ON TOPOLOGICAL $AG$-GROUPOIDS AND PARAMEDIAL QUASIGROUPS WITH MULTIPLE IDENTITIES

Natalia Bobeica, Liubomir Chiriac
Tiraspol State University, Chișinău, Republic of Moldova
nbobeica1978@gmail.com, llchiriac@gmail.com

Abstract
This paper studies some properties of $(n,m)$-homogeneous isotopies of topological $AG$-groupoids and paramedial quasigroups with $(n,m)$-identities. We study properties of paramedial groupoids with multiple identities. We extend some affirmations of the theory of topological groups on the class of topological $(n,m)$-homogeneous quasigroups.

Keywords: $AG$-groupoid, quasigroups, $(n,m)$-identity, $(n,m)$-homogeneous isotope.


1. INTRODUCTION

In this work we study the $(n,m)$-homogeneous isotopies of $AG$-topological groupoid with multiple identities and some topological properties of topological paramedial quasigroups. The results established in this paper are related to the results of M. Choban and L. Kiriyak in [2] and to the research papers [6-14]. In section 3 we expand on the notions of multiple identities and $(n,m)$-homogeneous isotopies introduced in [5]. This concept facilitates the study of topological groupoids with $(n,m)$-identities. In this section we prove that if $(G,+)$ is an $AG$-topological groupoid and $e$ is an $(1,p)$-zero, then every $(n,1)$-homogeneous isotope $(G,\cdot)$ of $(G,+)$ is an $AG$-groupoid, with $(1,np)$-identity $e$ in $(G,\cdot)$. In section 4 we study the paramedial groupoids with multiple identities. We present some interesting properties of a class of paramedial groupoids with $(2,1)$-identities. We prove that if $P$ is an open compact set of a right identity of a topological right paramedial loop $G$, then $P$ contains an open compact topological right paramedial subloop $Q$. This result was obtained by L.S. Pontrjagin for topological groups in (Theorem 16, [13]) and by L.L. Chiriac for topological left medial quasigroups in [14]. We shall use the notations and terminology from [1-3, 13].

2. BASIC NOTIONS

A non-empty set $G$ is said to be a groupoid relatively to a binary operation denoted by “$\cdot$”, if for every ordered pair $(a,b)$ of elements of $G$ there is a unique element $ab = a \cdot b \in G$. 

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If the groupoid $G$ is a topological space and the binary operation $(a, b) \rightarrow a \cdot b$ is continuous, then $G$ is called a topological groupoid.

An element $e \in G$ is called an identity if $ex = xe = x$ for every $x \in X$.

A quasigroup with an identity is called a loop. A groupoid $G$ is called medial if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

A groupoid $G$ is called paramedial if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

If a paramedial quasigroup $G$ contains an element $e$ such that $e \cdot x = x(x \cdot e = x)$ for all $x$ in $G$, then $e$ is called a left (right) identity element of $G$ and $G$ is called a left (right) paramedial loop.

A groupoid $(G, \cdot)$ is called a groupoid Abel-Grassmann or AG-groupoid if it satisfies the left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.

A groupoid $G$ is said to be hexagonal if it is idempotent, medial and semisymmetric, i.e. the equalities $x \cdot x = x$, $xy \cdot zt = xz \cdot yt$, $x \cdot zx = xz \cdot x = z$ hold for all of its elements. The last identities (semisymmetric law) can be represented in the form of an equivalence $x \cdot z = t \Leftrightarrow x = z \cdot t$.

A groupoid $G$ is called bicommutative if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

3. GROUPOIDS WITH MULTIPLE IDENTITIES

Consider a groupoid $(G, +)$. For every two elements $a, b$ from $(G, +)$ we denote:

$$1(a, b, +) = (a, b, +)1 = a + b,$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set $G$, then we shall use the symbols $n(a, b)$ and $(a, b)n$ instead of $n(a, b, +)$ and $(a, b, +)n$.

**Definition 3.1.** Let $(G, +)$ be a groupoid and let $n, m \geq 1$. The element $e$ of the groupoid $(G, +)$ is called:

* an $(n, m)$-zero of $G$ if $e + e = e$ and $n(e, x) = (x, e)m = x$ for every $x \in G$;
* an $(n, \infty)$-zero if $e + e = e$ and $n(e, x) = x$ for every $x \in G$;
* an $(\infty, m)$-zero if $e + e = e$ and $(x, e)m = x$ for every $x \in G$.

Clearly, if $e \in G$ is both an $(n, \infty)$-zero and an $(\infty, m)$-zero, then it is also an $(n, m)$-zero. If $(G, \cdot)$ is a multiplicative groupoid, then the element $e$ is called an $(n, m)$-identity. The notion of $(n, m)$-identity was introduced in [5].

**Example 3.1.** Let $(G, \cdot)$ be a paramedial groupoid, $e \in G$ and $ex = x$ for every $x \in G$. Then $(G, \cdot)$ is a paramedial groupoid with $(1, 2)$-identity $e$ in $G$. Indeed, if $x \in G$, then $xe \cdot e = xe \cdot ee = ee \cdot ex = e \cdot ex = e \cdot x = x$.

**Example 3.2.** Let $G = \{1, 2, 3, 4, 5\}$. We define the binary operation $\cdot'$.
Then \((G, \cdot)\) is a non-commutative, hexagonal and AG-quasigroup and each element from \((G, \cdot)\) is a \((2, 4)\)-identity in \(G\).

**Definition 3.2.** Let \((G, +)\) be a topological groupoid. A groupoid \((G, \cdot)\) is called a homogeneous isotope of the topological groupoid \((G, +)\) if there exist two topological automorphisms \(\varphi, \psi : (G, +) \to (G, +)\) such that \(x \cdot y = \varphi(x) + \psi(y)\), for all \(x, y \in G\).

For every mapping \(f : X \to X\) we denote \(f^1(x) = f(x)\) and \(f^{n+1}(x) = f(f^n(x))\) for any \(n \geq 1\).

**Definition 3.3.** Let \(n, m \leq \infty\). A groupoid \((G, \cdot)\) is called an \((n, m)\)-homogeneous isotope of a topological groupoid \((G, +)\) if there exist two topological automorphisms \(\varphi, \psi : (G, +) \to (G, +)\) such that:

1. \(x \cdot y = \varphi(x) + \psi(y)\) for all \(x, y \in G\);
2. \(\varphi\psi = \psi\varphi\);
3. If \(n < \infty\), then \(\varphi^n(x) = x\) for all \(x \in G\);
4. If \(m < \infty\), then \(\psi^m(x) = x\) for all \(x \in G\).

**Definition 3.4.** A groupoid \((G, \cdot)\) is called an isotope of a topological groupoid \((G, +)\), if there exist two homeomorphisms \(\varphi, \psi : (G, +) \to (G, +)\) such that \(x \cdot y = \varphi(x) + \psi(y)\) for all \(x, y \in G\).

Under the conditions of Definition 3.4 we shall say that the isotope \((G, \cdot)\) is generated by the homeomorphisms \(\varphi, \psi\) of the topological groupoids \((G, +)\) and write \((G, \cdot) = g(G, +, \varphi, \psi)\).

**Example 3.3.** Let \((R, +)\) be the topological Abelian group of real numbers.

1. If \(\varphi(x) = 5x, \psi(x) = x\) and \(x \cdot y = 5x + y\), then \((R, \cdot) = g(R, +, \varphi, \psi)\) is a locally compact medial quasigroup. By virtue of Theorem 7 from [2], there exists a left invariant Haar measure on \((R, \cdot)\).

2. If \(\varphi(x) = 5x, \psi(x) = 7x\) and \(x \cdot y = 5x + 7y\), then \((R, \cdot) = g(R, +, \varphi, \psi)\) is a locally compact medial quasigroup and on \((R, \cdot)\) as above, by virtue of Theorem 7 from [2], does not exist any left or right invariant Haar measure.

**Example 3.4.** Denote by \(Z_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p - 1\}\) the cyclic Abelian group of order \(p\). Consider the commutative group \((G, +) = (Z_7, +)\), \(\varphi(x) = 3x, \psi(x) = 4x\) and...
Consider the commutative group \( (G, \cdot) \). Then \( (G, \cdot) = g(G, +, \varphi, \psi) \) is a medial, paramedial and bicommutative quasigroup and the neutral element of \((G, +)\) is a \((3, 6)\)-identity in \((G, \cdot)\).

**Example 3.5.** Consider the commutative group \((G, +) = (Z_5, +)\), \(\varphi(x) = 3x, \psi(x) = 2x\) and \(x \cdot y = 3x + 2y\). Then \((G, \cdot) = g(G, +, \varphi, \psi)\) is a medial, paramedial and bicommutative quasigroup and the neutral element of \((G, +)\) is a \((4, 4)\)-identity in \((G, \cdot)\).

**Example 3.6.** Consider the commutative group \((G, +) = (Z_2, +)\), \(\varphi(x) = 2x, \psi(x) = 5x\) and \(x \cdot y = 2x + 5y\). Then \((G, \cdot) = g(G, +, \varphi, \psi)\) is a medial, paramedial and bicommutative quasigroup and the neutral element of \((G, +)\) is a \((6, 3)\)-identity in \((G, \cdot)\).

**Example 3.7.** Consider the commutative group \((G, +) = (Z_7, +)\), \(\varphi(x) = 3x, \psi(x) = 5x\) and \(x \cdot y = 3x + 5y\). Then \((G, \cdot) = g(G, +, \varphi, \psi)\) is a medial and hexagonal quasigroup and each element of \((G, \cdot)\) is a \((6, 6)\)-identity.

**Theorem 3.1.** If \((G, +)\) is an AG-groupoid and \(e \in G\) is an \((1, p)\)-zero, then every \((n, 1)\)-homogeneous isotope \((G, \cdot)\) of the topological groupoid \((G, +)\) is AG-groupoid with \((1, np)\)-identity \(e\) in \((G, \cdot)\) and \(a + bc = b \cdot (a + c)\) for all \(a, b, c \in G\) and \(n, p \in N\).

**Proof.** We will prove that \(e\) is an \((1, np)\)-identity in \((G, \cdot)\) by the method described in [2]. Let \((G, \cdot)\) be an \((n, 1)\)-homogeneous isotope of the groupoid \((G, +)\) and \(e\) be an \((1, p)\)-zero in \((G, +)\). We mention that \(\varphi^q(e) = \psi^q(e) = e\) for every \(q \in N\). Then in \((G, +)\) we have \(q \cdot 1(e, x, +) = x\) for each \(x \in G\) and for every \(q \in N\). Since we have \((n, 1)\)-homogeneous isotope \((G, \cdot)\) then \(m = 1\) and \(\psi(x) = x\) for all \(x \in G\). In this case

\[
1(e, x, \cdot) = 1(e, \psi(x), +)
\]

and

\[
q(e, x, \cdot) = q(e, \psi^q(x), +)
\]

for every \(q \geq 1\). Therefore

\[
1(e, x, \cdot) = 1(e, \psi(x), +) = 1(e, x, +) = x.
\]

Analogously we obtain that

\[
(e, x, \cdot)np = (e, \varphi^np(x), +)np = (e, x, +)np = x.
\]

Hence \(e\) is an \((1, np)\)-identity in \((G, \cdot)\). The medality of the AG-topological groupoid \((G, +)\) follows from [4]. Really,

\[
(x + y) + (z + t) = ((z + t) + y) + x = ((y + t) + z) + x = (x + z) + (y + t).
\]
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Since $e$ is an $(1, p)$-zero in $(G, +)$, hence $e$ is left zero and $e + x = x$ for every $x \in (G, +)$. In this case for each $AG$-groupoid $(G, +)$ with $(1, p)$-zero we get

$$x + (z + t) = (e + x) + (z + t) = (e + z) + (x + t) = z + (x + t).$$

We will prove that $(n, 1)$-homogeneous isotope $(G, \cdot)$ of the topological groupoid $(G, +)$ is $AG$-groupoid and $x \cdot (zt) = z \cdot (xt)$. Since $(G, \cdot)$ is $(n, 1)$-homogeneous isotope of $(G, +)$, then $\psi(x) = x$. Hence

$$x \cdot zt = \varphi(x) + \psi(zt) = \varphi(x) + (\varphi(z) + \psi(t)) = \varphi(z) + (\varphi(x) + \psi(t)) = \varphi(z) + xt = \varphi(z) + \psi(xt) = z \cdot xt.$$

Therefore the identity $x \cdot zt = z \cdot xt$ holds in groupoid $(G, \cdot)$. We will show that $a + bc = b \cdot (a + c)$. Really,

$$a + bc = (ea) + (bc) = (\varphi(e) + \psi(a)) + (\varphi(b) + \psi(c)) = (\varphi(e) + \varphi(b)) + (\psi(a) + \psi(c)) = \varphi(e + b) + \psi(a + c) = \varphi(b) + \psi(a + c) = b \cdot (a + c).$$

The proof is complete.

**Corollary 3.1.** If $(G, +)$ is an $AG$-groupoid and $e$ is a left zero, then every $(1, 1)$-homogeneous isotope $(G, \cdot)$ of the topological groupoid $(G, +)$ is $AG$-groupoid with left identity $e$ in $(G, \cdot)$ and $a + bc = b \cdot (a + c)$, for all $a, b, c \in G$.

4. **SOME PROPERTIES OF PARAMEDIAL QUASIGROUPS**

We provide an example of a paramedial quasigroup which is not medial.

**Example 4.1.** Let $G = \{1, 2, 3, 4\}$. We define the binary operation $\cdot$:

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Then $(G, \cdot)$ is a paramedial quasigroup but it is not medial. Indeed, for example

$$(2 \cdot 3) \cdot (1 \cdot 4) \neq (2 \cdot 1) \cdot (3 \cdot 4).$$
Theorem 4.1. If $(G, \cdot)$ is a multiplicative groupoid, $e \in G$ and the following conditions hold:

1. $xe = x$ for every $x \in G$,
2. $x^2 = x \cdot x = e$ for every $x \in G$,
3. $xy \cdot z = xz \cdot y$ for all $x, y, z \in G$,
4. for every $a, b \in G$ there exists an unique point $y \in G$ such that $ya = b$,
then $e$ is a $(2, 1)$-identity in $G$.

Proof. Fix $x \in G$. Pick $y \in G$ such that $y \cdot ex = x$. By conditions 2 of Theorem 4.1 we have

$$(y \cdot ex) \cdot x = x \cdot x = e.$$  \hspace{1cm} (1)

By condition 3 of this theorem

$$(y \cdot ex) \cdot x = yx \cdot ex.$$  \hspace{1cm} (2)

From (1) and (2) we obtain

$$yx \cdot ex = e.$$  \hspace{1cm} (3)

It is clear that

$$ex \cdot ex = e.$$  \hspace{1cm} (4)

Thus, from (3) and (4)

$$yx \cdot ex = ex \cdot ex.$$

Hence $yx = ex$ and $y = e$. Therefore $y \cdot (e \cdot x) = e(ex) = x$ and $e$ is a $(2, 1)$-identity in $G$. The proof is complete. \hfill \blacksquare

Theorem 4.2. If $(G, \cdot)$ is a multiplicative groupoid, $e \in G$ and the following conditions hold:

1. $xe = x$ for every $x \in G$,
2. $x^2 = x \cdot x = e$ for every $x \in G$,
3. $x \cdot zt = t \cdot xz$ for all $x, y, z \in G$,
4. for every $a, b \in G$ there exists a unique point $y \in G$ such that $ya = b$,
then $e$ is a $(2, 1)$-identity in $G$.

Proof. Similar to Theorem 4.1. \hfill \blacksquare

Remark 4.1. The proofs of Theorems 4.1. and 4.2. are based on a rather general method. Professor I. Burdujan noticed that there is an easier way to prove Theorem 4.2. He observed that $e \cdot ex = x \cdot ee = xe = x$ and $e$ is a $(2, 1)$-identity in $G$.

Theorem 4.3. If $(G, \cdot)$ is a multiplicative groupoid, $e \in G$ and the following conditions hold:
1. \( xe = x \) for every \( x \in G \),
2. \( x^2 = x \cdot x = e \) for every \( x \in G \),
3. \( xy \cdot uv = vy \cdot ux \) for all \( x, y, u, v \in G \),
4. if \( xa = ya \) then \( x = y \) for all \( x, y, a \in G \),
then \( (G, \cdot) \) is a paramedial quasigroup with a \((2, 1)\)-identity \( e \).

**Proof.** If \( x \in G \), then
\[
e \cdot ex = ee \cdot ex = xe \cdot ee = xe \cdot e = x \cdot e = x.
\]
Thus \( e \) is \((2, 1)\)-identity. Consider the equation \( xa = b \). Then
\[
xa \cdot e = b \cdot e,
xa \cdot ee = be,
ea \cdot ex = be,
(\(ea \cdot ex\)) \cdot be = be \cdot be.
\]
Thus, \((ea \cdot ex) \cdot (be) = e\). By condition (3) of theorem 4.3 we have
\[
(e \cdot ex) \cdot (b \cdot ea) = e. \tag{5}
\]
From (5) we obtain
\[
x \cdot (b \cdot ea) = e. \tag{6}
\]
By condition (2) of theorem 4.3
\[
(b \cdot ea) \cdot (b \cdot ea) = e. \tag{7}
\]
From (6) and (7) have \( x = b \cdot ea \). Since \( xa = b \) we can verify that \((b \cdot ea) \cdot a = (b \cdot ea) \cdot (ae) = (e \cdot ea) \cdot (ab) = a \cdot ab = ae \cdot ab = be \cdot aa = be \cdot e = b\).

In this case the element \( x = b \cdot ea \) is a unique solution of the equation \( xa = b \). Now we consider the equation \( ay = b \). We have
\[
e \cdot b = e \cdot ay = ee \cdot ay = ye \cdot ae = y \cdot a.
\]
Thus \( ya = eb \) and, by considering the solution of equation \( xa = b \) we have
\[
y = eb \cdot ea = ab \cdot ee = ab \cdot e = ab.
\]
It follows that \( y = ab \) is a unique solution of the equation \( ay = b \). The proof is complete.

**Corollary 4.1.** If \((G, \cdot)\) is a paramedial quasigroup with a \((2, 1)\)-identity \( e \), then solutions of the equations \( xa = b \) and \( ay = b \) are respectively \( x = b \cdot ea \) and \( y = ab \) for every \( a, b \in G \).

**Lemma 4.1.** Let \( P \) be a subset of topological right paramedial loop \((G, \cdot)\) and \( e \in P \). If \( P_1 = P \cap eP \), then
The mapping.

It is obvious that $eP_1 = P_1$.

1. If $P$ is open, then $P_1$ is open.
2. If $P$ is closed, then $P_1$ is closed.
3. If $P$ is compact, then $P_1$ is compact.

**Proof.** The mapping $f : G \to G$, where $f(x) = ex$, is a homeomorphism and $P_1 = P \cap eP$. That proved the assertions 2, 3 and 4. For every $x \in G$ we have $e \cdot ex = x$. Therefore, $eP_1 = eP \cap (e \cdot eP) = eP \cap P = P_1$. The proof is complete. $lacksquare$

**Proposition 4.1.** Let $(G, \cdot)$ be a right paramedial loop. Then the mapping

$$f : G \to G,$$

where $f(x) = ex$,

is an involutive mapping, i.e. $f = f^{-1}$.

**Proof.** It is obvious that $f$ is an one-to-one mapping. The solution of a equation $ay = b$ is $y = ab$. Hence $a \cdot ab = b$ for every $y \in G$. In particular $e \cdot ex = x$ and $f(f(x)) = x$. Hence $f = f^{-1}$. The proof is complete. $lacksquare$

**Theorem 4.4.** Let $(G, \cdot)$ be a topological right paramedial loop with the identity $x^2 = e$. If $P$ is an open compact subset such that $e \in P$, then $P$ contains an open compact right paramedial subloop $(Q, \cdot)$ of $(G, \cdot)$.

**Proof.** In virtue of Lemma 4.1 we consider that $eP = P$. Note $Q = \{q \in G : qP \cap Pq \subset P\}$. We prove that $Q$ is an open compact right paramedial subloop. Now we show that $Q$ is the open set. Let $q$ be a fixed point of $Q$ and $x$ be an arbitrary point of $P$. Since $qx \in P$ and $P$ is an open set, then there exists such neighborhoods $U_x \ni x$ and $V_x \ni q$, such that $U_xV_x \subset P$. In this case we have $P = \bigcup_{i=1}^{\infty} U_{x_i}$. Because $P$ is a compact set we can extract an open finitely subcovering $U_{x_1}, ..., U_{x_k}$ such that $P = \bigcup_{i=1}^{k} U_{x_i}$. Note $V = \bigcap_{i=1}^{k} V_{x_i}$. Then $PV \subset P$. Let us consider $qx \in P$. By analogy we prove that there exists such neighborhood $W \ni q$ so that $WP \subset P$. Note $V \cap W = \emptyset$. Then we have that $UP \subset P$ and $PU \subset P$. Hence for the open set $U \ni q$ we have that $U \subset Q$. Therefore $Q$ is the open set.

Let us show that $Q$ is a closed set. Suppose that $p \notin Q$. Then for some $q \in P$ we have that $pq \notin P$ or $qp \notin P$. Let us assume that $pq \notin P$. Then there exists an open set $U$ such that $p \in U$ and $Uq \subset G \setminus P$. Therefore $U \cap Q = \emptyset$ and $q$ is not a limit point of a set $Q$. It follows that the set $Q$ is closed.

We show that $Q \subset P$. If $q \in Q$, then $q \in qP \cap Pq$. Since $qP \cup Pq \subset P$, then $q \in P$. Therefore $Q \subset P$. By condition of theorem $eP \cup Pe = P \cup P = P \subset P$. Hence $e \in Q$.

We will prove that $Q$ is a right paramedial loop. Fix $a, b \in Q$. Then

$$P \cdot ab = Pe \cdot ab = be \cdot aP = b \cdot aP = b \cdot P \subset P$$
and

\[ ab \cdot P = ab \cdot eP = Pb \cdot ea = P \cdot ea = Pe \cdot ea = ae \cdot eP = a \cdot P \subset P.\]

Therefore \(ab \in Q\) and \(Q\) is a subgroupoid of \(G\). If \(a, b \in Q\) then for equation \(xa = b\) his solution \(x = b \cdot (ea)\) is in \(Q\). Really, since \(e, a \in Q\) we have \(ea \in Q\), and \(b \cdot ea \in Q\).

For equation \(ay = b\) his solution \(y = ab\) is also in \(Q\). Hence \(Q\) is a right paramedial subloop of \(G\). The proof is complete.

\textbf{Theorem 4.5.} Let \((G, \cdot)\) be a topological paramedial quasigroup with a \((2, 1)\)-identity \(e\) and \(x^2 = e\) for every \(x \in G\). If \(P\) is an open compact subset from \(G\) such that \(e \in P\), then \(P\) contains an open compact paramedial subquasigroup \((Q, \cdot)\) with a \((2, 1)\)-identity \(e\).

\textit{Proof.} It follows from Theorem 4.3, Lemma 4.1 and Theorem 4.4. The proof is complete.

\textbf{Remark 4.2.} For topological groups a series of properties that are based on the notion of open compact are proved (see [3, 13]).

\section{5. SOME REMARKS OF MEDIAL QUASIGROUPS}

The ideas used in the proofs of Theorem 4.3, Lemma 4.1, Theorem 4.4 can be adopted for topological right medial loops. We mention the following properties.

\textbf{Remark 5.1.} If \((G, \cdot)\) is a right medial loop, then the mapping \(f : G \rightarrow G\), where \(f(x) = ex\), is an involutive automorphism, i.e. \(f = f^{-1}\) and \(f(x \cdot y) = f(x) \cdot f(y)\) for every \(x, y \in G\).

\textbf{Remark 5.2.} If \((G, \cdot)\) is a right medial loop, \(e \in G\) and \(x^2 = e\) for every \(x \in G\), then \(e\) is a \((2, 1)\)-identity.

\textbf{Remark 5.3.} Let \((G, \cdot)\) be a right medial loop and \(x^2 = e\) for every \(x \in G\). The related operation \(x \circ y = ex \cdot ey\) in \(G\) satisfies the following properties:

1. \((G, \circ)\) is a medial quasigroup.
2. \(e \circ x = x\) for every \(x \in G\).
3. \(H = \{x \in G : ex = x\}\) is a commutative group and is a subloop of the loops \((G, \cdot)\) and \((G, \circ)\).

\textbf{Example 5.1.} Let \((G, +)\) be a commutative group. We define in \(G\) the operation \(" \cdot \): \(x \cdot y = x - y\) for every \(x, y \in G\). Then \((G, \cdot)\) is a right medial loop and the identity element \(e\) of \((G, +)\) is a \((2, 1)\)-identity for it.

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