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FAMILY OF ANALYTIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract In this paper a new class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ of analytic functions is introduced and defined in the open unit disc. Coefficient inequalities, distortion bounds, closure theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to this class are obtained. Partial sums $f_n(z)$ of functions $f(z)$ in this class are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f_n'(z)$ are also determined.

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1. INTRODUCTION

Let $\mathcal{S}$ be the class of functions f *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are *analytic* and *univalent* in the *open* unit disc $\mathcal{U} = \{z : |z| < 1\}$. Let the functions $\Phi(z) = z + \sum\limits_{n=2}^{\infty} \lambda_n z^n$ and $\Psi(z) = z + \sum\limits_{n=2}^{\infty} \mu_n z^n$ be analytic in $\mathcal{U}$, where $\lambda_n \geq \mu_n \geq 0$. Then we define $\mathcal{S}(\Phi, \Psi, \eta, \gamma)$ be the subclass of $\mathcal{S}$ consisting of functions $f(z)$ of the form (1) and satisfying the analytic criteria

$$\text{Re} \left\{ \frac{(f * \Phi)(z)}{(1-\eta)(f * \Psi)(z) + \eta(f * \Phi)(z)} \right\} \geq \gamma, \quad z \in \mathcal{U}, \tag{2}$$

where $0 \leq \eta < 1$, $0 \leq \gamma < 1$ and the operator $(*)$ stands for the convolution of two power series $f(z) = z \pm \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = z \pm \sum\limits_{n=2}^{\infty} b_n z^n$ defined by

$$(f * g)(z) = f(z) * g(z) = z \pm \sum_{n=2}^{\infty} a_n b_n z^n.$$

We also let

$$\mathcal{ST}(\Phi, \Psi, \eta, \gamma) = \mathcal{S}(\Phi, \Psi, \eta, \gamma) \cap \mathcal{T} \tag{3}$$

where $\mathcal{T}$, the subclass of $\mathcal{S}$, consists functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{4}$$

was introduced and studied by Silverman [4].

The families $\mathcal{S}(\Phi, \Psi, \eta, \gamma)$ and $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ are comprehensive classes consisting of various well-know classes of analytic functions as well as many new ones. For example, for suitable choices of Φ, Ψ, and η we obtain the following subclasses of $\mathcal{T}$ studied by various authors:

1 $\mathcal{ST}\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, 0, \ \gamma\right) = \mathcal{T}^\star(\gamma)$ (Silverman [4].);

2 $\mathcal{ST}\left(\frac{z+z^2}{(1-z)^3}, \ \frac{z}{(1-z)^2}, \ 0, \ \gamma\right) = \mathcal{C}(\gamma)$ (Silverman [4].);

3 $\mathcal{ST}\left(\frac{z}{(1-z)^2}, z, 0, \ \gamma\right) = \mathcal{P}^\star(\gamma)$ (Sarangi and Uralegaddi [3]);

4 $\mathcal{ST}\left(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}, 0, \ \gamma\right) = \mathcal{R}[\gamma]$ (Silverman and Silvia [6]) ;

5 $\mathcal{ST}(\Phi, \Psi; 0, \gamma) = \mathcal{E}_{\mathcal{T}}(\Phi, \Psi; \gamma)$; (Juneja *et al.*[2]).

In this paper, we obtain coefficient inequalities, distortion bounds, closure theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. Finally, we consider partial sums of functions in this class and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f_n'(z)$.

In the following section we obtain coefficient inequalities and distortion bounds for functions in the class $\mathcal{S}(\Phi, \Psi, \eta, \gamma)$.

2. COEFFICIENT ESTIMATES AND DISTORTION BOUNDS

First we obtain the coefficient inequalities for $f \in \mathcal{S}(\Phi, \Psi, \eta, \gamma)$.

Theorem 2.1. *If $f \in \mathcal{S}$ satisfies*

$$\sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma] \, |a_n| \leq 1 - \gamma \tag{5}$$

then $f \in \mathcal{S}(\Phi, \Psi, \eta, \gamma)$.

Proof. Assume that the inequality (5) holds and let $|z| = 1$. Then,

$$\left|\frac{(f*\Phi)(z)}{(1-\eta)(f*\Psi)(z)+\eta(f*\Phi)(z)}-1\right| = \left|\frac{z+\sum\limits_{n=2}^{\infty}\lambda_n a_n z^n}{z+\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]a_n z^n}-1\right|$$
$$\leq \frac{\sum\limits_{n=2}^{\infty}\lambda_n\,|a_n|\,|z|^{n-1}-\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]\,|a_n|\,|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]\,|a_n|\,|z|^{n-1}}.$$

Letting $z \longrightarrow 1$ we have

$$\frac{\sum\limits_{n=2}^{\infty}\lambda_n\,|a_n|-\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]\,|a_n|}{1-\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]\,|a_n|}.$$

The above expression is bounded above by $1-\gamma$ that is ,

$$\frac{\sum\limits_{n=2}^{\infty}\lambda_n\,|a_n|-\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]\,|a_n|}{1-\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]\,|a_n|}\leq 1-\gamma.$$

This shows that $\dfrac{(f*\Phi)z}{(1-\eta)(f*\Psi)(z)+\eta(f*\Phi)(z)}-1$ lies in a circle centered at $w=1$ whose radius is $1-\gamma$, that completes the proof. ∎

A necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{ST}(\Phi,\Psi,\eta,\gamma)$ is given by

Theorem 2.2. *Let the function f be defined by (4). Then $f \in \mathcal{ST}(\Phi,\Psi,\eta,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}[\lambda_n-(\mu_n+\eta(\lambda_n-\mu_n))\gamma]a_n \leq 1-\gamma. \tag{6}$$

The result is sharp for the function

$$f(z)=z-\frac{(1-\gamma)}{[\lambda_n-(\mu_n+\eta(\lambda_n-\mu_n))\gamma]}z^n,\quad n\geq 2. \tag{7}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f(z) \in \mathcal{ST}(\Phi,\Psi,\eta,\gamma)$ and z is real then assume that the inequality (6) holds and let $|z| = 1$.

Then,

$$\begin{aligned} &\mathrm{Re}\left\{\frac{(f*\Phi)(z)}{(1-\eta)(f*\Psi)(z)+\eta(f*\Phi)(z)}\right\} \\ =\ &\mathrm{Re}\left\{\frac{z-\sum\limits_{n=2}^{\infty}\lambda_n a_n z^n}{z-\sum\limits_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]a_n z^n}-1\right\}\geq\gamma, \end{aligned}$$

$z\in\mathcal{U}$. Choose the values of z on the real axis so that $\frac{(f*\Phi)z}{(1-\eta)(f*\Psi)(z)+\eta(f*\Phi)(z)}$ is real and upon clearing the denominator in the above inequality and letting $z=1^-$, through the real values we get

$$1-\sum_{n=2}^{\infty}\lambda_n a_n|z|^{n-1}\geq\gamma\left\{1-\sum_{n=2}^{\infty}[\mu_n+\eta(\lambda_n-\mu_n)]a_n|z|^{n-1}\right\}$$

which yields (6). The result is sharp for the function given by (7). ∎

Corollary 2.1. *If $f(z)$ of the form (4) is in $\mathcal{ST}(\Phi,\Psi,\eta,\gamma)$, then*

$$a_n\leq\frac{1-\gamma}{\lambda_n-(\mu_n+\eta(\lambda_n-\mu_n))\gamma},\quad n\geq 2, \tag{8}$$

with equality only for functions of the form (7).

Theorem 2.3. ***(Distortion Bounds)*** *Let the function $f(z)$ defined by (4) belong to $\mathcal{ST}(\Phi,\Psi,\eta,\gamma)$. If $\{\lambda_n-(\mu_n+\eta(\lambda_n-\mu_n)\}_{n=2}^{\infty}$ is nondecreasing sequences , then for $|z|\leq r$,*

$$r-\frac{1-\gamma}{\lambda_2-(\mu_2+\eta(\lambda_2-\mu_2))\gamma}r^2\leq|f(z)|\leq r+\frac{1-\gamma}{\lambda_2-(\mu_2+\eta(\lambda_2-\mu_2))\gamma}r^2 \tag{9}$$

and if $\left\{\frac{\lambda_n-(\mu_n+\eta(\lambda_n-\mu_n)}{n}\right\}_{n=2}^{\infty}$ is nondecreasing sequences , then for

$$1-\frac{2(1-\gamma)}{\lambda_2-(\mu_2+\eta(\lambda_2-\mu_2))\gamma}r\leq|f'(z)|\leq 1+\frac{2(1-\gamma)}{\lambda_2-(\mu_2+\eta(\lambda_2-\mu_2))\gamma}r \tag{10}$$

for $z\in\mathcal{U}$.

Proof. In view of Theorem 2.2, we note that

$$\lambda_2-(\mu_2+\eta(\lambda_2-\mu_2))\gamma\sum_{n=2}^{\infty}a_n\leq\sum_{n=2}^{\infty}\lambda_n-(\mu_n+\eta(\lambda_n-\mu_n)a_n\leq 1-\gamma, \tag{11}$$

also, we have

$$\frac{\lambda_2-(\mu_2+\eta(\lambda_2-\mu_2))\gamma}{2}\sum_{n=2}^{\infty}na_n\leq\sum_{n=2}^{\infty}\lambda_n-(\mu_n+\eta(\lambda_n-\mu_n)a_n\leq 1-\gamma. \tag{12}$$

Using (4) and the inequalities (11) and (12), we obtain

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{1-\gamma}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r^2. \tag{13}$$

Similarly,

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1-\gamma}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r^2.$$

Also we have

$$\left|f'(z)\right| \geq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} n a_n \geq 1 - \frac{2(1-\gamma)}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r$$

and

$$\left|f'(z)\right| \leq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n a_n \leq 1 + \frac{2(1-\gamma)}{\lambda_2 - (\mu_2 + \eta(\lambda_2 - \mu_2))\gamma} r.$$

∎

3. EXTREME POINTS AND CLOSURE THEOREMS

In this section we discuss the closure properties.

Theorem 3.1. ***(Extreme Points)*** *Let*

$$\begin{aligned} f_1(z) &= z \quad \text{and} \\ f_n(z) &= z - \frac{(1-\gamma)}{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]} z^n, \quad n \geq 2, \end{aligned} \tag{14}$$

for $0 \leq \gamma < 1, \eta \geq 0$. *Then* $f(z)$ *is in the class* $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z), \tag{15}$$

where $\mu_n \geq 0$ *and* $\sum_{n=1}^{\infty} \theta_n = 1$.

Proof. Suppose $f(z)$ can be written as in (15). Then

$$f(z) = z - \sum_{n=2}^{\infty} \theta_n \frac{(1-\gamma)}{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]a_n}{(1-\gamma)} \theta_n \frac{(1-\gamma)}{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]a_n} = \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \leq 1.$$

Thus $f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. Conversely, let us have $f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. Then by using (15), we set

$$\theta_n = \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)} a_n, \quad n \geq 2$$

and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$. Then we have $f(z) = \sum_{n=1}^{\infty} \theta_n f_n(z)$ and hence this completes the proof of Theorem 3.1. ∎

Theorem 3.2. ***(Closure Theorem)*** *Let the functions $f_j(z)$ be defined for $j = 1, 2, \ldots, m$ defined by*

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0, \ \ z \in \mathcal{U} \tag{16}$$

be in the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma_j)$ $(j = 1, 2, \ldots m)$ respectively. Then the function $h(z)$ defined by

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} a_{n,j} \right) z^n$$

is in the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$, where $\gamma = \min_{1 \leq j \leq m} \{\gamma_j\}$ where $0 \leq \gamma_j < 1$.

Proof. Since $f_j(z) \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma_j)$ $(j = 1, 2, 3, \ldots m)$ by applying Theorem 2.2, to (16) we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n)\gamma] \left(\frac{1}{m} \sum_{j=1}^{m} a_{n,j} \right) \\
&= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma] a_{n,j} \right) \\
&\leq \frac{1}{m} \sum_{j=1}^{m} (1 - \gamma_j) \leq 1 - \gamma
\end{aligned}$$

which in view of Theorem 2.2, implies that $h(z) \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ and so the proof is complete. ∎

Theorem 3.3. *The class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ is a convex set.*

Proof. Let the functions defined by (16) be in the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \nu f_1(z) + (1 - \nu) f_2(z), \quad 0 \leq \nu \leq 1,$$

is in the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. Since

$$h(z) = z - \sum_{n=2}^{\infty} [\nu a_{n,1} + (1-\nu) a_{n,2}] z^n,$$

an easy computation with the aid of Theorem 2.2 gives,

$$\begin{aligned} &\sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma] \nu a_{n,1} + \sum_{n=2}^{\infty} [\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n)\gamma] (1-\nu) a_{n,2} \\ \leq\ & \nu(1-\gamma) + (1-\nu)(1-\gamma) \\ \leq\ & 1-\gamma, \end{aligned}$$

which implies that $h \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. Hence $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ is convex. ∎

3.1. RADII OF STARLIKENESS AND CONVEXITY

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$.

Theorem 3.4. *Let the function $f(z)$ defined by (4) belong to the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \inf_{n \geq 2} \left[\frac{(1-\rho)}{n} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)} \right]^{\frac{1}{n-1}}. \tag{17}$$

The result is sharp, with extremal function $f(z)$ given by (14).

Proof. Given $f \in \mathcal{T}$, and f is close-to-convex of order ρ, we have

$$|f'(z) - 1| < 1 - \rho. \tag{18}$$

For the left hand side of (18) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1-\rho$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\rho} a_n |z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)} a_n \leq 1,$$

we can say (18) is true if

$$\frac{n}{1-\rho}|z|^{n-1} \leq \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\rho}{n}\right)\frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)}\right]$$

which completes the proof. ∎

Theorem 3.5. *Let $f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$. Then*

(i) f is starlike of order $\rho(0 \leq \rho < 1)$ in the disc $|z| < r_2$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho$, $(|z| < r_2\ ;\ 0 \leq \rho < 1)$, where

$$r_2 = \inf_{n\geq 2}\left[\left(\frac{1-\rho}{n-\rho}\right)\frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)}\right]^{\frac{1}{n-1}}, \tag{19}$$

(ii) f is convex of order ρ $(0 \leq \rho < 1)$ in the unit disc $|z| < r_3$, that is $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho$, $(|z| < r_3; 0 \leq \rho < 1)$, where

$$r_3 = \inf_{n\geq 2}\left[\left(\frac{1-\sigma}{n(n-\rho)}\right)\frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)}\right]^{\frac{1}{n-1}}. \tag{20}$$

Each of these results are sharp for the extremal function $f(z)$ given by (14).

Proof. (i) Given $f \in \mathcal{T}$, and f is starlike of order ρ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \rho. \tag{21}$$

For the left hand side of (21) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n\,|z|^{n-1}}.$$

The last expression is less than $1-\rho$ if

$$\sum_{n=2}^{\infty}\frac{n-\rho}{1-\rho}a_n\,|z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)} a_n \leq 1.$$

We can say (21) is true if

$$\frac{n-\rho}{1-\rho}|z|^{n-1} < \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)},$$

or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\rho}{n-\rho}\right) \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)}\right]$$

which yields the starlikeness of the family.

(ii) Given $f \in \mathcal{T}$, and f is convex of order ρ, we have

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 - \rho. \tag{22}$$

For the left hand side of (22) we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n\,|z|^{n-1}}{1 - \sum_{n=2}^{\infty} na_n\,|z|^{n-1}}.$$

The last expression is less than $1-\rho$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\rho)}{1-\rho} a_n\,|z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)} a_n \leq 1.$$

We can say (22) is true if

$$\frac{n(n-\rho)}{1-\rho}|z|^{n-1} < \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\rho}{n(n-\rho)}\right) \frac{[\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma]}{(1-\gamma)}\right]$$

which yields the convexity of the family. ∎

4. PARTIAL SUMS

Following the earlier works by Silverman [5] and Silvia [7] on partial sums of analytic functions. We consider in this section partial sums of functions in the class $\mathcal{ST}(\Phi, \Psi, \eta, \gamma)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f_k'(z)$.

Theorem 4.1. *Define the partial sums $f_1(z)$ and $f_k(z)$, by*

$$f_1(z) = z; \text{ and } f_k(z) = z + \sum_{n=2}^{k} a_n z^n, \ (k \geq 2). \tag{23}$$

If $f(z)$ of the form (1) satisfies the condition (5) and

$$c_n \geq \begin{cases} 1, & n = 2, 3, \ldots, k \\ c_{k+1}, & n = k+1, k+2, \ldots, \end{cases} \tag{24}$$

where, for convenience,

$$c_n := \frac{\lambda_n - (\mu_n + \eta(\lambda_n - \mu_n))\gamma}{1 - \gamma}. \tag{25}$$

Then

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} > 1 - \frac{1}{c_{k+1}}, \quad z \in \mathcal{U}, k \in \mathbb{N} \tag{26}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} > \frac{c_{k+1}}{1 + c_{k+1}}. \tag{27}$$

where

$$c_n \geq \begin{cases} 1, & n = 2, 3, \ldots, k+1 \\ c_{k+1}, & n = k+1, k+2, \ldots . \end{cases} \tag{28}$$

Proof. From (24) it follows that

$$\sum_{n=2}^{k} |a_n| + c_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} c_n |a_n| \leq 1. \tag{29}$$

By setting

$$\begin{aligned} g_1(z) &= c_{k+1}\left\{\frac{f(z)}{f_k(z)} - \left(1 - \frac{1}{c_{k+1}}\right)\right\} \\ &= 1 + \frac{c_{k+1} \sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{k} a_n z^{n-1}} \end{aligned} \tag{30}$$

and applying (29), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le \frac{c_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{n}|a_n| - c_{k+1}\sum\limits_{n=k+1}^{\infty}|a_n|}. \tag{31}$$

Now $\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \le 1$ if

$$\sum_{n=2}^{k}|a_n| + \sum_{n=k+1}^{\infty} c_{k+1}\,|a_n| \le 1.$$

From the condition (5), it is sufficient to show that

$$\sum_{n=2}^{k}|a_n| + \sum_{n=k+1}^{\infty} c_{k+1}\,|a_n| \le \sum_{n=2}^{\infty} c_n\,|a_n|$$

which is equivalent to

$$\sum_{n=2}^{k}(c_n-1)\,|a_n| + \sum_{n=k+1}^{\infty}(c_n-c_{k+1})\,|a_n| \ge 0. \tag{32}$$

which readily yields the assertion (26) of Theorem 4.1. In order to see that

$$f(z) = z + \frac{z^{k+1}}{c_{k+1}} \tag{33}$$

gives sharp result, we observe that for $z = re^{i\pi/k}$ that $\frac{f(z)}{f_k(z)} = 1 + \frac{z^k}{c_{k+1}} \to 1 - \frac{1}{c_{k+1}}$ as $z \to 1^-$. Similarly, if we take

$$\begin{aligned} g_2(z) &= (1+c_{k+1})\left\{\frac{f_k(z)}{f(z)} - \frac{c_{k+1}}{1+c_{k+1}}\right\} \\ &= 1 - \frac{(1+c_{n+1})\sum\limits_{n=k+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} a_n z^{n-1}} \end{aligned} \tag{34}$$

and making use of (29), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \le \frac{(1+c_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{k}|a_n| - (1-c_{k+1})\sum\limits_{n=k+1}^{\infty}|a_n|}$$

which leads us immediately to the assertion (27) of Theorem 4.1. The bound in (27) is sharp for each $k \in \mathbb{N}$ with the extremal function $f(z)$ given by (33). The proof of the Theorem 4.1, is thus complete. ∎

Remark 4.1. *Letting $\eta = 0$ in Theorem4.1, we obtain Theorem 1 in [1].*

Taking $\Phi(z) = z/(1-z)^2$, $\Psi(z) = z/(1-z)$ and $\eta = 0$ in Theorem 4.1, we obtain

Corollary 4.1. *([5]) . Let the function $f(z)$ be defined by (1). If*

$$\sum_{n=2}^{\infty}(n-\gamma)\,|a_n| \leq 1-\gamma \tag{35}$$

then

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{k}{k+1-\gamma} \qquad \text{(for all } z \in \mathfrak{U}) \tag{36}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{k+1-\gamma}{k+2-2\gamma} \qquad \text{(for all } z \in \mathfrak{U}). \tag{37}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\gamma}{k+1-\gamma} z^{k+1}. \tag{38}$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$, $\Psi(z) = z/(1-z)^2$ and $\eta = 0$ in Theorem 4.1, we obtain

Corollary 4.2. *([5]). Let the function $f(z)$ be defined by (1). If*

$$\sum_{n=2}^{\infty} n(n-\gamma)\,|a_n| \leq 1-\gamma \tag{39}$$

then

$$Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{k(k+2-\gamma)}{(k+1)(k+1-\gamma)} \qquad \text{(for all } z \in \mathfrak{U}) \tag{40}$$

and

$$Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(k+1)(k+1-\gamma)}{(k+1)[(k+1)-\gamma]+1-\gamma} \qquad \text{(for all } z \in \mathfrak{U}). \tag{41}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\alpha}{(k+1)^2-\alpha(k+1)} z^{k+1}. \tag{42}$$

Theorem 4.2. *If $f(z)$ of the form (1) satisfies the condition (5). Then*

$$Re\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq 1 - \frac{k+1}{c_{k+1}}. \tag{43}$$

and

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{c_{k+1}}{k+1+c_{k+1}}. \tag{44}$$

where c_n defined as in (25) and satisfies the condition

$$c_n \geq \begin{cases} n, & if\ n = 2, 3, \ldots, k+1 \\ \frac{c_{k+1}}{k+1}n & if\ n = k+1, k+2, \ldots. \end{cases} \tag{45}$$

The results are sharp with the function given by (33).

Proof. By setting

$$\begin{aligned} g(z) &= c_{k+1}\left\{\frac{f'(z)}{f_k'(z)} - \left(1 - \frac{k+1}{c_{k+1}}\right)\right\} \\ &= \frac{1+\frac{c_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} na_n z^{n-1} + \sum\limits_{n=2}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{k} na_n z^{n-1}} \\ &= 1 + \frac{\frac{c_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{k} na_n z^{n-1}}. \end{aligned} \tag{46}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{c_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \frac{c_{k+1}}{k+1}\sum\limits_{n=k+1}^{\infty} n|a_n|}.$$

Now

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$$

if

$$\sum_{n=2}^{k} n|a_n| + \frac{c_{k+1}}{k+1}\sum_{n=k+1}^{\infty} n|a_n| \leq 1 \tag{47}$$

since the left hand side of (47) is bounded above by $\sum\limits_{n=2}^{k} c_n|a_n|$ if

$$\sum_{n=2}^{k}(c_n - n)|a_n| + \sum_{n=k+1}^{\infty}\left(c_n - \frac{c_{k+1}}{k+1}n\right)|a_n| \geq 0, \tag{48}$$

and the proof of (43) is complete.

To prove the result (44), define the function $g(z)$ by

$$\begin{aligned} g(z) &= [(k+1)+c_{k+1}]\left\{\frac{f_k'(z)}{f'(z)} - \frac{c_{k+1}}{k+1+c_{k+1}}\right\} \\ &= 1 - \frac{\left(1+\frac{c_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} na_n z^{n-1}}{1+\sum\limits_{n=2}^{k} na_n z^{n-1}} \end{aligned}$$

and making use of (48), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{c_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{k} n|a_n| - \left(1+\frac{c_{k+1}}{k+1}\right)\sum\limits_{n=k+1}^{\infty} n|a_n|} \leq 1.$$

which leads us immediately to the assertion (44) of the Theorem 4.2. ∎

Remark 4.2. *Letting $\eta = 0$ in Theorem 4.2, we obtain Theorem 2 in [1].*

Taking $\Phi(z) = z/(1-z)^2$, $\Psi(z) = z/(1-z)$ and $\eta = 0$ in Theorem 4.2, we obtain

Corollary 4.3. *([5]). Let the function $f(z)$ be defined by (1). If*

$$\sum_{n=2}^{\infty}(n-\gamma)\,|a_n| \leq 1-\gamma \tag{49}$$

then

$$Re\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq \frac{k\gamma}{k+1-\gamma} \qquad (\text{for all } z \in \mathcal{U}) \tag{50}$$

and

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{k+1-\gamma}{(k+1)(2-\gamma)-\gamma} \qquad (\text{for all } z \in \mathcal{U}). \tag{51}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\gamma}{k+1-\gamma} z^{k+1}. \tag{52}$$

Taking $\Phi(z) = (z+z^2)/(1-z)^3$ and $\Psi(z) = z/(1-z)^2$ and $\eta = 0$ in Theorem 4.2, we obtain

Corollary 4.4. *([5]). Let the function $f(z)$ be defined by (1). If*

$$\sum_{n=2}^{\infty} n(n-\gamma)\,|a_n| \leq 1-\gamma \tag{53}$$

then

$$Re\left\{\frac{f'(z)}{f_k'(z)}\right\} \geq \frac{k}{k+1-\gamma} \qquad (\text{for all } z \in \mathcal{U}) \tag{54}$$

and

$$Re\left\{\frac{f_k'(z)}{f'(z)}\right\} \geq \frac{k+1-\gamma}{k+2-2\gamma} \qquad (\text{for all } z \in \mathcal{U}). \tag{55}$$

The results are sharp with the function given by

$$f(z) = z + \frac{1-\gamma}{(k+1)^2 - \gamma(k+1)} z^{k+1}. \tag{56}$$

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