INTEGRAL PROPERTIES OF A CERTAIN CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

Adriana Cătaş, Emilia Borșa
Department of Mathematics and Computer Sciences, University of Oradea, Romania
acatas@gmail.com, eborsa@uoradea.ro

Abstract
In this paper integral properties of a certain class of analytic functions with negative coefficients are studied. This class is defined using a generalized Sălăgean operator. The obtained results are sharp and they improve known results.

Keywords: generalized Sălăgean operator, extremal functions.

2010 MSC: 30C45.

Presented at CAIM 2011.

1. INTRODUCTION

We will denote by \( \mathcal{H}(U) \) the set of analytic functions in the unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and for \( a \in \mathbb{C}, n \in \mathbb{N}^* \) we consider the following class

\[ A_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots \}. \]

Let \( \mathbb{N} \) denote the set of nonnegative integers \( \{0, 1, \ldots, n, \ldots\} \), \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \) and let \( \mathbb{N}_j, j \in \mathbb{N}^* \), be the class of functions of the form

\[ f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad a_k \geq 0, \ k \in \mathbb{N}, \ k \geq j+1, \quad (1) \]

that are analytic in the open unit disc.

For a function \( f \) belonging to the class \( A_n \), F.M. Al-Oboudi defined, in paper [1], the following differential operator:

\[
D^0 f(z) = f(z) \quad \quad D^1 f(z) = D_\lambda f(z) = (1 - \lambda)f(z) + \lambda z f'(z) \quad \quad D^m f(z) = D_\lambda (D^{m-1}_\lambda f(z)), \quad \lambda > 0.
\]

When \( \lambda = 1 \), we get the Sălăgean operator [5].

For a function \( f \in \mathbb{N}_j \) we have

\[
D^m f(z) = z - \sum_{k=j+1}^{\infty} c_k(m, \lambda) a_\lambda z^k
\]
where
\[ c_k(m, \lambda) = [1 + (k - 1) \lambda]^{m}, \quad \lambda > 0, \; m = 0, 1, 2, \ldots \] (2)

**Definition 1.1.** [4] Let \( \alpha, \gamma \in [0, 1), \; m \in \mathbb{N}, \; j \in \mathbb{N}^* \). A function \( f \) belonging to \( \mathcal{N}_j \) is said to be in class \( \mathcal{T}_j(m, \gamma, \alpha, \lambda) \) if and only if
\[ R \frac{D^{m+1}_\lambda f(z)}{(1 - \gamma)D^{m+1}_\lambda f(z) + \gamma D^{m+1}_\lambda f(z)} > \alpha, \quad z \in U. \] (3)

**Remark 1.1.** The class \( \mathcal{T}_j(m, \gamma, \alpha, \lambda) \) contains the following subclasses
i) \( \mathcal{T}_1(0, 0, \alpha, 1) = \mathcal{T}^*(\alpha) \) and \( \mathcal{T}_1(1, 0, \alpha, 1) = C(\alpha) \) defined and studied by Silverman [11] (these classes are the class of starlike functions of order \( \alpha \) with negative coefficients and the class of convex functions of order \( \alpha \) with negative coefficients respectively);
ii) \( \mathcal{T}_j(0, 0, \alpha, 1) \) and \( \mathcal{T}_j(1, 0, \alpha, 1) \) studied by Chatterjea [5] and Srivastava et al. [12];
iii) \( \mathcal{T}_1(0, \gamma, \alpha, 1) = \mathcal{T}(\gamma, \alpha) \) and \( \mathcal{T}_1(1, \gamma, \alpha, 1) = C(\gamma, \alpha) \) studied by Altintaş and Owa [2]

The next characterization theorem of the class \( \mathcal{T}_j(m, \gamma, \alpha, \lambda) \) was given in [4].

**Theorem 1.1.** [4] Let the function \( f \) be defined by (1). Then \( f \) belongs to the class \( \mathcal{T}_j(m, \gamma, \alpha, \lambda) \) if and only if
\[ \sum_{k=j+1}^{\infty} c_k(m, \lambda)[1 + (k - 1) \lambda - \alpha[1 + \gamma(k - 1) \lambda]]a_k \leq 1 - \alpha \] (4)
where \( c_k(m, \lambda) \) is given in (2).

The result is sharp and the extremal functions are
\[ f_k(z) = z - \frac{1 - \alpha}{c_k(m, \lambda)[1 + (k - 1) \lambda - \alpha[1 + \gamma(k - 1) \lambda]]} \cdot z^k \] (5)
with \( k \geq j + 1 \).

Let \( I_c : \mathcal{N}_j \to \mathcal{N}_j, \; j \in \mathbb{N}^* \) be the integral operator defined by \( F = I_c(f) \), where \( c \in (-1, \infty) \), \( f \in \mathcal{N}_j \) and
\[ F(z) = I_c(f)(z) = \frac{c + 1}{z} \int_0^z t^{-1} f(t) dt. \] (6)

From the representation (6) it follows that
\[ F(z) = I_c(f)(z) = z - \sum_{k=j+1}^{\infty} b_k z^k \] (7)
where
\[ b_k = \left( \frac{c + 1}{c + k} \right) a_k. \] (8)
2. **INTEGRAL PROPERTIES OF THE CLASS**  
\( \mathcal{T}_j(M, \gamma, \alpha, \lambda) \) **CONCERNING PARAMETER** \( \alpha \)

In this section we will find the parameter \( \beta \) such that \( F \in \mathcal{T}_j(m, \gamma, \beta, \lambda) \) for \( f \in \mathcal{T}_j(m, \gamma, \alpha, \lambda) \) and \( F = I_c(f) \).

**Theorem 2.1.** Let \( m \in \mathbb{N} \), \( j \in \mathbb{N}^* \), \( \alpha, \gamma \in [0, 1) \) and let \( c \in (-1, \infty) \).

If \( f \in \mathcal{T}_j(m, \gamma, \alpha, \lambda) \) and \( F = I_c(f) \), then \( F \in \mathcal{T}_j(m, \gamma, \beta, \lambda) \), where

\[
\beta = \beta(\gamma, \alpha, \lambda, c; j + 1) = 1 - \frac{(c + 1)(1 - \alpha)(1 - \gamma)}{\lambda((1 - \alpha \gamma)(c + j + 1) - \gamma(c + 1)(1 - \alpha)) + 1 - \alpha}
\]

and \( \alpha < \beta(\gamma, \alpha, \lambda, c; j + 1) < 1 \). The result is sharp.

**Proof.** From Theorem 1.1 and from (7) we have \( F \in \mathcal{T}_j(m, \gamma, \beta, \lambda) \) if and only if

\[
\sum_{k=j+1}^{\infty} c_k(m, \lambda)[1 + (k - 1)\lambda - \beta[1 + \gamma(k - 1)\lambda]](c + 1)\frac{a_k}{(c + k)(1 - \beta)} \leq 1. \tag{10}
\]

We find the largest \( \beta \) such that (10) holds. We note that the inequalities

\[
\frac{1 + (k - 1)\lambda - \beta[1 + \gamma(k - 1)\lambda]}{(c + k)(1 - \beta)} \leq 1 - \frac{(c + 1)(1 - \alpha)}{1 - \alpha}, \quad k \geq j + 1 \tag{11}
\]

imply (10), because \( f \in \mathcal{T}_j(m, \gamma, \alpha, \lambda) \) and it satisfies (4). But the inequalities (11) are equivalent to

\[
\beta A(\gamma, \alpha, \lambda, c; k) \leq B(\gamma, \alpha, \lambda, c; k) \tag{12}
\]

where

\[
A(\gamma, \alpha, \lambda, c; k) = \lambda(k - 1)((1 - \alpha \gamma)(c + k) - \gamma(c + 1)(1 - \alpha)) + (1 - \alpha)(k - 1)
\]

and

\[
B(\gamma, \alpha, \lambda, c; k) = A(\gamma, \alpha, \lambda, c; k) - \lambda(k - 1)(c + 1)(1 - \alpha)(1 - \gamma). \tag{13}
\]

Since \( 1 - \alpha \gamma > 1 - \alpha \) and \( (c + k) > \gamma(c + 1) \), we have \( A(\gamma, \alpha, \lambda, c; k) > 0 \) and from (12) we deduce

\[
\beta \leq \frac{B(\gamma, \alpha, \lambda, c; k)}{A(\gamma, \alpha, \lambda, c; k)}, \quad k \geq j + 1. \tag{14}
\]

We define

\[
\beta(\gamma, \alpha, \lambda, c; k) := \frac{B(\gamma, \alpha, \lambda, c; k)}{A(\gamma, \alpha, \lambda, c; k)}. \tag{15}
\]
We show now that $\beta(\gamma, \alpha, \lambda, c; k)$ is an increasing function of $k$, $k \geq j + 1$. Indeed,

$$
\beta(\gamma, \alpha, \lambda, c; k) = 1 - \frac{\lambda(k - 1)(c + 1)(1 - \alpha)(1 - \gamma)}{A(\gamma, \alpha, \lambda, c; k)}
$$

$$
= 1 - \frac{\lambda(c + 1)(1 - \alpha)(1 - \gamma)}{E(\gamma, \alpha, \lambda, c; k)}
$$

where

$$
E(\gamma, \alpha, \lambda, c; k) = \frac{A(\gamma, \alpha, \lambda, c; k)}{k - 1}
$$

and $\beta(\gamma, \alpha, \lambda, c; k)$ increases when $k$ increases if and only if $E(\gamma, \alpha, \lambda, c; k)$ is also an increasing function of $k$.

We define the function

$$
g(x) := E(\gamma, \alpha, \lambda, c; x), \quad x \in [j + 1, \infty) \subset [2, \infty)
$$

$$
g(x) = \frac{\lambda((1 - \alpha)(c + x) - \gamma(c + 1)(1 - \alpha)] + 1 - \alpha. \quad (16)
$$

One obtains

$$
g'(x) = \lambda(1 - \alpha) \geq 0, \quad \forall \ x \in [j + 1, \infty).
$$

We have obtained $g(j + 1) \leq g(k), \ k \geq j + 1$ and this implies

$$
\beta := \beta(\gamma, \alpha, \lambda, c; j + 1) \leq \beta(\gamma, \alpha, \lambda, c; k), \quad k \geq j + 1. \quad (17)
$$

$$
\beta = 1 - \frac{\lambda(c + 1)(1 - \alpha)(1 - \gamma)}{\lambda[(1 - \alpha)(c + j + 1) - \gamma(c + 1)(1 - \alpha)] + 1 - \alpha.
$$

The result is sharp because

$$
J_c(f_\alpha) = f_\beta \quad (18)
$$

where

$$
f_\alpha(z) = z - \frac{1 - \alpha}{(1 + j\lambda)[1 + j\lambda - \alpha(1 + j\lambda)]} \cdot z^{j+1}, \quad (19)
$$

$$
f_\beta(z) = z - \frac{1 - \beta}{(1 + j\lambda)[1 + j\lambda - \beta(1 + j\lambda)]} \cdot z^{j+1} \quad (20)
$$

are the extremal functions of $\mathcal{T}_j(m, \gamma, \alpha, \lambda)$ and $\mathcal{T}_j(m, \gamma, \beta, \lambda)$ respectively and $\beta = \beta(\gamma, \alpha, \lambda, \lambda, c; j + 1)$.

Indeed, we have

$$
J_c(f_\alpha)(z) = z - \frac{(1 - \alpha)(c + 1)}{(1 + j\lambda)[c + j + 1][1 + j\lambda - \alpha(1 + j\lambda)]} \cdot z^{j+1}.
$$

If we use the notations

$$
A := A(\gamma, \alpha, \lambda, c; j + 1) \quad \text{and} \quad B := B(\gamma, \alpha, \lambda, c; j + 1)
$$
one obtains
\[
\frac{1 - \beta}{(j\lambda + 1)(1 - \beta\gamma) - \beta(1 - \gamma)} = \frac{(1 - \alpha)(c + 1)}{(c + j + 1)[1 + \lambda j - \alpha(1 + \lambda j\gamma)]}
\]
and this implies (18).

From the relation \(\beta = 1 - \frac{j\lambda(c + 1)(1 - \alpha)(1 - \gamma)}{A}\) and because \(A > 0\), we deduce \(\beta < 1\). We also have \(\beta > \alpha\). Indeed,
\[
\beta - \alpha > \frac{(1 - \alpha)(1 - \alpha\gamma)j}{\lambda(1 - \alpha\gamma) + (c + 1)(1 - \gamma)} > 0.
\]

3. INTEGRAL PROPERTIES CONCERNING THE PARAMETER \(\gamma\)

In this section we will find the parameter \(\eta^*\) such that \(F \in \mathcal{T}_j(m, \eta^*, \alpha, \lambda)\) for \(f \in \mathcal{T}_j(m, \gamma, \alpha, \lambda)\) and \(F = I_c(f)\).

**Theorem 3.1.** Let \(m \in \mathbb{N}\), \(j \in \mathbb{N}^+, \alpha, \gamma \in (0, 1), \lambda > 0\) and let \(c\) a real number such that
\[
c > \max \left\{ -1, \frac{\lambda(j + 1)(1 - \alpha\gamma) + 1 - \alpha - \lambda}{\alpha\gamma\lambda} \right\}. \tag{21}
\]
If \(f \in \mathcal{T}_j(m, \gamma, \alpha, \lambda)\) and \(F = I_c(f)\) then \(F \in \mathcal{T}_j(m, \eta^*, \alpha, \lambda)\), where
\[
\eta^* = \frac{\lambda(c + 1) - (c + j + 1)(1 - \alpha\gamma)\lambda - (1 - \alpha)}{\lambda\alpha(c + 1)} \tag{22}
\]
and \(0 < \eta^* < \gamma\). The result is sharp.

**Proof.** From Theorem 1.1 and from (7) we have \(F \in \mathcal{T}_j(m, \eta, \alpha, \lambda)\) if and only if
\[
\sum_{k=j+1}^{\infty} c_k(m, \lambda)[1 + (k - 1)\lambda - \alpha[1 + \eta(k - 1)\lambda]](c + 1) \cdot a_k \leq 1. \tag{23}
\]

We find the smallest \(\eta\) such that (23) holds. We note that the inequalities
\[
\frac{(1 + (k - 1)\lambda - \alpha[1 + \eta(k - 1)\lambda])(c + 1)}{c + k} \leq 1 + (k - 1)\lambda - \alpha[1 + \gamma(k - 1)\lambda], \quad k \geq j + 1
\]
imply (23), because \(f \in \mathcal{T}_j(m, \gamma, \alpha, \lambda)\) and it satisfies (4). But the inequalities (24) can be rewritten for all \(k \geq j + 1\)
\[
[(k - 1)(1 - \alpha\eta)\lambda + 1 - \alpha](c + 1) \leq [(k - 1)(1 - \alpha\gamma)\lambda + 1 - \alpha](c + k)
\]
or
\[ \eta \geq \frac{\lambda(c+1) - (c+k)(1 - a\gamma)\lambda - (1 - \alpha)}{\lambda a(c+1)}, \quad k \geq j + 1. \]  

(25)

If we let
\[ h(\gamma, \alpha, \lambda; c) = \frac{\lambda(c+1) - (c+k)(1 - a\gamma)\lambda - (1 - \alpha)}{\lambda a(c+1)}, \quad k \geq j + 1 \]  

(26)

then function $h$ can be rewritten in the form
\[ h(\gamma, \alpha, \lambda; c) = \left[ 1 - \frac{g(k) - \lambda(c+1)(1 - \gamma)}{\lambda a\gamma(c+1)} \right] \cdot \gamma \]

where the function $g$ is given in (16).

Since
\[ g(k) - \lambda(c+1)(1 - \gamma) \] is an increasing function of $k$, we deduce that $h(\gamma, \alpha, \lambda; c; k)$ is a decreasing function of $k$, $(k \geq j + 1)$ and $h(\gamma, \alpha, \lambda; c; k) \leq h(\gamma, \alpha, \lambda; j + 1)$. For $k = j + 1$ one obtains
\[ \eta^* = \left[ 1 - \frac{\lambda j(1 - a\gamma) + 1 - \alpha}{\lambda a\gamma(c+1)} \right] \cdot \gamma \]
and for $c$ given in (21) we can see that $0 < \eta^* < \gamma$.

In order to prove that the result is sharp we show that
\[ I_c(f_\gamma) = f_{\eta^*}, \]  

(27)

where
\[ f_\gamma(z) = z - \frac{1 - \alpha}{(1 + j\lambda)^m[1 + j\lambda - a(1 + j\lambda\gamma)]} \cdot z^{j+1} \]  

(28)

and
\[ f_{\eta^*}(z) = z - \frac{1 - \alpha}{(1 + j\lambda)^m[1 + j\lambda - a(1 + j\lambda\eta^*)]} \cdot z^{j+1} \]  

(29)

are the extremal functions of $T_j(m, \gamma, \alpha, \lambda)$ and $T_j(m, \eta^*, \alpha, \lambda)$ respectively and $\eta^* = h(\gamma, \alpha, \lambda; c; j + 1)$.

Indeed, we have
\[ I_c(f_\gamma)(z) = z - \frac{(c+1)(1 - \alpha)}{(1 + j\lambda)^m(c+j+1)[1 + j\lambda - a(1 + j\lambda\gamma)]} \cdot z^{j+1}. \]

One obtains
\[ \frac{1}{1 + j\lambda - a(1 + j\lambda\eta^*)} = \frac{(c+1)}{(c+1)} \frac{(c+1)(1 - \alpha) + (c + j + 1)(1 - a\gamma)\lambda j - (1 - \alpha)j}{(c+1)} \]
\[ = \frac{(c+1)}{(c+1)[1 + j\lambda - a(1 + j\lambda\gamma)]} \]
and this implies (27).
Integral properties of a certain class of analytic functions with negative coefficients

References


