SET-NORM CONTINUITY
OF SET MULTIFUNCTIONS
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Abstract
In this paper, we present different types of continuous set multifunctions with respect to a set norm (such as uniformly autocontinuous or autocontinuous from above), their relationships with non-additive set multifunctions and some properties of atoms and pseudo-atoms for null-null-additive set multifunctions.

Keywords: set-norm, autocontinuous from above, uniformly autocontinuous, null-additive, null-null-additive, sn-continuous, sn-exhaustive, atom, pseudo-atom.


1. INTRODUCTION
Non-additive measures have been lately studied by many authors (see for example Asahina [1], Choquet [2], Dempster [4], Denneberg [5], Dobrakov [6], Drewnowski [7], Li [15], Liginal and Ow [16], Pap [17], Precupanu [18], Shafer [20], Sugeno [21], Suzuki [22], Wu Congxin and Wu Cong [23]) due to their applications in statistics, economy, theory of games, human decision making, medicine. In non-additive measure theory, some continuity conditions are used to prove important results with respect to non-additive measures (for example, Theorem of Egoroff in Li [15]). In Precupanu and Croitoru [19], Gavriluț [10], Gavriluț and Croitoru [11, 12], Croitoru et al. [3] we extended some classical measure or integral concepts to set multifunctions, proposing a framework for the set-valued case.

In this paper, we introduce and study different types of set-norm continuous set multifunctions, such as uniformly autocontinuous or autocontinuous from above. We also establish some properties of atoms and pseudo-atoms for null-null-additive set multifunctions.

2. PRELIMINARIES
In the sequel, X will be a real linear space and \( \mathcal{P}_0(X) \) the family of non-empty subsets of X. On \( \mathcal{P}_0(X) \) we consider an order relation denoted by \( \leq \) and we shall write \( (\mathcal{P}_0(X), \leq) \). We write \( E < F \) if \( E \leq F \) and \( E \neq F \), for \( E, F \in \mathcal{P}_0(X) \). For convenience, the notation \( F \geq E \) will be used instead of \( E \leq F \).
If $X$ is a normed space, then $\mathcal{P}_f(X)$ is the family of non-empty closed subsets of $X$ and $\mathcal{P}_{bf}(X)$ is the family of non-empty closed bounded subsets of $X$.

For every $M, N \in \mathcal{P}_0(X)$, we denote
\[
h(M, N) = \max \{ e(M, N), e(N, M) \},
\]
where $e(M, N) = \sup_{x \in M} d(x, N)$ is the excess of $M$ over $N$ and $d(x, N)$ is the distance from $x$ to $N$. It is known that $h$ becomes an extended metric on $\mathcal{P}_f(X)$ (i.e. it is a metric which can also take the value $+\infty$) and $h$ becomes a metric (called Pompeiu-Hausdorff) on $\mathcal{P}_{bf}(X)$ (Hu and Papageorgiou [13]).

We denote $\mathbb{N} = \mathbb{N}\setminus\{0\}$, $\mathbb{R}_+ = \{0, +\infty\}$ and $\overline{\mathbb{R}}_+ = \{0, +\infty\}$.

**Example 2.1.**

1. The usual set inclusion "$\subseteq$" is an order relation on $\mathcal{P}_0(X)$ and we write $(\mathcal{P}_0(X), \subseteq)$.

2. Let $(X, \leq)$ be a real ordered vector space. For every $E, F \in \mathcal{P}_0(X)$ and $\alpha \in \mathbb{R}$ let
\[
E + F = \{ x + y | x \in E, y \in F \},
\]
\[
\alpha E = \{ \alpha x | x \in E \},
\]
\[
E - F = \{ x - y | x \in E, y \in F \}.
\]
Let $P$ be the cone of positive elements of $X$, that is $P = \{ x \in X | x \geq 0 \}$. For every $E, F \in \mathcal{P}_0(X)$, we set $E < F$ if and only if $E \subseteq F - P$ and $F \subseteq E + P$. The relation "$<"$ is reflexive and transitive.

For instance, let $E \in \mathcal{P}_0(\mathbb{R})$. Then $E > \{0\}$ if and only if $E \subseteq [0, +\infty)$.

If $X = \mathbb{R}$, then the relation "$<"$ is an order relation on $\mathcal{P}_c(\mathbb{R})$, the family of non-empty compact convex subsets of $\mathbb{R}$.

**Definition 2.1.** A function $|\cdot| : \mathcal{P}_0(X) \to [0, +\infty]$ is called a set-norm on $\mathcal{P}_0(X)$ if it satisfies the conditions:

(i) $|E| = 0 \Leftrightarrow E = \{0\}$, for $E \in \mathcal{P}_0(X)$.
(ii) $|\alpha E| = |\alpha| \cdot |E|$, $\forall \alpha \in \mathbb{R}$, $\forall E \in \mathcal{P}_0(X)$.
(iii) $|E + F| \leq |E| + |F|$, $\forall E, F \in \mathcal{P}_0(X)$.

**Definition 2.2.** A set-norm $|\cdot|$ on $(\mathcal{P}_0(X), \subseteq)$ is called monotone if $|E| \leq |F|$ for every sets $E, F \in \mathcal{P}_0(X)$, so that $E \leq F$.

**Example 2.2.** Let $(X, \|\cdot\|)$ be a real normed space and $|E|_h = h(E, \{0\}) = \sup_{x \in E} \|x\|$, for every $E \in \mathcal{P}_0(X)$, where $h$ is the Pompeiu-Hausdorff metric.

Then the function $|\cdot|_h$ is a monotone set-norm on $(\mathcal{P}_0(X), \subseteq)$, called the set-norm induced by $h$.

We now recall some well-known results. In the sequel, $T$ is a nonempty set and $\mathcal{C}$ is a ring of subsets of $T$. 
Definition 2.3. Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a set function.

(i) A set $A \in \mathcal{C}$ is said to be an atom of $\nu$ if $\nu(A) > 0$ and for every $B \in \mathcal{C}$, with $B \subseteq A$, we have either $\nu(B) = 0$ or $\nu(A \setminus B) = 0$.

(ii) A set $A \in \mathcal{C}$ is called a pseudo-atom of $\nu$ if $\nu(A) > 0$ and $B \in \mathcal{C}$, $B \subseteq A$ implies either $\nu(B) = 0$ or $\nu(B) = \nu(A)$.

Definition 2.4. A set function $\nu : \mathcal{C} \to \mathbb{R}_+$, so that $\nu(\emptyset) = 0$, is said to be:

(i) monotone if $\nu(A) \leq \nu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

(ii) null-monotone if for every $A, B \in \mathcal{C}$, $A \subseteq B$ and $\nu(B) = 0 \Rightarrow \nu(A) = 0$.

(iii) a submeasure (in the sense of Drewnowski [7]) if $\nu$ is monotone and subadditive, that is, $\nu(A \cup B) \leq \nu(A) + \nu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

(iv) finitely additive if $\nu(A \cup B) = \nu(A) + \nu(B)$, for every $A, B \in \mathcal{C}$, so that $A \cap B = \emptyset$.

(v) exhaustive if $\lim_{n \to \infty} \nu(A_n) = 0$, for every sequence of pairwise disjoint sets $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}$.

(vi) order-continuous (shortly o-continuous) if $\lim_{n \to \infty} \nu(A_n) = 0$, for every sequence of sets $(A_n)_{n \in \mathbb{N}^*} \subseteq \mathcal{C}$, so that $A_n \searrow \emptyset$ (i.e. $A_n \supseteq A_{n+1}$, and $\bigcap_{n=1}^{\infty} A_n = \emptyset$, $\forall n \in \mathbb{N}^*$).

(vii) uniformly autocontinuous if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$, so that for every $B \in \mathcal{C}$, with $\nu(B) < \delta(\varepsilon)$, we have $\nu(A \cup B) < \nu(A) + \varepsilon$, for every $A \in \mathcal{C}$.

(viii) null-additive if $\nu(A \cup B) = \nu(A)$, whenever $A, B \in \mathcal{C}$ and $\nu(B) = 0$.

(ix) null-null-additive if $\nu(A \cup B) = 0$, whenever $A, B \in \mathcal{C}$ and $\nu(A) = \nu(B) = 0$.

3. SET-NORM CONTINUOUS SET MULTIFUNCTIONS

In this section, we present different types of set-norm continuity for set multifunctions and relationships among them. In the sequel, $T$ is a nonempty set and $\mathcal{C}$ is a ring of subsets of $T$.

Definition 3.1. (Gavrițu [8,9])

Let $\mu : \mathcal{C} \to \mathcal{P}_0(X)$ be a set multifunction. $\mu$ is said to be:

(i) monotone if $\mu(A) \leq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

(ii) null-monotone (shortly, n-mon) if for every $A, B \in \mathcal{C}$, so that $A \subseteq B$ we have $\mu(B) = \{0\} \Rightarrow \mu(A) = \{0\}$.

(iii) a multisubmeasure (shortly, msm) if $\mu$ is monotone, $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$ (or, equivalently, for every $A, B \in \mathcal{C}$).

(iv) a multimeasure if $\mu(\emptyset) = \{0\}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, for every $A, B \in \mathcal{C}$ with $A \cap B = \emptyset$.

(v) null-additive (shortly, n-add) if for every $A, B \in \mathcal{C}$,

$$\mu(B) = \{0\} \Rightarrow \mu(A \cup B) = \mu(A).$$
(vi) null-null-additive (shortly, n-n-add) if for every $A, B \in \mathcal{C}$,

$$\mu(A) = \mu(B) = [0] \Rightarrow \mu(A \cup B) = [0].$$

**Remarks 3.1.** I. Let $\nu : \mathcal{C} \to \mathbb{R}_+$ be a set function with $\nu(0) = 0$ and the set multifunction $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ defined by $\mu(A) = [0, \nu(A)]$, for every $A \in \mathcal{C}$. Then $\mu$ is monotone (null-additive, null-null-additive, respectively) if and only if $\nu$ monotone (null-additive, null-null-additive, respectively). Moreover, $\mu$ is a multimeasure (a multisubmeasure, respectively) if and only if $\nu$ is finitely additive (a submeasure in the sense of Drewnowski [7], respectively).

II. In Definition 3.1, some known notions are extended to the set-valued case. The difficulty arises here since we have to consider an order relation on $\mathcal{P}_0(\mathbb{X})$ and many classical measure theory proof methods fail.

For instance, if $\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), \subseteq)$ is single-valued and monotone, then $\mu$ reduces in fact to a constant function defined by $\mu(A) = \{\mu(0)\}$, for every $A \in \mathcal{C}$. So, the definitions in set-valued case do not reduce to the usual single-valued case.

III. The definition of a $\mathcal{P}_0(\mathbb{X})$-valued (or even $\mathcal{P}_f(\mathbb{X})$-valued) multimeasure (or multisubmeasure) can not be reduced to that of the single-valued case because $\mathcal{P}_0(\mathbb{X})$ (and also $\mathcal{P}_f(\mathbb{X})$) is not a linear space: indeed, $\mathcal{P}_0(\mathbb{X})$ is not a group with respect to the addition ”+” defined by $M + N = \{x + y | x \in M, y \in N\}$, for every $M, N \in \mathcal{P}_0(\mathbb{X})$.

IV. If $\nu_1, \nu_2$ are two finite measures defined on an algebra $\mathcal{C}$, so that $\nu_1 \leq \nu_2$ and $\nu_2$ is a probability measure, then one obtains a particular multimeasure $\mu : \mathcal{C} \to \mathcal{P}_0([0, 1])$, defined by $\mu(A) = [\nu_1(A), \nu_2(A)]$, for every $A \in \mathcal{C}$, which is the simplest example of a probability multimeasure. We recall that a multimeasure $M : \mathcal{C} \to \mathcal{P}_0([0, 1])$ is said to be a probability multimeasure if $1 \in M(T)$. These probability multimeasures are used in control, robotics and decision theory (in Bayesian estimation).

In Gavriliţ [8,9] and Croitoru et al. [3], some types of continuous (with respect to the Pompeiu-Hausdorff metric) set multifunctions were introduced. We now extend and study these continuity definitions to the set-norm case.

**Definition 3.2.** Let $|\cdot|$ be a set-norm on $\mathcal{P}_0(\mathbb{X})$. If $\mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{X})$ is a set multifunction, then $\mu$ is said to be:

(i) set-norm autocontinuous from above (shortly, sn-ac-ab) if for every $(B_n)_{n \in \mathbb{N}} \subset \mathcal{C}$ so that $\lim_{n \to \infty} |\mu(B_n)| = 0$, we have

$$\lim_{n \to \infty} |\mu(A \cup B_n)| = |\mu(A)|, \quad \forall A \in \mathcal{C}.$$

(ii) set-norm uniformly autocontinuous (shortly, sn-u-ac) if for every $\varepsilon > 0$, there is $\delta(\varepsilon) = \delta > 0$, so that for every $A \in \mathcal{C}$ and every $B \in \mathcal{C}$, with $|\mu(B)| < \delta$, we have $|\mu(A \cup B)| < |\mu(A)| + \varepsilon$.  


Theorem 3.1. Let \(| \cdot |\) be a monotone set-norm on \((\mathcal{P}(X), \subseteq)\) and \(\mu : \mathcal{C} \to \mathcal{P}_0(X)\) a set multifunction. Then the following statements hold:

I. If \(\mu\) is a multisubmeasure, then \(\mu\) is sn-u-ac.

II. If \(\mu\) is sn-u-ac, then \(\mu\) is sn-ac-ab.

III. If \(\mu\) is sn-ac-ab, then \(\mu\) is null-null-additive.

IV. If \(\mu\) is a multisubmeasure, then \(\mu\) is null-additive.

V. If \(\mu\) is null-additive, then \(\mu\) is null-null-additive and null-monotone.

So, the following schema is working:

\[
\begin{array}{c c c c c}
\text{msm} & \Rightarrow & \text{n-mon} \\
\Downarrow & \searrow & \uparrow \\
\text{sn-u-ac} & \Rightarrow & \text{n-add} \\
\Downarrow & \rightarrow & \Uparrow \\
\text{sn-ac-ab} & \Rightarrow & \text{n-n-add}
\end{array}
\]

Proof. I. Let \(\varepsilon > 0\) and \(B \in \mathcal{C}\) such that \(|\mu(B)| < \varepsilon\). Since \(\mu\) is monotone, it results \(|\mu(A)| \leq |\mu(A \cup B)|\), for all \(A \in \mathcal{C}\).

Then \(|\mu(A \cup B)| \leq |\mu(A) + \mu(B)| \leq |\mu(A)| + |\mu(B)| < |\mu(A)| + \varepsilon\), which proves that \(\mu\) is sn-u-ac.

II. Let \(A \in \mathcal{C}\) and \((B_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}\), so that \(|\mu(B_n)| \to 0\). Since \(\mu\) is sn-u-ac, for every \(\varepsilon > 0\), there is \(d > 0\), so that for every \(A \in \mathcal{C}\) and every \(B \in \mathcal{C}\), with \(|\mu(B)| < d\), we have

\[
|\mu(A \cup B)| < |\mu(A)| + \varepsilon.
\]

Since \(|\mu(B_n)| \to 0\), there is \(n_0 \in \mathbb{N}\), such that \(|\mu(B_n)| < d\), for every \(n \in \mathbb{N}, n \geq n_0\).

From (1) it follows \(|\mu(A \cup B_n)| < |\mu(A)| + \varepsilon\), for every natural \(n \geq n_0\).

By the monotonicity of \(\mu\), we have

\[
|\mu(A)| \leq |\mu(A \cup B_n)|, \quad \forall n \geq n_0.
\]

From (1) and (2) it follows that \(\mu\) is sn-ac-ab.

III. Let \(A, B \in \mathcal{C}\), such that \(\mu(A) = \mu(B) = \{0\}\) and let \(B_n = B\), for every \(n \in \mathbb{N}\).

Then \(|\mu(B_n)| \to 0\). Since \(\mu\) is sn-ac-ab, we have \(\lim_{n \to \infty} |\mu(A \cup B_n)| = |\mu(A)| = 0\). This implies \(|\mu(A \cup B)| = 0\), so \(\mu(A \cup B) = \{0\}\) and thus \(\mu\) is null-null-additive.

IV and V result straightforward. \(\blacksquare\)

The following examples show that the converses of the statements in Theorem 3.1 are false.

Example 3.1. 1. Let \(T = \mathbb{N}, \mathcal{C} = \mathcal{P}(\mathbb{N})\) and \(\mu : \mathcal{C} \to (\mathcal{P}_0(\mathbb{R}), | \cdot |_b)\) defined for every \(A \in \mathcal{C}\) by \(\mu(A) = \{0\}\) if \(A\) is finite and \(\mu(A) = [1, \infty)\), if \(A\) is countable. Then \(\mu\) is sn-u-ac and it is not a multisubmeasure.

II. Let \(T = [a, b], \mathcal{C} = \mathcal{P}(T)\) and \(\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})\) defined by \(\mu(T) = [0, 1], \mu([a]) = \mu([b]) = [0, 1], \mu([a]) = [0, 1]\) and \(\mu(\emptyset) = \{0\}\). Then \(\mu\) is null-additive, but it is not a multisubmeasure.
III. Let \( \nu : \mathcal{C} \to \mathbb{R}_0 \) be a set function with \( \nu(\emptyset) = 0 \) and \( \mu : \mathcal{C} \to (\mathcal{P}(\mathbb{R}_+), |\cdot|_h) \) defined by \( \mu(A) = \{\nu(A)\} \), for every \( A \in \mathcal{C} \). Then \( \mu \) is sn-ac-ab (sn-u-ac, respectively) if and only if \( \nu \) is sn-ac-ab (sn-u-ac, respectively).

IV. Let \( \nu : \mathcal{C} \to \mathbb{R}_+ \) be a set function with \( \nu(\emptyset) = 0 \) and \( \mu : \mathcal{C} \to (\mathcal{P}(\mathbb{R}_+), |\cdot|_h) \) defined by \( \mu(A) = [0, \nu(A)] \), for every \( A \in \mathcal{C} \). Then \( \mu \) is sn-ac-ab (sn-u-ac, respectively) if and only if \( \nu \) is sn-ac-ab (sn-u-ac, respectively).

V. Let \( T = [0, 1], \mathcal{C} \) the Borel \( \sigma \)-algebra on \( T \), \( 1 : \mathcal{C} \to \mathbb{R}_+ \) the Lebesgue measure and \( \mu : \mathcal{C} \to (\mathcal{P}(\mathbb{R}_+), |\cdot|_h) \) defined by \( \mu(A) = \{\nu(A)\} \), where \( \nu(A) = \text{tg}(\frac{\pi}{4}A) \), for every \( A \in \mathcal{C} \).

According to Example 4-[14], \( \nu \) is ac-ab, but it is not u-ac. From the above III, it results that \( \mu \) is sn-ac-ab, but it is not sn-u-ac.

VI. Let \( T = (a, b), \mathcal{C} = \mathcal{P}(T) \) and \( \mu : \mathcal{C} \to \mathcal{P}(\mathbb{R}) \) defined by \( \mu(T) = [0, 1], \mu([b]) = [0, \frac{1}{2}] \) and \( \mu([a]) = \mu(\emptyset) = \{0\} \). Then \( \mu \) is null-monotone and null-null-additive, but it is not a multisubmeasure and it is not null-additive.

VII. Let \( T = (a, b), \mathcal{C} = \mathcal{P}(T) \) and \( \mu : \mathcal{C} \to \mathcal{P}(\mathbb{R}) \) defined by \( \mu(A) = \{1, 2, 3\} \) if \( A = T \) and \( \mu(A) = \{0\} \) otherwise. Then \( \mu \) is null-monotone, but \( \mu \) is not null-null-additive and not null-additive.

VIII. Let \( T = (a, b), \mathcal{C} = \mathcal{P}(T) \) and \( \mu : \mathcal{C} \to \mathcal{P}(\mathbb{R}) \) defined by \( \mu(A) = \{1, 2\} \) if \( A = [a] \) or \( A = [b] \), \( \mu(\emptyset) = \{3\} \) and \( \mu([a, b]) = \{0\} \). Then \( \mu \) is null-null-additive, but not null-monotone.

**Definition 3.3.** Let \( |\cdot| \) be a monotone set-norm on \((\mathcal{P}(\mathbb{R}), \leq)\). A set multifunction \( \mu : \mathcal{C} \to \mathcal{P}(\mathbb{R}) \) is said to be:

(i) set-norm exhaustive (shortly, sn-exhaustive) if \( \lim_{n \to \infty} |\mu(A_n)| = 0 \), for every pairwise disjoint sequence of sets \((A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \).

(ii) set-norm continuous (shortly, sn-continuous) if \( \lim_{n \to \infty} |\mu(A_n)| = 0 \), for every sequence of sets \((A_n)_{n \in \mathbb{N}^*} \subset \mathcal{C} \) such that \( A_n \searrow \emptyset \).

**Remarks 3.2.** Let \( |\cdot| \) be a monotone set-norm on \((\mathcal{P}(\mathbb{R}), \leq)\). If \( \mathcal{C} \) is finite, then every set multifunction \( \mu : \mathcal{C} \to \mathcal{P}(\mathbb{R}) \) with \( \mu(\emptyset) = \{0\} \) is sn-exhaustive and sn-continuous.

**Theorem 3.2.** Let \( |\cdot| \) be a monotone set-norm on \((\mathcal{P}(\mathbb{R}), \leq)\). If \( \mathcal{C} \) is a \( \sigma \)-ring, \( \mu : \mathcal{C} \to \mathcal{P}(\mathbb{R}) \) is monotone, sn-continuous and \( \mu(\emptyset) = \{0\} \), then \( \mu \) is sn-exhaustive.

**Proof.** Let \((A_n)_{n \in \mathbb{N}^*} \) be a sequence of mutually disjoint sets of \( \mathcal{C} \) and let \( B_n = \bigcup_{k=n}^{\infty} A_k \), for every \( n \in \mathbb{N}^* \). Then \( B_n \in \mathcal{C} \), for every \( n \in \mathbb{N}^* \) and \( B_n \searrow \emptyset \). Since \( \mu \) is sn-continuous, it results \( |\mu(B_n)| \to 0 \), which implies \( |\mu(A_n)| \to 0 \). So, \( \mu \) is sn-exhaustive. \( \blacksquare \)
4. ATOMS AND PSEUDO-ATOMS

This section contains several properties of atoms and pseudo-atoms for null-null-additive set multifunctions.

Definition 4.1. ([10,11,12]) Let \( \mu : \mathcal{C} \to \mathcal{P}_0(X) \) be a set multifunction.

(i) A set \( A \in \mathcal{C} \) is said to be an atom of \( \mu \) if \( \mu(A) \neq \{0\} \) and for every \( B \in \mathcal{C} \), with \( B \subseteq A \), we have either \( \mu(B) = \{0\} \) or \( \mu(A \setminus B) = \{0\} \).

(ii) A set \( A \in \mathcal{C} \) is called a pseudo-atom of \( \mu \) if \( \mu(A) \neq \{0\} \) and for every \( B \in \mathcal{C} \), with \( B \subseteq A \), we have either \( \mu(B) = \{0\} \) or \( \mu(B) = \mu(A) \).

(iii) \( \mu \) is said to be non-atomic (non-pseudo-atomic, respectively) if it has no atoms (no pseudo-atoms, respectively).

Remarks 4.1. Let \( \mu : \mathcal{C} \to \mathcal{P}_0(X) \) be a set multifunction, with \( \mu(\emptyset) = \{0\} \).

I. If \( \mu \) is monotone, then \( \mu \) is non-atomic (non-pseudo-atomic, respectively) if for every \( A \in \mathcal{C} \), with \( \mu(A) \supseteq \{0\} \), there is a \( B \in \mathcal{C} \) so that \( B \subseteq A \), \( \mu(B) \supseteq \{0\} \) and \( \mu(A \setminus B) \supseteq \{0\} \) (\( \mu(A) \supseteq \mu(B) \), respectively).

II. If \( \mu \) is null-monotone, then \( A \in \mathcal{C} \) is an atom of \( \mu \) if and only if \( A \) is an atom of \( \mu \).

III. If \( \mu \) is null-additive, then every atom of \( \mu \) is a pseudo-atom of \( \mu \) (as we shall see in Examples 4.1-I, the converse is not valid).

Example 4.1. I. Let \( T = \{a,b,c\} \), \( \mathcal{C} = \mathcal{P}(T) \) and \( \mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R}) \) defined by \( \mu(A) = [0,1] \) if \( A \neq \emptyset \) and \( \mu(A) = \{0\} \) if \( A = \emptyset \). Then \( \mu \) is null-additive, \( A = \{a,b\} \) is a pseudo-atom of \( \mu \), but not an atom of \( \mu \).

II. Let \( T = 2\mathbb{N} = \{0,2,4,\ldots\} \), \( \mathcal{C} = \mathcal{P}(T) \) and \( \mu : \mathcal{C} \to \mathcal{P}_0(\mathbb{R}) \) defined for every \( A \in \mathcal{C} \) by:

\[
\mu(A) = \begin{cases} 
\{0\}, & \text{if } A = \emptyset \\
\frac{1}{2} A \cup \{0\}, & \text{if } A \neq \emptyset
\end{cases}
\]

where \( \frac{1}{2} A = \{\frac{x}{2} \mid x \in A\} \), \( \mu \) is a multisubmeasure.

If \( A \in \mathcal{C} \), with \( \text{card}A = 1 \) and \( A \neq \{0\} \) or \( A \in \mathcal{C}, A = \{0,2n\}, n \in \mathbb{N}^* \), then \( A \) is an atom of \( \mu \) (and a pseudo-atom of \( \mu \), too, according to Remark 4.1-III). By \( \text{card}A \) we mean the cardinal of \( A \).

If \( A \in \mathcal{C} \), with \( \text{card}A \geq 2 \) and there exist \( a,b \in A \) such that \( a \neq b \) and \( ab \neq 0 \), then \( A \) is not a pseudo-atom of \( \mu \) (and not an atom of \( \mu \), according to Remark 4.1-III).

III. Let \( \mathcal{C} = \mathcal{P}(\mathbb{N}) \) and \( \mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}) \) defined for every \( A \in \mathcal{C} \) by:

\[
\mu(A) = \begin{cases} 
\{0\}, & \text{if } A \text{ is finite} \\
\{0\} \cup [n_A, +\infty), & \text{if } A \text{ is infinite and } n_A = \min A.
\end{cases}
\]

Then \( \mu \) is monotone and non-pseudo-atomic.

Remarks 4.2. Let \( \mu : \mathcal{C} \to \mathcal{P}_0(X) \) be a set multifunction, with \( \mu(\emptyset) = \{0\} \).
I. If $A \in \mathcal{C}$ is a pseudo-atom of $\mu$ and $B \in \mathcal{C}$, $B \subseteq A$ is such that $\mu(B) \supseteq \{0\}$, then $B$ is a pseudo-atom of $\mu$ and $\mu(B) = \mu(A)$.

II. Suppose $\mu$ is null-monotone. If $A \in \mathcal{C}$ is an atom of $\mu$ and $B \in \mathcal{C}$, $B \subseteq A$ is such that $\mu(B) \supseteq \{0\}$, then $B$ is an atom of $\mu$ and $\mu(A \setminus B) = \{0\}$.

**Theorem 4.1.** Suppose $\mu : \mathcal{C} \rightarrow \mathcal{P}_0(X)$ is monotone, so that $\mu(\emptyset) = \{0\}$ and $A, B \in \mathcal{C}$ are pseudo-atoms of $\mu$. Then the following statements hold:

I. $\mu(A) \neq \mu(B) \Rightarrow \mu(A \cap B) = \{0\}$.

II. Suppose $\mu$ is null-null-additive. If $\mu(A \cap B) = \{0\}$, then $A \setminus B$ and $B \setminus A$ are pseudo-atoms of $\mu$ and $\mu(A \setminus B) = \mu(A)$, $\mu(B \setminus A) = \mu(B)$.

**Proof.** I) Suppose $\mu(A \cap B) \supseteq \{0\}$. According to Remark 4.2-I, we have $\mu(A \cap B) = \mu(A) = \mu(B)$, which is false.

II. Let us prove that $\mu(A \setminus B) \supseteq \{0\}$. Suppose on the contrary that $\mu(A \setminus B) = \{0\}$. Since $\mu$ is null-null-additive, we have $\mu(A) \leq \mu((A \setminus B) \cup (A \setminus B)) = \{0\}$, which is false. So, $\mu(A \setminus B) \supseteq \{0\}$ and from Remark 4.2-I, it results that $A \setminus B$ is a pseudo-atom of $\mu$ and $\mu(A \setminus B) = \mu(A)$. Analogously, $B \setminus A$ is a pseudo-atom of $\mu$ and $\mu(B \setminus A) = \mu(B)$.

**Theorem 4.2.** Suppose $\mu : \mathcal{C} \rightarrow \mathcal{P}_0(X)$ is monotone and null-null-additive, so that $\mu(\emptyset) = \{0\}$ and $A, B \in \mathcal{C}$ are pseudo-atoms of $\mu$. Then there exist pairwise disjoint sets $E_1, E_2, E_3 \in \mathcal{C}$, with $A \cup B = E_1 \cup E_2 \cup E_3$, such that, for every $i \in \{1, 2, 3\}$, either $E_i$ is a pseudo-atom of $\mu$ or $\mu(E_i) = \{0\}$.

**Proof.** Let $E_1 = A \cap B$, $E_2 = A \setminus B$, $E_3 = B \setminus A$. We have the following cases:

(i) $\mu(E_1) = \{0\}$. According to Theorem 4.1-II, $E_2$ and $E_3$ are pseudo-atoms of $\mu$ and $\mu(E_2) = \mu(A)$, $\mu(E_3) = \mu(B)$.

(ii) $\mu(E_1) \supseteq \{0\}$, $\mu(E_2) \supseteq \{0\}$, $\mu(E_3) \supseteq \{0\}$. By Remark 4.2-I, $E_1$ is a pseudo-atom of $\mu$ and $\mu(E_1) = \mu(A) = \mu(B)$. Analogously, $E_2$ and $E_3$ are pseudo-atoms of $\mu$.

(iii) $\mu(E_1) \supseteq \{0\}$, $\mu(E_2) = \{0\}$, $\mu(E_3) \supseteq \{0\}$. From Remark 4.2-I, it results that $E_1$ is a pseudo-atom of $\mu$ and $\mu(E_1) = \mu(A) = \mu(B)$. Analogously, $E_3$ is a pseudo-atom of $\mu$ and $\mu(E_3) = \mu(B)$.

The last two cases are similar to (iii).

(iv) $\mu(E_1) \supseteq \{0\}$, $\mu(E_2) \supseteq \{0\}$, $\mu(E_3) = \{0\}$.

(v) $\mu(E_1) = \{0\}$, $\mu(E_2) = \{0\}$, $\mu(E_3) = \{0\}$. □

**Remarks 4.3.** Let $\mu : \mathcal{C} \rightarrow \mathcal{P}_0(X)$ be monotone, null-null-additive, such that $\mu(\emptyset) = \{0\}$.

I. By Theorem 4.2, the union of two pseudo-atoms $A, B$ of $\mu$ either is a pseudo-atom of $\mu$ or is equal to an union of two pairwise disjoint pseudo-atoms $E_1, E_2$ of $\mu$. In the latter case, we set $E_1 = A, E_2 = B \setminus A$.

II. By induction, it is easy to show that for any sequence of pseudo-atoms of $\mu$, $(A_i)_{i=1}^n$, $n \in \mathbb{N} \cup \{+\infty\}$, there is a sequence $(E_j)_{j=1}^m$, $m \in \mathbb{N} \cup \{+\infty\}$, $m \leq n$, of pairwise disjoint pseudo-atoms of $\mu$, such that $\bigcup_{i=1}^n A_i = \bigcup_{j=1}^m E_j$. 


Theorem 4.3. Let \(|\cdot|\) be a monotone set-norm on \((\mathcal{P}_0(X), \subseteq)\). Suppose \(\mathcal{C}\) is a \(\sigma\)-ring and \(\mu : \mathcal{C} \to \mathcal{P}_0(X)\) is monotone, null-null-additive, sn-continuous, \(\mu(\emptyset) = \{0\}\) and \(\mu\) has atoms. Then there exists a finite or countable family of pairwise disjoint atoms of \(\mu\), \((B_n)_{n=1}^\infty, n \in \mathbb{N} \cup \{+\infty\}\), satisfying the conditions:

(i) \(|\mu(B_{i-1})| > |\mu(B_i)|\), \(\forall 1 < i \leq n\),

(ii) \(\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^+\), such that \(|\mu(\bigcup_{k \geq n_0} B_k)| < \varepsilon\).

(iii) for every atom \(A\) of \(\mu\), there is \(B_k\) such that \(\mu(A \Delta B_k) = \{0\}\).

Proof. We remark that (ii) is verified for every infinite sequence \((B_n)_{n=1}^\infty\) of pairwise disjoint sets of \(\mathcal{C}\). We have \(\bigcup_{n \geq 1} B_k \in \mathcal{C}\), and let \(I\) be the set of all indices \(i\) such that there is an atom \(C\) of \(\mu\) which is pairwise disjoint with all \(B_k\), \(1 \leq k \leq p_1\) and so that \(\frac{1}{rn} \leq |\mu(B_k)| < \frac{1}{r+1}\).

Then we rearrange the sets \(B_k\) by non-increasing of \(|\mu(B_k)|\) to satisfy (i). By Remark 4.2-II, if \(E, F\) are any two atoms of \(\mu\), then either \(\mu(E \cap F) = \{0\}\) or \(\mu(E \Delta F) = \{0\}\). If an atom \(A\) of \(\mu\) is so that \(\mu(A \cap B_k) = \{0\}\) for every \(k\), then \((B_k)_{k=p_{i-1}+1}^{p_i}\) is not maximal at the \(i\)-th step for \(i \in \mathbb{N}^+\), \(\frac{1}{i+1} \leq |\mu(A)| < \frac{1}{i}\).

So there is an element \(B_k\) that satisfies (iii). [\]

Theorem 4.4. Let \(\mathcal{C}\) be a \(\sigma\)-ring and \(\mu : \mathcal{C} \to \mathcal{P}_f(X)\) is monotone, sn-continuous, null-null-additive, \(\mu(\emptyset) = \{0\}\) and \(\mu\) has atoms. Then there exists a finite or infinite family \((B_i)_{i \in \mathbb{N}}\) of pairwise disjoint atoms of \(\mu\), such that for every \(A \in \mathcal{C}\) and every \(\varepsilon > 0\), there are: a subsequence \((B_{i_n})_n\) of \((B_i)_{i \in \mathbb{N}}\), \(i_0 \in \mathbb{N}\) and \(F, E \in \mathcal{C}\) such that \(A = (\bigcup_{i \in I} B_{i_n}) \cup E, |\mu(\bigcup_{i \in I} B_{i_n})| \leq \varepsilon, \mu(F) = \{0\}\), and \(E\) contains no atoms of \(\mu\).

Proof. Let \((B_i)_{i=1}^\infty, n \in \mathbb{N} \cup \{+\infty\}\), be the family as in the proof of Theorem 4.3, and let \(I\) be the set of all indices \(i\) such that there is an atom \(C \subset A, C \in \mathcal{C}\) so that \(\mu(C \Delta B_i) = \{0\}\). Then the subsequence \((B_{i_n})_n\) and the sets \(F = \bigcup_{i \in I} B_{i_n}\), \(E = A\setminus \bigcup_{i \in I} B_{i_n}\) satisfy the conclusion of the theorem. [\]

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References


