ON SURVIVAL AND RUIN PROBABILITIES IN A PERTURBED RISK MODEL

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Abstract

We analyze the ruin probability in infinite and finite time horizon for some risk models. This is the probability that an insurer will face ruin when it starts with some initial reserve and is subjected to independent and identical distributed claims over time. Closed form expressions for this probability are available only in few cases, therefore actuaries dwell with approximations. In this paper, we consider a perturbed risk model in which a current premium rate will be adjusted after a claim occurs and the adjusted rate is determined by the amount of the claim. At the same time, in this risk model the surplus of the insurer is perturbed by a standard Brownian motion which is independent of the number of claims process and of claim sizes. We focus on an integro-differential equation for the survival probabilities and on a discrete-time model for the ruin probabilities. We give a numerical illustration on the latter risk model.

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1. INTRODUCTION

An actuarial risk model has two main components: one characterizing the frequency of events and another describing the claim size resulting from the occurrence of a catastrophic event. In examining the nature of the risk associated with a portfolio of policies, it is often of interest to assess how the portfolio performs over an extended period of time. One approach focuses on the use of ruin theory, which is concerned with the insurer’s surplus, i.e. the excess of the income over the outgo, or claims paid, with respect to a portfolio of business. Ruin is said to occur if the insurer’s surplus reaches a specified lower bound. The ruin probability is the probability of suchlike event.

There are various ways to model the surplus process of an insurance company and to define the ruin probability as well as the survival probability.

In this paper, we consider a perturbed risk model in which a current premium rate will be adjusted after a claim occurs and the adjusted rate is determined by the
amount of the claim. At the same time, in this risk model the surplus of the insurer is perturbed by a standard Brownian motion which is independent of the number of claims process and of claim sizes. We focus on an integro-differential equation for the survival probabilities and on a discrete-time model for the ruin probabilities, as well as for the level of the insurer’s surplus process.

The paper is organized as follows. Section 2 discusses the Cramer-Lundberg risk model using a martingale approach. Section 3 describes the diffusion approximation for the maximum of a random walk. Section 4 is devoted to perturbed risk models: in continuous time, involving survival probabilities, and in discrete time, giving the ruin probability after a certain number of periods. Section 5 contains a numerical illustration on the latter risk model.

2. MARTINGALE APPROACH IN THE CRAMER-LUNDBERG RISK MODEL

The theory of martingales provides a quick way of calculating the ruin probability. We shall assume that the evolution of the capital of a insurance company takes place in a probability space \((\Omega, K, P)\) as follows.

The initial capital (initial reserve) is \(U(0) = u > 0\). Insurance premiums are cashed continuously at a constant rate \(c > 0\) and claims are received at random times \((\text{moments}) T_1, T_2, \ldots (0 = T_0 < T_1 < T_2 < \ldots)\) and the amounts to be paid out at these moments are described by the nonnegative random variables \(X_1, X_2, \ldots\). Thus, taking into account receipts and claims, the capital \(U(t)\) at time \(t \geq 0\) is

\[
U(t) = u + ct - S(t),
\]

where \(S(t) = \sum_{i \geq 1} X_i \cdot I(T_i \leq t)\), with \(S(0) = 0\). Let \(\Theta_i = T_i - T_{i-1}, i \geq 1\), be the inter-ocurrence times; \(N(t) = \sum_{i \geq 1} I(T_i \leq t), N = \{N(t); t \geq 0\}\), with \(N(0) = 0\), is the claim arrival process.

The first time the insurance company’s capital becomes less than zero is the time of ruin

\[
\tau = \inf \{t \geq 0 : U(t) < 0\},
\]

and \(\tau = \infty\), if \(U(t) \geq 0, \forall t \geq 0\). The probability of ruin is

\[
\Psi(u) = P(\tau < \infty),
\]

and the probability of ruin before some moment \(T\) is

\[
\Psi(u, T) = P(\tau \leq T).
\]

The Cramer-Lundberg model is characterized by the following assumptions:

i) the random variables \(\Theta_i, i \geq 1\) are independent and identically distributed (iid) having an exponential distribution \(\text{Exp}(\lambda)\), expectation \(1/\lambda\);
ii) the random variables $X_i, i \geq 1$ are iid with cumulative distribution function (cdf) $B(x)$ such that $B(0) = 0, \mu_B = \int_{\mathbb{R}_+} x dB(x) < \infty$;

iii) the sequences $(X_i)_{i \geq 1}$ and $(T_i)_{i \geq 1}$ are independent.

Since $(T_k > t) = (N(t) < k)$, the stochastic process $N$ is a homogenous Poisson process with parameter rate $\lambda > 0$. Let $\hat{B}(\omega) = E[e^{\omega X}] = \int_{\mathbb{R}_+} e^{\omega x} dB(x)$ be the moment generating function (mgf), $\tilde{B}(\omega)$ be the Laplace-Stieltjes transform, $k(\omega) = \log \hat{B}(\omega)$ be the cumulant generating function (cdf), $h(\omega) = \int_{\mathbb{R}_+} (e^{\omega x} - 1) dB(x)$ and $g(\omega) = \lambda \cdot h(\omega) - c \cdot \omega$. We have $h(\omega) = \tilde{B}(-\omega) - 1$, for all $\omega \geq 0$.

We consider the exponential family $\{B_\omega\}$ generated by $B$, i.e. $B_\omega(dx) = e^{\omega x - k(\omega)} B(dx)$ or equivalently, in terms of the cdf of $B_\omega$, $k_\omega'(\alpha) = k(\alpha + \omega) - k(\omega)$.

It is natural to consider the models which have the property that there exists a constant $\rho$ such that $\frac{1}{t} \sum_{i=1}^{N(t)} X_i \overset{a.s.}\longrightarrow \rho, t \to \infty$, $\rho$ is the average amount of claim per unit of time. In this model, it is easy to see that $\rho = \lambda \mu$, i.e. on average, $\lambda$ claims arrive per unit of time with $\mu$ the mean of a single claim, and also $\lim_{t \to \infty} E[\frac{1}{t} \sum_{i=1}^{N(t)} X_i] = \rho$. We consider the safety loading or the loading $\theta$ defined as the relative amount by which the premium rate $c$ exceeds $\rho$; it is necessary for the net profit condition. Indeed, from assumption (iii) we find that

$$E[U(t) - U(0)] = ct - E[S(t)] = ct - \sum_i E[X_i] E[I(T_i \leq t)] = ct - \mu \sum_i P(T_i \leq t) = ct - \mu \sum_i P(N(t) \geq i) = ct - \mu E[N(t)] = t(c - \lambda \mu).$$

Thus, a natural requirement for an insurance company to operate with a clear profit is that $c > \lambda \mu$.

Taking $X_0 = 0$, we find for $r > 0$ with $h(r) < \infty$,

$$E[e^{-r(U(t) - U(0))}] = e^{-rct} E\left[e^{\sum_{i=1}^{N(t)} X_i}\right] = e^{-rct} \sum_{n=0}^{\infty} E\left[e^{\sum_{i=1}^{n} X_i}\right] P(N(t) = n) = e^{-rct} \sum_{n=0}^{\infty} (1 + h(r))^n \frac{e^{-\lambda t} (\lambda t)^n}{n!} = e^{\theta(h(r) - cr)} = e^{\theta g(r)}.$$
Analogously, it can be shown that for any \( s < t \), \( E[e^{-r(U(t) - U(s))}] = e^{(t-s)g(r)} \).

Let \( F_s = \sigma(U(s), s \leq t) \). Since the stochastic process \( U = \{U(t); t \geq 0\} \) is a process with independent increments, we have

\[
E[e^{-r(U(t) - U(s))} | F_s] = E[e^{-r(U(t) - U(s))}] = e^{(t-s)g(r)},
\]

then

\[
E[e^{-r(U(t) - U(s))} | F_s] = e^{-rU(s) - sg(r)}.
\]

Denoting \( Z_t = e^{-rU(t) - tg(r)} \), \( t \geq 0 \), we have \( E[Z_t | F_s] = Z_s, s \leq t \) is a continuous analogue of the martingale property.

The stochastic process \( Z = \{Z_t; t \geq 0\} \) is nonnegative with \( E[Z_t] = e^{-rt} < \infty \), thus the stochastic process with continuous time is a martingale. Therefore, \( E[Z_{t\wedge r}] = E[Z_0] \) for any Markov time \( r \) (in particular, for \( r \) stopping time). For time \( r = \tau \), we have

\[
e^{-ru} = E[e^{-r(U(\tau) - U(\tau))} g(\tau)]
\]

\[
\geq E[e^{-r(U(\tau) - (t \wedge \tau))} | \tau \leq t] \tau \leq t \)\]

\[
= E[e^{-rU(t) - rg(\tau)} | \tau \leq t] \tau \leq t \)
\]

\[
\geq E[e^{-rg(\tau)} | \tau \leq t] \tau \leq t \)
\]

\[
\geq \inf_{0 \leq s \leq t} e^{-rg(s)} \tau \leq t \).
\]

So,

\[
P(\tau \leq t) \leq \frac{e^{-ru}}{\inf_{0 \leq s \leq t} e^{-rg(s)}} = e^{-ru} \sup_{0 \leq s \leq t} e^{rg(r)}.
\]

Clearly, \( g(0) = 0, g'(0) = \lambda \mu - c < 0, \) and \( g''(r) = \lambda h''(r) \geq 0 \). There exists a unique positive value \( r = \gamma \) so that \( g(\gamma) = 0 \). Because

\[
\int_0^\infty e^{rx}(1 - B(x)) \, dx = \int_0^\infty \int_0^\infty e^{rx} dB(y) \, dx
\]

\[
= \int \left( \int_0^\infty e^{rx} dx \right) dB(y)
\]

\[
= \frac{1}{r} h(r),
\]

\( \gamma \) may be asserted to be the unique root of the equation

\[
\lambda \int_0^\infty e^{rx}(1 - B(x)) \, dx = c.
\]
Let \( r = y \) (\( y \) is called the adjustment coefficient), then for any \( t > 0 \),

\[ P(\tau \leq t) \leq e^{-\gamma t}, \]

whence \( P(\tau < \infty) \leq e^{-\gamma u} \). But \( \Psi(u) = P(\tau < \infty) \) is the ruin probability, so the Lundberg’s inequality is obtained.

**Proposition 2.1. (Lundberg’s Inequality)** For all \( u \geq 0 \),

\[ \Psi(u) \leq e^{-\gamma u}. \tag{5} \]

**Proposition 2.2. (Cramer’s asymptotic ruin formula)** If the adjustment coefficient \( \gamma \) exists, then

\[ \Psi(u) \sim \Psi_{CL}(u) = C \cdot e^{-\gamma u}, \quad u \to \infty, \tag{6} \]

where \( C = \frac{e^{-\rho}}{\lambda \hat{B}^\gamma} \).

**Remark 1.** If \( B \sim \text{Exp}(\alpha) \), then \( \Psi_{CL}(u) = \Psi(u) \).

### 3. DIFFUSION APPROXIMATION FOR THE MAXIMUM OF A RANDOM WALK

Closed form expressions for the ruin probability are available only in few cases, therefore actuaries are interested in approximations. There is a huge amount of research in this direction, and in this paper we focus on the diffusion approximation.

The idea behind the diffusion approximation is first to approximate the claim surplus process by a Brownian motion with drift by matching the two first moments.

Let \( \{X_n; n \geq 1\} \) be a sequence of iid random variables and let \( S = \{S_n; n \geq 0\} \) be its associated random walk with drift \( \mu \). The aim is to develop high accuracy approximations for the distribution of the maximum random variable \( M = \max\{S_n; n \geq 0\} \), which can be thought as the maximum of the aggregate loss or claim.

Clearly, \( -\mu = E(X_1) \) must be negative in order that \( M \) is finite-valued. For \( u > 0 \), \( \{M > u\} = \{\tau(u) < \infty\} \), where \( \tau(u) = \inf\{n \geq 1 : S_n > u\} \), so that calculating the tail of \( M \) is equivalent to calculating a level crossing probability for the random walk \( S \).

In insurance risk theory, \( P(\tau(u) < \infty) \) is the probability that an insurer will face ruin when the initial reserve is \( u \) and is subjected to iid claims over time. One important approximation holds as \( \mu \downarrow 0 \). This asymptotic regime corresponds in risk theory to the setting in which the safety loading \( \theta \) is small (i.e. the premium charged is close to the typical pay-out for claims). The approximation

\[ P(M > u) \approx \exp\left(-2\mu u/\sigma^2\right) \tag{7} \]

is valid as \( \mu \downarrow 0 \), where \( \sigma^2 = \text{Var}(X_1) \). Because the right hand side of (7) is the exact value of the level crossing probability for the natural Brownian approximation to the random walk \( S \), (7) is often called the diffusion approximation to the distribution of \( M \).
There are applications for which the diffusion approximation gives poor results. Siegmund (1979) suggested a so-called “corrected diffusion approximation” (CDA) that reflects information in the increment distribution beyond the mean and variance. Blanchet and Glynn (2006) developed this method to the full asymptotic expansion initiated by Siegmund.

The first problem considered by Siegmund is to find the expected value of the maximum of a random walk with small, negative drift, and the second problem is to find the distribution of the same quantity.

The result in the first case is the following: consider an exponential family $P_{\omega}$, $\omega$ belongs to a neighborhood of 0, such that under $P_{\omega}$, $X_1, X_2, \ldots$ are independent random variables with density function $\exp(\omega x - k(\omega))$ relative to a non-arithmetic distribution $F$, where $k(\omega)$ is the cumulant generating function. We assume that the problem is normalized such that $E_0[X_1] = k'(0) = 0$, $\text{Var}_0[X_1] = k''(0) = 1$.

Let $S_n = \sum_{i=1}^n X_i$, $\tau(u) = \inf \{n : S_n > u\}$, $\tau_+ = \tau(0)$ and $M = \sup_n \{S_n\}$, which is almost surely finite if $\omega < 0$. Then, as $\omega \not\to 0$, $E_{\omega}[M] = \frac{1}{\Delta} - \frac{E_0[S_{\tau_+}^2]}{2E_0[S_{\tau_+}]} + o(\Delta)$, where $\Delta = \omega_1 - \omega$, and $\omega_1 > 0$ is such that $k(\omega_1) = k(\omega)$. The random walk is assumed to belong to a translation family, i.e., $P_{\omega}(X_1 \in A) = P_0(X_1 - \omega \in A)$, where $E_0[X_1^2] < \infty$. Then, we have $E_{\omega}[M] = \frac{1}{\Delta} - \frac{E_0[S_{\tau_+}^2]}{2E_0[S_{\tau_+}]} + o(1)$, which is the result of Siegmund in the form he gave it. From the Wiener-Hopf factorization it is not hard to show that

$$\frac{E_0[S_{\tau_+}^2]}{E_{\omega}S_{\tau_+}^2} + \frac{E_0\Delta}{E_{\omega}\Delta} = \frac{E_0X_1^2}{\Delta^2}.\tag{8}$$

The distribution of the maximum is given considering such probabilities as $P_{\omega}(\tau(u) < \infty) = P_{\omega}(M > u)$. The appropriate normalization in the exponential family case is to take $u = \frac{2\epsilon}{\Delta}$, in which case it was showed that as $\omega \not\to 0$,

$$P_{\omega}(\tau\left(\frac{2\epsilon}{\Delta}\right) < \infty) = e^{-2\epsilon} \left(1 - \frac{E_0[S_{\tau_+}^2]}{2E_0[S_{\tau_+}]} + o(\Delta)\right).\tag{8}$$

The probability $\Psi(u)$ of ruin in a compound Poisson risk process $U = \{U(t) : t \geq 0\}$, with initial reserve $u$, is defined as $\Psi(u) = P(\inf_{t \geq 0} U(t) < 0)$, assuming the conditions of the Cramer-Lundberg model. Also, the net premium is considered to be received at a constant rate $c$ over time, $c = (1 + \theta)\lambda\mu_B$, where $\theta > 0$ is the relative safety loading. Thus the insurance surplus at time $t$ is

$$U(t) = u + ct - S(t), t \geq 0.\tag{9}$$

The standard diffusion approximation (Grandell 1977) is

$$\Psi(u) \approx \Psi_D(u) = \exp\left(-2\theta u \frac{\mu_B}{\mu_B^2 + \sigma_B^2}\right),$$

where $\sigma_B^2$ denotes the variance of $B$. For light-tailed random walk problems, Siegmund (1979) derived a correction which was adapted to ruin probabilities by As-
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mussen and Binswanger (1997). An alternative covering also certain heavy-tailed cases was given in Hogan (1986). The result will be the corrected diffusion approximation

\[ \Psi(u) \approx \Psi_{CD}(u) = \left(1 + \frac{4\theta^2 u^2 m_3}{m_2^3} - \frac{2\theta m_1 m_3}{m_2^2}\right) \exp \left(-2\theta u \frac{m_1}{m_2}\right), \quad (11) \]

when \( m_3 < \infty \), where \( m_i \) is the \( i \)-th moment of \( B \).

We remind that if \( \xi \) is a random variable with cumulative distribution function \( B \) and cumulant generating function \( k(\omega) = \ln E[e^{\omega \xi}] = \ln \hat{B}(\omega) \), the standard definition of the exponential family \( \{B_\omega\} \) generated by \( B \) is

\[ B_\omega(dx) = e^{\omega x - k(\omega)} B(dx) \quad (12) \]

or equivalently

\[ k(\omega)(\alpha) = k(\alpha + \omega) - k(\omega). \quad (13) \]

The question that naturally arises is whether \( k(\omega) \) is the cgf corresponding to a compound Poisson risk process in the sense that for a suitable arrival intensity \( \lambda_\omega \) and a suitable claim-size distribution \( B_\omega \), we have \( k(\omega)(\alpha) = \lambda(\hat{B}_\omega(\omega) - 1) - \alpha \). The answer is yes, the solution is

\[ \lambda_\omega = \lambda \hat{B}(\omega), \]
\[ B_\omega(dx) = \frac{\exp(\alpha x)}{\hat{B}(\omega)} B(dx) \quad (14) \]

or equivalently

\[ \hat{B}_\omega(\omega) = \frac{\hat{B}(\omega + \alpha)}{\hat{B}(\omega)}. \quad (15) \]

In the following, we formalize this for the purpose of studying the whole process. Let \( P \) be the probability measure on \( D(0, \infty) \) governing a given compound Poisson risk process with arrival intensity \( \lambda \) and claim size distribution \( B \), and define \( \lambda_\omega, B_\omega \) by (14). Then \( P_\omega \) denotes the probability measure governing the compound Poisson risk process with arrival intensity \( \lambda_\omega \) and claim size distribution \( B_\omega \); the corresponding expectation operator is \( E_\omega \).

Since Brownian motion is skip-free, the idea to replace the risk process by a Brownian motion ignores the presence of the overshoot and other things. The objective of the corrected diffusion approximation is to take this and other deficits into consideration. The set-up is the exponential family of compound risk processes with parameters \( \lambda_\omega \) and \( B_\omega \). However, if we let the given risk process with safety loading \( \theta > 0 \) correspond to \( \omega = 0 \), it is more convenient here to use some value \( \omega_0 < 0 \) and let \( \omega = 0 \) correspond to \( \theta = 0 \) (zero drift). This is because in the regime of the diffusion approximation, \( \theta \) is close to zero, and we want to consider the limit \( \theta \nearrow 0 \) corresponding to \( \omega_0 \nearrow 0 \).
In terms of the given risk process with Poisson intensity $\lambda$, claim size distribution $B$, $k(\alpha) = \lambda \left( \hat{B}(\alpha) - 1 \right) - \alpha$ and $\rho = \lambda \mu_B < 1$, $\eta = \frac{1}{\rho} - 1$, this means the following:

1) Determine $\gamma_m > 0$ such that $k'(\gamma_m) = 0$ and let $\omega_0 = -\gamma_m$.

2) Let $P_0$ refer to the risk process with parameters $\lambda_0 = \lambda \hat{B}(\omega_0)$, $B_0(dx) = \frac{\exp(-\omega_0 x)}{B_0(\omega_0)} B(dx)$. Then $E_0[X^n(\tau)] = \frac{\hat{B}^{(0)}(0)}{\hat{B}(\omega_0)}$ and $k_0(r) = k(r - \omega_0) - k(\omega - \omega_0)$, $k_0''(0) = 0$.

3) For each $\omega$, let $P_{\omega}$ refer to the risk process with parameters $\lambda_\omega = \lambda_0 \hat{B}(\omega)$, $B_\omega(dx) = \frac{\exp(\omega x)}{B_0(\omega)} B_0(dx) = \frac{\exp(\omega x)}{B_0(\omega)}B(dx)$. Then $k_\omega(r) = k_0(r + \omega) - k_0(\omega) = k(r + \omega - \omega_0) - k(\omega - \omega_0)$ and the given risk process corresponds to $P_{\omega_0}$ where $\omega_0 = \gamma_m$. In this set-up we are studying $\Psi(u, T) = P_{\omega_0}(\tau(u) \leq T)$ for $\omega_0 < 0, \omega_0 \not< 0$.

Recall that $IG(x; \zeta, u)$ (inverse Gaussian) denotes the distribution function of the passage time of the Brownian motion $\{W_\zeta(t)\}$ with unit variance and drift $\zeta$ from level 0 to level $u > 0$. One has $IG(x; \zeta, u) = IG(\frac{x}{\sqrt{u}}, \zeta, u, 1)$. The corrected diffusion approximation to be derived is

$$\Psi(u, T) \approx \Psi_{CD}(u, T) = IG\left(\frac{T v_1}{u^2} + \frac{v_2}{u}; -\frac{\gamma u}{2}, 1 + \frac{v_2}{u}\right),$$

where $\gamma$ is the adjustment coefficient (i.e. $k(\gamma) = 0$) and $v_1 = \lambda \hat{B}''(\gamma_m)$, $v_2 = \frac{E_0(X^n)}{2E_0(X^n)} = \frac{B''(\gamma_m)}{B(\gamma_m)}$. The initial reserve $u$ for the given risk process is written as $u = \zeta / \omega_0$ and, for brevity, we write $\tau = \tau(u)$, $\xi = \xi(u) = S_\tau - u$.

Note that $\mu = k_\omega'(\omega_0) \sim \omega_0 k_\omega''(\omega_0) = v_1$, $\text{Var}_0(S_1) \sim \text{Var}_0(S_1) = v_1$, $\omega_0 \not< 0$,

$$\left\{\frac{1}{u \sqrt{v_1}}\right\}_{t \geq 0} \xrightarrow{D} \left\{W_\xi \sqrt{v_1}(t)\right\}_{t \geq 0}.$$

$$\Psi(u; u, \tau^2) \longrightarrow IG(t; \zeta \sqrt{v_1}, \frac{1}{\xi \sqrt{v_1}}) = IG(tv_1; \zeta, 1).$$

Let $\{W_t, t \geq 0\}$ denote the Wiener process with drift with mean $\mu t$ function and variance function $\sigma^2 t$. We consider the probability of ruin in a time interval $(0, T)$. Let $\tau = \inf_{t > 0} \{t : u + W_t < 0\}$. The probability of ruin before $T$ is $\Psi(u, T) = P(\tau < T) = P(\inf_{0 < t < T} W_t < -u)$.

**Proposition 3.1.**

$$\Psi(u, T) = \Phi\left(-\frac{u + \mu T}{\sigma \sqrt{T}}\right) + e^{-\frac{u^2}{\sigma^2 T}}\Phi\left(-\frac{u - \mu T}{\sigma \sqrt{T}}\right).$$

Letting $T \rightarrow \infty$, the ultimate ruin probability is $\Psi(u) = e^{-\frac{u^2}{\sigma^2}}$ (i.e. the diffusion approximation).

**Proposition 3.2.** The probability density function of the time length until ruin is given by

$$f_T(t) = \frac{u}{\sigma \sqrt{2\pi}} t^{\frac{3}{2}} e^{-\frac{(u - \mu)^2}{2\sigma^2 t^2}}, t > 0.$$
4. PERTURBED RISK MODEL

4.1. INTEGRO-DIFFERENTIAL EQUATION

In the classical risk model, the premium rate \( c \) is a positive constant which does not depend on the history of claims. However, in practice, especially in car insurance, the premium rate is adjusted according to previous claims. Thereby, the constant premium rate hypothesis is quite restrictive and departs the model from reality. Lately, there have been developed risk models in which the premium rate depends on stochastic elements of the insurer’s surplus process. For example, in Dufresne and Gerber (1991) a diffusion is added at the Poisson compound process, the diffusion describes the uncertainty on the total income from premiums or on the aggregate claim. Asmussen (2000) worked out a model in which the premium rates are adjusted according to the current level of the insurer’s surplus. Albrecher and Asmussen (2006) studied a premium rate which is dynamically adjusted in accordance to the history of claims. Albrecher and Boxma (2004) considered a model in which the next claim arrival is induced by the size of the previous claim. They extended their model to a new one with Markov dependence such that the arrival rates as well as the claim size distributions are determined by the state of a Markov chain in continuous time. Albrecher and Teugels (2006) studied risk models with dependence between the inter-occurrence times and claims size.

In the following, we will study a perturbed risk model with dependence between the premium rates and claim sizes. We will consider that the current premium rate is adjusted after the occurrence of a claim and the adjusted rate is determined by the claim size. Moreover, the diffusion coefficient of the model will be modified adequately. Therefore, the premium rate in period \([\tau_n, \tau_{n+1})\) is a random variable depending on the random variable \(X_n\) which gives the size of the claim at the moment \(\tau_n\), \(n = 1, 2, \ldots\). We will denote by \(c(X_n)\) this premium rate suitable for the time interval \([\tau_n, \tau_{n+1})\). In addition, it is considered that the insurer’s surplus is perturbed by a standard brownian motion \(\{W_t, t \geq 0\}\) which is independent of \(N = \{N(t) ; t \geq 0\}\) and \(\{X_i\}_{i=1,2,\ldots}\). The diffusion coefficient for the time period \([\tau_n, \tau_{n+1})\) is denoted by \(\sigma(X_n)\), as it depends also on \(X_n\).

The insurer’s surplus at moment \(t \in [\tau_n, \tau_{n+1})\), \(n \geq 1\), is \(U(t)\), given by:

\[
U(t) = U(\tau_n) + c(X_n)(t - \tau_n) + \sigma(X_n)(W_t - W_{\tau_n})
\]

and

\[
U(\tau_{n+1}) = U(\tau_n) + c(X_n)T_{n+1} - X_{n+1} + \sigma(X_n)(W_{\tau_{n+1}} - W_{\tau_n}).
\]

Regarding the claims sizes, we will consider that there exists a threshold \(b > 0\), such that when \(X_n > b\), the premium rate is \(c(X_n) = c_2\), and when \(X_n \leq b\), the premium rate is \(c(X_n) = c_1\). Thus, it is considered that the base premium rate of the portfolio is \(c_1\), and when there occurs a high loss, the rate premium will be increased to the level \(c_2\). Obviously, it is reasonable to assume that \(c_2 \geq c_1 > 0\). In this
interpretation, $c_1$ can be regarded as an acceptable income rate, while $c_2$ is a penalty for a high demand or loss. We have:

$$c(x) = \begin{cases} 
  c_1, & x \leq b \\
  c_2, & x > b 
\end{cases},$$

and appropriately

$$\sigma(x) = \begin{cases} 
  \sigma_1, & x \leq b \\
  \sigma_2, & x > b 
\end{cases},$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$ are constants that describe changes of diffusion coefficients due to changes in the premium rate.

We assume that the conditions for safety loading are met, i.e. $c_1 > \lambda E[X_n | X_n \leq b]$ and $c_2 > \lambda E[X_n | X_n > b]$.

Let $\{U_i(t), t \geq 0\}, i = 1, 2,$ be the surplus process with the initial premium rate $c_i$, the initial diffusion coefficient $\sigma_i$, over the first period between claims $[0, \tau_i)$. The functions $f_1(x) = f(x) \cdot I[x \leq b]$ and $f_2(x) = f(x) \cdot I[x > b], \ x \geq 0,$ are introduced, where $f$ is the pdf of the distribution function $B$ of the claim sizes.

The survival probability of the process $\{U_i(t), t \geq 0\}$ is

$$\phi_i(u) = P(U_i(t) \geq 0, \forall t \geq 0 | U_i(0) = u), i = 1, 2.$$ 

Let us assume that $\phi_i(u), i = 1, 2,$ are twice differentiable.

**Theorem 4.1.** For any $u > 0, \phi_i(u), i = 1, 2,$ satisfy the following system of equations:

$$\frac{1}{2} \sigma_1^2 \phi_1''(u) + c_1 \phi_1'(u) = \lambda \phi_1(u) - \lambda \int_0^u (\phi_1(u - x) f_1(x) + \phi_2(u - x) f_2(x)) \, dx \quad (22)$$

and

$$\frac{1}{2} \sigma_2^2 \phi_2''(u) + c_2 \phi_2'(u) = \lambda \phi_2(u) - \lambda \int_0^u (\phi_1(u - x) f_1(x) + \phi_2(u - x) f_2(x)) \, dx. \quad (23)$$

with the frontier conditions $\phi_1(0) = 0, \phi_1(\infty) = 1,$ and $\phi_i''(0) = -2c_i \phi_i'(0) / \sigma_i^2, i = 1, 2.$

**Proof.** We pursue Zhou M. and Cai J. (2009). Considering $U_1(t)$ in a small time interval $(0, t]$ we have that

$$\Phi_1(u) = \lambda t E[\int_0^{s(t)} (\Phi_1(s(t) - x) I(x \leq b) + \Phi_2(s(t) - x) I(x > b)) \, dB(x)]$$

$$+ (1 - \lambda t) E[\Phi_1(s(t))] + o(t), \quad (24)$$
where \( s(t) = u + c_1 \cdot t + \sigma_1 \cdot W_t \). By Itô’s formula, we obtain

\[
\Phi_1(s(t)) = \Phi_1(u) + \int_0^t \left( \frac{1}{2} \sigma_1^2 \Phi_1''(s(y)) + c_1 \Phi_1'(s(y)) \right) dy + \sigma_1 \int_0^t \Phi_1'(s(y)) dW_y.
\]

Because \( \left\{ \int_0^t \Phi_1'(s(y)) dW_y, t \geq 0 \right\} \) is a martingale, note that

\[
\mathbb{E}\left[ \int_0^t \Phi_1'(s(y)) dW_y \right] = 0.
\]

Using (24) and then dividing by \( t \) on both sides, and letting \( t \to 0 \), we obtain

\[
\frac{1}{2} \sigma_1^2 \Phi_1''(u) + c_1 \Phi_1'(u) = \lambda \Phi_1(u) - \lambda \int_0^u \Phi_1(u - x) f_1(x) + \Phi_2(u - x) f_2(x) dx.
\]

Analogously, we obtain the other equation. The boundary condition \( \Phi_i(0) = 0 \) comes from the oscillation of the diffusion, and the boundary condition \( \Phi_i(\infty) = 1 \) holds because of the positive loading condition.

4.2. DISCRETE-TIME PERTURBED RISK MODEL

Let us now consider a discrete-time insurance model, suitable for applications. Let the increment in the surplus process in period \( t \), usually year, be defined as

\[
W_t = P_t + A_t - S_t, \quad t \geq 1 \tag{25}
\]

where: \( P_t \) is the premium collected in the \( t \)-th period, \( S_t \) stands for the losses paid in the \( t \)-th period, \( A_t \) is any cash flow other than the premium and the payment of claims. The most significant cash flow is the earning of investment income on the surplus available at the beginning of the period. The surplus at the end of the \( t \)-th period is then

\[
U_t = U_{t-1} + W_t = U_{t-1} + P_t + A_t - S_t = u + \sum_{j=1}^{t} (P_j + A_j - S_j). \tag{26}
\]

Let us assume that given \( U_{t-1} \), the random variable \( W_t \) depends only upon \( U_{t-1} \) and not upon any other previous experience.

In order to calculate ruin probabilities, we consider a process defined as follows:

\[
W_t^* = \begin{cases} 
0, & U_{t-1}^* < 0 \\
W_t, & U_{t-1}^* \geq 0
\end{cases}, \quad U_t^* = U_{t-1}^* + W_t^*, \quad t \geq 1, \tag{27}
\]

where \( U_0^* = u \).

We evaluate the ruin probability using convolutions. The calculation is recursively, using the distribution of \( U_t^* \). Let us suppose that we obtained the discrete probability function (pf) of \( U_{t-1}^* \). Then the ruin probability is

\[
\tilde{\Psi}(u, t-1) = P(U_{t-1}^* < 0) \tag{28}
\]
and the distribution of nonnegative surplus is \( f_j = P(U^*_{t-1} = u_j), j = 1, 2, \ldots \), where \( u_j > 0, \forall j \) and \( u_n \) is the largest possible value of \( U^*_{t-1} \). Let \( g_{jk} = P(W_t = w_{jk} | U^*_{t-1} = u_j) \).

**Theorem 4.2.** The ruin probability after \( t \) periods of time is

\[
\Psi(u, t) = \Psi(u, t-1) + \sum_{j=1}^{\infty} \sum_{w_{jk}+u_j<0} g_{jk} \cdot f_j \quad (29)
\]

and

\[
P(U^*_{t-1} = a) = \sum_{j=1}^{\infty} \sum_{w_{jk}+u_j=a} g_{jk} \cdot f_j. \quad (30)
\]

5. **Numerical Illustration**

Let us assume that annual losses can take the values 0, 2, and 10 monetary units, with probabilities 0.6, 0.3, and 0.1, respectively. Also, suppose that the initial surplus is \( u \), and a premium of 1 monetary unit is collected at the beginning of each year, that interest is earned at 5% on any surplus available at the beginning of the year because claims are paid at the end of the year. In addition, we introduce a rebate of 0.1 m.u. which is given in any year in which there are no losses. We determine the ruin probability at the end of each of the first three years for some values of initial capital \( u \) using (29). We present our results in Table 1, in Table 2, and Figure 1.

Table 1. The ruin probability \( \Psi(u, t) \).

<table>
<thead>
<tr>
<th>( u ) ( \backslash ) ( t )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.28</td>
<td>0.352</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.19</td>
<td>0.298</td>
</tr>
<tr>
<td>4</td>
<td>0.1</td>
<td>0.19</td>
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</tr>
<tr>
<td>8</td>
<td>0.1</td>
<td>0.13</td>
<td>0.163</td>
</tr>
<tr>
<td>10</td>
<td>0.0</td>
<td>0.01</td>
<td>0.037</td>
</tr>
</tbody>
</table>

Table 2. The surplus process for \( u = 8 \) m.u. at the end of the 3rd year.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( u_j )</th>
<th>( f_j )</th>
<th>0</th>
<th>2</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8675</td>
<td>0.6</td>
<td>1.860875</td>
<td>-0.039125</td>
<td>-8.039125</td>
</tr>
<tr>
<td>2</td>
<td>6.8725</td>
<td>0.09</td>
<td>8.166125</td>
<td>6.266125</td>
<td>-1.733875</td>
</tr>
<tr>
<td>3</td>
<td>8.7725</td>
<td>0.18</td>
<td>10.161125</td>
<td>8.261125</td>
<td>0.261125</td>
</tr>
<tr>
<td>4</td>
<td>8.8675</td>
<td>0.18</td>
<td>10.260875</td>
<td>8.360875</td>
<td>0.360875</td>
</tr>
<tr>
<td>5</td>
<td>10.7675</td>
<td>0.36</td>
<td>12.255875</td>
<td>10.355875</td>
<td>2.355875</td>
</tr>
</tbody>
</table>
Fig. 1. The evolution of the ruin probability on the first three years, depending on the amount of the initial capital.

References


