FAMILY OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

Gangadharan Murugusundaramoorthy\textsuperscript{1}, Basem Aref Frasin\textsuperscript{2}
\textsuperscript{1}School of Science and Humanities, V I T University, Vellore, India.
\textsuperscript{2}Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq, Jordan (corresponding author)
gmsmoorthy@yahoo.com, bafrasin@yahoo.com

Abstract
In this paper a new class \( ST(\Phi, \Psi, \eta, \gamma) \) of analytic functions is introduced and defined in the open unit disc. Coefficient inequalities, distortion bounds, closure theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to this class are obtained. Partial sums \( f_n(z) \) of functions \( f(z) \) in this class are considered and sharp lower bounds for the ratios of real part of \( f(z) \) to \( f_n(z) \) and \( f'(z) \) to \( f'_n(z) \) are also determined.

Keywords: Univalent, starlike, convex, Hadamard product, partial sums.
2000 MSC: 30C45.
Received on April 12, 2010.

1. INTRODUCTION

Let \( S \) be the class of functions \( f \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]  

which are analytic and univalent in the open unit disc \( \mathbb{U} = \{ z : |z| < 1 \} \). Let the functions \( \Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \) and \( \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n \) be analytic in \( \mathbb{U} \), where \( \lambda_n \geq \mu_n \geq 0 \). Then we define \( ST(\Phi, \Psi, \eta, \gamma) \) be the subclass of \( S \) consisting of functions \( f(z) \) of the form (1) and satisfying the analytic criteria

\[
\text{Re} \left\{ \frac{(f * \Phi)(z)}{(1 - \eta)(f * \Psi)(z) + \eta(f * \Phi)(z)} \right\} \geq \gamma, \quad z \in \mathbb{U}, \tag{2}
\]

where \( 0 \leq \eta < 1, \ 0 \leq \gamma < 1 \) and the operator \((*)\) stands for the convolution of two power series \( f(z) = z \pm \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) = z \pm \sum_{n=2}^{\infty} b_n z^n \) defined by

\[
(f * g)(z) = f(z) * g(z) = z \pm \sum_{n=2}^{\infty} a_n b_n z^n.
\]
We also let
\[ S \mathcal{J}(\Phi, \Psi, \eta, \gamma) = S(\Phi, \Psi, \eta, \gamma) \cap \mathcal{J} \] (3)
where \( \mathcal{J} \), the subclass of \( S \), consists functions of the form
\[ f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \] (4)
was introduced and studied by Silverman [4].

The families \( S(\Phi, \Psi, \eta, \gamma) \) and \( S \mathcal{J}(\Phi, \Psi, \eta, \gamma) \) are comprehensive classes consisting of various well-know classes of analytic functions as well as many new ones. For example, for suitable choices of \( \Phi, \Psi, \) and \( \eta \) we obtain the following subclasses of \( \mathcal{J} \) studied by various authors:

1. \( S \mathcal{J}\left( \frac{z-\gamma}{1-z}, 0, \gamma \right) = \mathcal{J}^*(\gamma) \) (Silverman [4]);
2. \( S \mathcal{J}\left( \frac{z+z^2}{2(1-z)}, 0, \gamma \right) = \mathcal{C}(\gamma) \) (Silverman [4]);
3. \( S \mathcal{J}\left( \frac{z}{1-z}, 0, \gamma \right) = \mathcal{P}^*(\gamma) \) (Sarangi and Uralegaddi [3]);
4. \( S \mathcal{J}\left( \frac{z+(1-2\alpha)z^2}{(1-z)^2}, 0, \gamma \right) = \mathcal{R}[\gamma] \) (Silverman and Silvia [6]);
5. \( S \mathcal{J}(\Phi, \Psi; 0, \gamma) = E_\mathcal{J}(\Phi, \Psi; \gamma) \) (Juneja et al.[2]).

In this paper, we obtain coefficient inequalities, distortion bounds, closure theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class \( S \mathcal{J}(\Phi, \Psi, \eta, \gamma) \). Finally, we consider partial sums of functions in this class and obtain sharp lower bounds for the ratios of real part of \( f(z) \) to \( f_n(z) \) and \( f'(z) \) to \( f'_n(z) \).

In the following section we obtain coefficient inequalities and distortion bounds for functions in the class \( S(\Phi, \Psi, \eta, \gamma) \).

### 2. COEFFICIENT ESTIMATES AND DISTORTION BOUNDS

First we obtain the coefficient inequalities for \( f \in S(\Phi, \Psi, \eta, \gamma) \).

**Theorem 2.1.** If \( f \in S \) satisfies
\[ \sum_{n=2}^{\infty} \left| \lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma \right| |a_n| \leq 1 - \gamma \] (5)
then \( f \in S(\Phi, \Psi, \eta, \gamma) \).
Proof. Assume that the inequality (5) holds and let $|z| = 1$. Then,

$$\left| \frac{(f \ast \Phi)(z)}{(1 - \eta)(f \ast \Psi)(z) + \eta(f \ast \Phi)(z)} - 1 \right| = \left| \frac{z + \sum_{n=2}^{\infty} \lambda_n a_n z^n}{\sum_{n=2}^{\infty} \lambda_n |a_n|^n} - 1 \right|$$

$$\leq \frac{\sum_{n=2}^{\infty} \lambda_n |a_n| |z|^{n-1} - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] |a_n| |z|^{n-1}}.$$

Letting $z \to 1$ we have

$$\sum_{n=2}^{\infty} \lambda_n |a_n| - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] |a_n|$$

$$1 - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] |a_n| \leq 1 - \gamma.$$

The above expression is bounded above by $1 - \gamma$ that is,

$$\frac{\sum_{n=2}^{\infty} \lambda_n |a_n| - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] |a_n|}{1 - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] |a_n|} \leq 1 - \gamma.$$

This shows that

$$\frac{(f \ast \Phi)z}{(1 - \eta)(f \ast \Psi)(z) + \eta(f \ast \Phi)(z)} - 1$$

lies in a circle centered at $w = 1$ whose radius is $1 - \gamma$, that completes the proof.

A necessary and sufficient condition for a function $f(z)$ to be in the class $\mathbb{S} \mathbb{T}(\Phi, \Psi, \eta, \gamma)$ is given by

**Theorem 2.2.** Let the function $f$ be defined by (4). Then $f \in \mathbb{S} \mathbb{T}(\Phi, \Psi, \eta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n)) \gamma] a_n \leq 1 - \gamma. \quad (6)$$

The result is sharp for the function

$$f(z) = z - \frac{(1 - \gamma)}{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n)) \gamma]} z^n, \quad n \geq 2. \quad (7)$$

**Proof.** In view of Theorem 2.1, we need only to prove the necessity. If $f(z) \in \mathbb{S} \mathbb{T}(\Phi, \Psi, \eta, \gamma)$ and $z$ is real then assume that the inequality (6) holds and let $|z| = 1.$
Then,

\[
\text{Re} \left\{ \frac{(f * \Phi)(z)}{(1 - \eta)(f * \Psi)(z) + \eta(f * \Phi)(z)} \right\}
\]

\[
= \text{Re} \left\{ \frac{z - \sum_{n=2}^{\infty} \lambda_n a_n z^n}{z - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] a_n z^n} - 1 \right\} \geq \gamma,
\]

\(z \in \mathbb{U}\). Choose the values of \(z\) on the real axis so that \(\frac{(f * \Phi)}{(1 - \eta)(f * \Psi)(z) + \eta(f * \Phi)(z)}\) is real and upon clearing the denominator in the above inequality and letting \(z = 1^+\), through the real values we get

\[
1 - \sum_{n=2}^{\infty} \lambda_n a_n |z|^n - 1 \geq \gamma \left( 1 - \sum_{n=2}^{\infty} [\mu_n + \eta(\lambda_n - \mu_n)] a_n |z|^n - 1 \right)
\]

which yields (6). The result is sharp for the function given by (7).

**Corollary 2.1.** If \(f(z)\) of the form (4) is in \(\mathcal{S}\mathcal{T}(\Phi, \Psi, \eta, \gamma)\), then

\[
a_n \leq \frac{1 - \gamma}{\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n)) \gamma}, \quad n \geq 2,
\]

with equality only for functions of the form (7).

**Theorem 2.3. (Distortion Bounds)** Let the function \(f(z)\) defined by (4) belong to \(\mathcal{S}\mathcal{T}(\Phi, \Psi, \eta, \gamma)\). If \(\{\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\}_{n=2}^{\infty}\) is nondecreasing sequences, then for \(|z| \leq r\),

\[
r - \frac{1 - \gamma}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2)) \gamma} r^2 \leq |f(z)| \leq r + \frac{1 - \gamma}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2)) \gamma} r^2 \tag{9}
\]

and if \(\{\frac{\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))}{n}\}_{n=2}^{\infty}\) is nondecreasing sequences, then for

\[
1 - \frac{2(1 - \gamma)}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2)) \gamma} r \leq |f'(z)| \leq 1 + \frac{2(1 - \gamma)}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2)) \gamma} r \tag{10}
\]

for \(z \in \mathbb{U}\).

**Proof.** In view of Theorem 2.2, we note that

\[
\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2)) \gamma \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} \lambda_n - (\mu_n + \eta(\lambda_n - \mu_n)) a_n \leq 1 - \gamma,
\]

also, we have

\[
\frac{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))}{2} \sum_{n=2}^{\infty} n a_n \leq \sum_{n=2}^{\infty} \lambda_n - (\mu_n + \eta(\lambda_n - \mu_n)) a_n \leq 1 - \gamma.
\]

\[128\]
Using (4) and the inequalities (11) and (12), we obtain

\[ |f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{1 - \gamma}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r^2. \]  

(13)

Similarly,

\[ |f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1 - \gamma}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r^2. \]

Also we have

\[ |f'(z)| \geq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} a_n \geq 1 - \frac{2(1 - \gamma)}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r, \]

and

\[ |f'(z)| \leq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} a_n \leq 1 + \frac{2(1 - \gamma)}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r. \]

3. EXTREME POINTS AND CLOSURE THEOREMS

In this section we discuss the closure properties.

**Theorem 3.1. (Extreme Points)** Let

\[ f_1(z) = z \quad \text{and} \quad f_n(z) = z - \frac{(1 - \gamma)}{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]} z^n, \quad n \geq 2, \]  

(14)

for \( 0 \leq \gamma < 1, \eta \geq 0 \). Then \( f(z) \) is in the class \( S\mathcal{J}(\Phi, \Psi, \eta, \gamma) \) if and only if it can be expressed in the form

\[ f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z), \]  

(15)

where \( \mu_n \geq 0 \) and \( \sum_{n=1}^{\infty} \theta_n = 1 \).

**Proof.** Suppose \( f(z) \) can be written as in (15). Then

\[ f(z) = z - \sum_{n=2}^{\infty} \theta_n \frac{(1 - \gamma)}{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]} z^n. \]
Now,
\[ \sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]\gamma}{(1 - \gamma)} a_n \theta_n \left[ \frac{1}{\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma} \right] a_n = \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \leq 1. \]
Thus \( f \in ST(\Phi, \Psi, \eta, \gamma) \). Conversely, let us have \( f \in ST(\Phi, \Psi, \eta, \gamma) \). Then by using (15), we set
\[ \theta_n = \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]\gamma}{(1 - \gamma)} a_n, \quad n \geq 2 \]
and \( \theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n \). Then we have \( f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z) \) and hence this completes the proof of Theorem 3.1.

**Theorem 3.2. (Closure Theorem)** Let the functions \( f_j(z) \) be defined for \( j = 1, 2, \ldots, m \) defined by
\[ f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \quad z \in U \quad \text{(16)} \]
be in the class \( ST(\Phi, \Psi, \eta, \gamma_j) (j = 1, 2, \ldots, m) \) respectively. Then the function \( h(z) \) defined by
\[ h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left( \sum_{j=1}^{m} a_{n,j} \right) z^n \]
is in the class \( ST(\Phi, \Psi, \eta, \gamma) \), where \( \gamma = \min_{1 \leq j \leq m} \{ \gamma_j \} \) where \( 0 \leq \gamma_j < 1 \).

**Proof.** Since \( f_j(z) \in ST(\Phi, \Psi, \eta, \gamma) \) \( (j = 1, 2, 3, \ldots, m) \) by applying Theorem 2.2, to (16) we observe that
\[ \sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]\gamma \left( \frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) \]
\[ = \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]\gamma a_{n,j} \right) \]
\[ \leq \frac{1}{m} \sum_{j=1}^{m} (1 - \gamma_j) \leq 1 - \gamma \]
which in view of Theorem 2.2, implies that \( h(z) \in ST(\Phi, \Psi, \eta, \gamma) \) and so the proof is complete.

**Theorem 3.3.** The class \( ST(\Phi, \Psi, \eta, \gamma) \) is a convex set.

**Proof.** Let the functions defined by (16) be in the class \( ST(\Phi, \Psi, \eta, \gamma) \). It sufficent to show that the function \( h(z) \) defined by
\[ h(z) = vf_1(z) + (1 - v)f_2(z), \quad 0 \leq v \leq 1, \]
is in the class $S\mathcal{T}(\Phi, \Psi, \eta, \gamma)$. Since
\[ h(z) = z - \sum_{n=2}^{\infty} [\nu a_{n,1} + (1 - \nu)a_{n,2}]z^n, \]
an easy computation with the aid of Theorem 2.2 gives,
\[
\sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]v a_{n,1} + \sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma](1 - \nu)a_{n,2} \\
\leq \nu(1 - \gamma) + (1 - \nu)(1 - \gamma) \\
\leq 1 - \gamma,
\]
which implies that $h \in S\mathcal{T}(\Phi, \Psi, \eta, \gamma)$. Hence $S\mathcal{T}(\Phi, \Psi, \eta, \gamma)$ is convex.}

### 3.1. RADIi OF STARLIKENESS AND CONVEXITY

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $S\mathcal{T}(\Phi, \Psi, \eta, \gamma)$.

**Theorem 3.4.** Let the function $f(z)$ defined by (4) belong to the class $S\mathcal{T}(\Phi, \Psi, \eta, \gamma)$. Then $f(z)$ is close-to-convex of order $\rho$ ($0 \leq \rho < 1$) in the disc $|z| < r_1$, where
\[
r_1 := \inf_{n \geq 2} \left[ \frac{(1 - \rho) [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{n} \right]^{\frac{1}{\nu(1 - \gamma)}}.
\]
The result is sharp, with extremal function $f(z)$ given by (14).

**Proof.** Given $f \in \mathcal{T}$, and $f$ is close-to-convex of order $\rho$, we have
\[
|f'(z) - 1| < 1 - \rho.
\]
For the left hand side of (18) we have
\[
|f'(z) - 1| \leq \sum_{n=2}^{\infty} na_{n}|z|^{n-1}.
\]
The last expression is less than $1 - \rho$ if
\[
\sum_{n=2}^{\infty} \frac{n}{1 - \rho} a_{n}|z|^{n-1} < 1.
\]
Using the fact, that $f \in S\mathcal{T}(\Phi, \Psi, \eta, \gamma)$ if and only if
\[
\sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1 - \gamma)} a_{n} \leq 1,
\]
we can say (18) is true if

\[
\frac{n}{1 - \rho} |z|^{n-1} \leq \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1 - \gamma)}.
\]

Or, equivalently,

\[
|z|^{n-1} = \left( \frac{1 - \rho}{n} \right) \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1 - \gamma)}^{n-1}
\]

which completes the proof.

**Theorem 3.5.** Let \( f \in ST(\Phi, \Psi, \eta, \gamma) \). Then

(i) \( f \) is starlike of order \( \rho \) \((0 \leq \rho < 1)\) in the disc \( |z| < r_2 \); that is,

*Re* \( \left\{ \frac{zf''(z)}{f'(z)} \right\} > \rho \), \((|z| < r_2; 0 \leq \rho < 1)\), where

\[
r_2 = \inf_{n \geq 2} \left( \frac{1 - \rho}{n(n - \rho)} \right) \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1 - \gamma)}^{n-1}, \tag{19}
\]

(ii) \( f \) is convex of order \( \rho \) \((0 \leq \rho < 1)\) in the unit disc \( |z| < r_3 \), that is

*Re* \( \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \rho \), \((|z| < r_3; 0 \leq \rho < 1)\), where

\[
r_3 = \inf_{n \geq 2} \left( \frac{1 - \sigma}{n(n - \rho)} \right) \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1 - \gamma)}^{n-1}. \tag{20}
\]

Each of these results are sharp for the extremal function \( f(z) \) given by (14).

**Proof.** (i) Given \( f \in T \), and \( f \) is starlike of order \( \rho \), we have

\[
\left| \frac{zf''(z)}{f'(z)} - 1 \right| < 1 - \rho. \tag{21}
\]

For the left hand side of (21) we have

\[
\left| \frac{zf''(z)}{f'(z)} - 1 \right| \leq \sum_{n=2}^{\infty} \frac{(n - 1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.
\]

The last expression is less than \( 1 - \rho \) if

\[
\sum_{n=2}^{\infty} \frac{n - \rho}{1 - \rho} a_n |z|^{n-1} < 1.
\]
Using the fact, that \( f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma) \) if and only if
\[
\sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]a_n}{(1 - \gamma)} \leq 1.
\]
We can say (21) is true if
\[
\frac{n - \rho}{1 - \rho} |z|^{n-1} < \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]}{(1 - \gamma)}
\]
or, equivalently,
\[
|z|^{n-1} = \left( \frac{1 - \rho}{n - \rho} \right) \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]a_n}{(1 - \gamma)}
\]
which yields the starlikeness of the family.

(ii) Given \( f \in \mathcal{T} \), and \( f \) is convex of order \( \rho \), we have
\[
\left| \frac{zf''(z)}{f'(z)} \right| < 1 - \rho. \quad (22)
\]
For the left hand side of (22) we have
\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq \sum_{n=2}^{\infty} \frac{n(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n |z|^{n-1}}.
\]
The last expression is less than \( 1 - \rho \) if
\[
\sum_{n=2}^{\infty} \frac{n(n-\rho)}{1 - \rho} a_n |z|^{n-1} < 1.
\]
Using the fact, that \( f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma) \) if and only if
\[
\sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]a_n}{(1 - \gamma)} \leq 1.
\]
We can say (22) is true if
\[
\frac{n(n - \rho)}{1 - \rho} |z|^{n-1} < \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]a_n}{(1 - \gamma)}.
\]
Or, equivalently,
\[
|z|^{n-1} = \left( \frac{1 - \rho}{n(n - \rho)} \right) \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))]a_n}{(1 - \gamma)}
\]
which yields the convexity of the family. \( \blacksquare \)
4. PARTIAL SUMS

Following the earlier works by Silverman [5] and Silvia [7] on partial sums of analytic functions. We consider in this section partial sums of functions in the class $ST(\Phi, \Psi, \eta, \gamma)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$.

Theorem 4.1. Define the partial sums $f_1(z)$ and $f_k(z)$, by

$$f_1(z) = z; \quad f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \quad (k \geq 2). \quad (23)$$

If $f(z)$ of the form (1) satisfies the condition (5) and

$$c_n \geq \left\{ \begin{array}{ll}
1, & n = 2, 3, \ldots, k \\
c_{k+1}, & n = k + 1, k + 2, \ldots,
\end{array} \right. \quad (24)$$

where, for convenience,

$$c_n := \frac{\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma}{1 - \gamma}. \quad (25)$$

Then

$$\text{Re} \left\{ \frac{f(z)}{f_k(z)} \right\} > 1 - \frac{1}{c_{k+1}}, \quad z \in \mathbb{U}, k \in \mathbb{N} \quad (26)$$

and

$$\text{Re} \left\{ \frac{f_k(z)}{f(z)} \right\} > \frac{c_{k+1}}{1 + c_{k+1}} \quad (27)$$

where

$$c_n \geq \left\{ \begin{array}{ll}
1, & n = 2, 3, \ldots, k + 1 \\
c_{k+1}, & n = k + 1, k + 2, \ldots.
\end{array} \right. \quad (28)$$

Proof. From (24) it follows that

$$\sum_{n=2}^{k} |a_n| + c_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} c_n |a_n| \leq 1. \quad (29)$$

By setting

$$g_1(z) = c_{k+1} \left\{ \frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{c_{k+1}}\right) \right\}$$

$$= 1 + \frac{c_{k+1} \sum_{n=2}^{k} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} \quad (30)$$

$$\therefore \text{Re} \left\{ \frac{f(z)}{f_k(z)} \right\} > 1 - \frac{1}{c_{k+1}} \quad (26)$$
and applying (29), we find that

\[
\left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq \frac{c_{k+1} \sum_{n=2}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{k} |a_n| - c_{k+1} \sum_{n=k+1}^{\infty} |a_n|}.
\]  

(31)

Now \( \left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| \leq 1 \) if

\[
\sum_{n=2}^{k} |a_n| + \sum_{n=k+1}^{\infty} c_{k+1} |a_n| \leq 1.
\]

From the condition (5), it is sufficient to show that

\[
\sum_{n=2}^{k} |a_n| + \sum_{n=k+1}^{\infty} c_{k+1} |a_n| \leq \sum_{n=2}^{\infty} c_n |a_n|
\]

which is equivalent to

\[
\sum_{n=2}^{k} (c_n - 1) |a_n| + \sum_{n=k+1}^{\infty} (c_n - c_{k+1}) |a_n| \geq 0.
\]  

(32)

which readily yields the assertion (26) of Theorem 4.1. In order to see that

\[ f(z) = z + \frac{c_{k+1}}{c_k} \]

(33)

gives sharp result, we observe that for \( z = re^{i\pi/k} \) that \( f(z) \) \( \frac{f(z)}{f'(z)} = 1 + \frac{c_k}{c_{k+1}} \to 1 - \frac{1}{c_{k+1}} \) as \( z \to 1^- \). Similarly, if we take

\[
g_2(z) = (1 + c_{k+1}) \left( \frac{f(z)}{f'(z)} - \frac{c_{k+1}}{1+c_{k+1}} \right)
\]

\[
= 1 - \frac{1 + \sum_{n=2}^{\infty} c_n e^{z-1}}{1 + \sum_{n=2}^{\infty} c_n e^{z-1}}
\]

(34)

and making use of (29), we can deduce that

\[
\left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| \leq \frac{(1 + c_{k+1}) \sum_{n=k+1}^{\infty} |a_n|}{2 - 2 \sum_{n=2}^{k} |a_n| - (1 - c_{k+1}) \sum_{n=k+1}^{\infty} |a_n|}
\]

which leads us immediately to the assertion (27) of Theorem 4.1. The bound in (27) is sharp for each \( k \in \mathbb{N} \) with the extremal function \( f(z) \) given by (33). The proof of the Theorem 4.1, is thus complete.  

Remark 4.1. Letting $\eta = 0$ in Theorem 4.1, we obtain Theorem 1 in [1].

Taking $\Phi(z) = z/(1-z)^2$, $\Psi(z) = z/(1-z)$ and $\eta = 0$ in Theorem 4.1, we obtain

Corollary 4.1. ([5]). Let the function $f(z)$ be defined by (1). If

$$\sum_{n=2}^{\infty} (n-\gamma) |a_n| \leq 1 - \gamma$$

then

$$\text{Re} \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{k}{k + 1 - \gamma} \quad \text{(for all } z \in U)$$

(36)

and

$$\text{Re} \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{k + 1 - \gamma}{k + 2 - 2\gamma} \quad \text{(for all } z \in U).$$

(37)

The results are sharp with the function given by

$$f(z) = z + \frac{1 - \gamma}{k + 1 - \gamma} z^{k+1}.$$  

(38)

Taking $\Phi(z) = (z + z^2)/(1-z)^3$, $\Psi(z) = z/(1-z)^2$ and $\eta = 0$ in Theorem 4.1, we obtain

Corollary 4.2. ([5]). Let the function $f(z)$ be defined by (1). If

$$\sum_{n=2}^{\infty} n(n-\gamma) |a_n| \leq 1 - \gamma$$

then

$$\text{Re} \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{k(k+2-\gamma)}{(k+1)(k+1-\gamma)} \quad \text{(for all } z \in U)$$

(40)

and

$$\text{Re} \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{(k+1)(k+1-\gamma)}{(k+1)((k+1)-\gamma)+1-\gamma} \quad \text{(for all } z \in U).$$

(41)

The results are sharp with the function given by

$$f(z) = z + \frac{1 - \alpha}{(k + 1)^2 - \alpha(k + 1)} z^{k+1}.$$  

(42)

Theorem 4.2. If $f(z)$ of the form (1) satisfies the condition (5). Then

$$\text{Re} \left\{ \frac{f'(z)}{f'_k(z)} \right\} \geq 1 - \frac{k + 1}{c_{k+1}}.$$  

(43)
and

$$\text{Re} \left\{ \frac{f_k^{(z)}}{f^{(z)}} \right\} \geq \frac{c_{k+1}}{k + 1 + c_{k+1}}.$$  \hspace{1cm} (44)

where \(c_n\) defined as in (25) and satisfies the condition

$$c_n \geq \begin{cases} n, & \text{if } n = 2, 3, \ldots, k + 1 \\ \frac{n}{k+1}, & \text{if } n = k + 1, k + 2, \ldots. \end{cases}$$  \hspace{1cm} (45)

The results are sharp with the function given by (33).

**Proof.** By setting

$$g(z) = c_{k+1} \left\{ \frac{f_k^{(z)}}{f^{(z)}} - \left(1 - \frac{k+1}{c_{k+1}} \right) \right\} = 1 + \frac{c_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n a_n z^{n-1}.$$

Then

$$\left| \frac{g(z)-1}{g(z)+1} \right| \leq 2 - 2 \sum_{n=2}^{k} n |a_n|.$$

Now

$$\left| \frac{g(z)-1}{g(z)+1} \right| \leq 1$$

if

$$\sum_{n=2}^{k} n |a_n| + \frac{c_{k+1}}{k+1} \sum_{n=k+1}^{\infty} n |a_n| \leq 1$$  \hspace{1cm} (47)

since the left hand side of (47) is bounded above by \(\sum_{n=2}^{k} c_n |a_n|\) if

$$\sum_{n=2}^{k} (c_n - n) |a_n| + \sum_{n=k+1}^{\infty} \left( c_n - \frac{c_{k+1}}{k+1} n \right) |a_n| \geq 0,$$  \hspace{1cm} (48)

and the proof of (43) is complete.

To prove the result (44), define the function \(g(z)\) by

$$g(z) = \left[(k+1) + c_{k+1}\right] \left\{ \frac{f_k^{(z)}}{f^{(z)}} - \frac{c_{k+1}}{k+1 + c_{k+1}} \right\} = 1 - \frac{1 + \frac{c_{k+1}}{k+1}}{1 + \sum_{n=k+1}^{\infty} n a_n z^{n-1}}.$$

and making use of (48), we deduce that
\[
\left| \frac{g(z) - 1}{g(z) + 1} \right| \leq \frac{\left( 1 + \frac{c_{k+1}}{k+1} \right) \sum_{n=k+1}^{\infty} n|a_n|}{2 - 2 \sum_{n=2}^{k} n|a_n| - \left( 1 + \frac{c_{k+1}}{k+1} \right) \sum_{n=k+1}^{\infty} n|a_n|} \leq 1.
\]
which leads us immediately to the assertion (44) of the Theorem 4.2.]

Remark 4.2. Letting \( \eta = 0 \) in Theorem 4.2, we obtain Theorem 2 in [1].

Taking \( \Phi(z) = z/(1-z)^2 \), \( \Psi(z) = z/(1-z) \) and \( \eta = 0 \) in Theorem 4.2, we obtain

Corollary 4.3. ([5]). Let the function \( f(z) \) be defined by (1). If
\[
\sum_{n=2}^{\infty} (n - \gamma)|a_n| \leq 1 - \gamma \quad (49)
\]
then
\[
Re \left\{ \frac{f'(z)}{f''(z)} \right\} \geq \frac{k \gamma}{k + 1 - \gamma} \quad (for \ all \ z \in \mathbb{U}) \quad (50)
\]
and
\[
Re \left\{ \frac{f_k'(z)}{f_k''(z)} \right\} \geq \frac{k + 1 - \gamma}{(k + 1)(2 - \gamma) - \gamma} \quad (for \ all \ z \in \mathbb{U}). \quad (51)
\]
The results are sharp with the function given by
\[
f(z) = z + \frac{1 - \gamma}{k + 1 - \gamma}z^{k+1}. \quad (52)
\]
Taking \( \Phi(z) = (z + z^2)/(1-z)^3 \) and \( \Psi(z) = z/(1-z)^2 \) and \( \eta = 0 \) in Theorem 4.2, we obtain

Corollary 4.4. ([5]). Let the function \( f(z) \) be defined by (1). If
\[
\sum_{n=2}^{\infty} n(n - \gamma)|a_n| \leq 1 - \gamma \quad (53)
\]
then
\[
Re \left\{ \frac{f'(z)}{f''(z)} \right\} \geq \frac{k}{k + 1 - \gamma} \quad (for \ all \ z \in \mathbb{U}) \quad (54)
\]
and
\[
Re \left\{ \frac{f_k'(z)}{f_k''(z)} \right\} \geq \frac{k + 1 - \gamma}{k + 2 - 2 \gamma} \quad (for \ all \ z \in \mathbb{U}). \quad (55)
\]
The results are sharp with the function given by

\[ f(z) = z + \frac{1 - \gamma}{(k + 1)^2 - \gamma(k + 1)} z^{k+1}. \]  \hspace{1cm} (56)

Acknowledgement. The authors would like to thank the referee for his helpful comments and suggestions.

References


