer was to try to make $X(t)$ remain in the interval $(-d, d)$ for as long as possible.

A possible application of the model considered in this paper is the following: assume that an airplane is moving from right to left, from $X(0) = x > 0$, as it approaching the runway. The initial height of the airplane is $Y(0) = y > 0$. The optimizer wants the plane to reach the ground, represented by the value $y = 0$, at time $\tau(x, y)$, with $X[\tau(x, y)] \geq 0$. That is, the value $x = 0$ denotes here the end of the runway.

Consider the controlled two-dimensional diffusion process defined by the system of stochastic differential equations

$$\begin{aligned} dX_1(t) &= -dt + b_1 u_1(t)dt + \sqrt{2}\, dB_1(t), \\ dX_2(t) &= b_2 u_2(t)dt + \sqrt{2}\, dB_2(t), \end{aligned}$$

where the constants b_1 and b_2 are different from zero.

Assume that the cost function, whose expected value we want to minimize, is given by

$$J_0(x, y) = \int_0^{\tau(x,y)} \frac{1}{2}[q_1 u_1^2(t) + q_2 u_2^2(t)]dt + X[\tau(x, y)] - \gamma \ln I(x, y),$$

where q_1, q_2 and γ are positive constants. Thus, the pilot should try to land his/her airplane as close as possible to the end of the runway, taking the quadratic control costs into account. Notice that we give an infinite penalty if the landing does not take place in finite time. In practice, we could replace $I(x, y)$ by

$$I_0(x, y) = \begin{cases} 1 & \text{if } \tau(x, y) < t_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $t_0 \in [0, \infty)$.

If the constant γ is such that

$$2 = \gamma \frac{b_i^2}{q_i} \quad \text{for } i = 1, 2,$$

then we can use the theorem in Whittle [5] to express the optimal control u_i^*, for $i = 1, 2$, in terms of the function $\phi(x, y)$ given in Proposition 2.1. More precisely, the optimal control would be given by

$$u_i^* = \gamma \frac{b_i}{q_i} \frac{\phi_{x_i}(x, y)}{\phi(x, y)} = \frac{2}{b_i} \frac{\phi_{x_i}(x, y)}{\phi(x, y)},$$

with a replaced by $1/\gamma$ in Proposition 2.1.

4. CONCLUSION

Thanks to the Fourier transform of the mathematical expectation $\phi(x, y)$ defined in (1) that was computed in Zwillinger [6] and Davies [1], we were able to obtain an explicit and exact expression for the function $\phi(x, y)$. In Section 3, we presented a possible application of the results to an optimization problem.

As we have already mentioned, the main difficulty in the computation of the function $\phi(x, y)$ is the fact that it is discontinuous on the boundary $y = 0$. We saw that the solution to such a problem, like the one found by Lachal [2], is generally quite complicated. The expression that we have given in Proposition 2.1 is rather involved, but it is still usable in an optimization context.

As a sequel, we could consider other first-passage problems for two-dimensional diffusion processes for which there is a discontinuity on the boundary. The Wiener-Hopf technique is well adapted to compute the Fourier transform of the function we want to determine in such a case. Then, the problem of inverting this Fourier transform will generally be very difficult. Therefore, we could again appeal to probabilistic arguments to solve this type of problems.

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