ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

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Abstract By means of a generalized Sălăgean differential operator we define a new class \( BO(n, \mu, \alpha, \lambda) \) involving functions \( f \in A \). Parallel results, for some related classes including the class of starlike and convex functions respectively, are obtained too.

Keywords: analytic function, starlike function, convex function, generalized Sălăgean operator.

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1. INTRODUCTION AND DEFINITIONS

Let \( A \) denote the class of functions of the form \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), which are analytic in the open unit disc \( U = \{ z : |z| < 1 \} \) and \( \mathcal{H}(U) \) the space of holomorphic functions in \( U \). Let

\[
A_n = \{ f \in \mathcal{H}(U), \ f(z) = z + a_{n+1}z^{n+1} + \ldots, \ z \in U \}
\]

with \( A_1 = A \) and

\[
\mathcal{H}[a, n] = \{ f \in \mathcal{H}(U), \ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots, \ z \in U \}
\]

for \( a \in \mathbb{C} \) and \( n \in \mathbb{N} \).

Let \( S \) denote the subclass of functions that are univalent in \( U \).

By \( S^s(\alpha) \) we denote a subclass of \( A \) consisting of starlike univalent functions of order \( \alpha \), \( 0 \leq \alpha < 1 \) which satisfies \( \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \), \( z \in U \).

Further, a function \( f \) belonging to \( S \) is said to be convex of order \( \alpha \) in \( U \), if and only if \( \text{Re} \left( \frac{zf''(z)}{f'(z)} + 1 \right) > \alpha \), \( z \in U \), for some \( \alpha \), \( 0 \leq \alpha < 1 \). We denote by
\(K(\alpha)\) the class of functions in \(S\) which are convex of order \(\alpha\) in \(U\) and denote by \(R(\alpha)\) the class of functions in \(A\) which satisfy Re \(f'(z) > \alpha, z \in U\).

It is well-known that \(K(\alpha) \subset S^*(\alpha) \subset S\).

If \(f\) and \(g\) are analytic functions in \(U\), we say that \(f\) is subordinate to \(g\), written \(f \prec g\), if there is a function \(w\) analytic in \(U\), with \(w(0) = 0, |w(z)| < 1\), for all \(z \in U\) such that \(f(z) = g(w(z))\) for all \(z \in U\). If \(g\) is univalent, then \(f \prec g\) if and only if \(f(0) = g(0)\) and \(f(U) \subseteq g(U)\).

Let \(D^n\) be a generalized Sălăgean operator introduced by Al-Oboudi in [1], \(D^n : A \rightarrow A, n \in \mathbb{N}\), defined as

\[
D^0 f(z) = f(z), \\
D^1 f(z) = (1 - \lambda) f(z) + \lambda zf'(z) = D_\lambda f(z), \quad \lambda > 0 \\
D^n f(z) = D_\lambda(D^{n-1} f(z)), \quad z \in U.
\]

We note that if \(f \in A\), then

\[
D^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j - 1) \lambda]^{n} a_j z^j, \quad z \in U.
\]

For \(\lambda = 1\), we get the Sălăgean operator [5].

In order to prove our main theorem we need the following lemma.

**Lemma 1.1.** [4]. Let \(p\) be analytic in \(U\) with \(p(0) = 1\) and suppose that

\[
\text{Re} \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha}, \quad z \in U.
\]

Then \(\text{Re} p(z) > \alpha\) for \(z \in U\) and \(1/2 \leq \alpha < 1\).

2. MAIN RESULTS

**Definition 1.** We say that a function \(f \in A\) is in the class \(\mathcal{B}\mathcal{O}(n, \mu, \alpha, \lambda)\), \(n \in \mathbb{N}, \mu \geq 0, \lambda \geq 0, \alpha \in [0, 1)\) if

\[
\left| \frac{D^{n+1}_\lambda f(z)}{z} \left( \frac{z}{D_\lambda^n f(z)} \right)^\mu - 1 \right| < 1 - \alpha, \quad z \in U.
\]

**Remark.** The family \(\mathcal{B}\mathcal{O}(n, \mu, \alpha, \lambda)\) is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, \(\mathcal{B}\mathcal{O}(0, 1, \alpha, 1) \equiv \mathcal{K}(\alpha)\).
If for the function \( f \in \mathcal{A} \), \( n \in \mathbb{N} \), \( \mu \geq 0 \), \( \lambda > 0 \), \( 1/2 \leq \alpha < 1 \) we have
\[
1 \frac{D_{\lambda}^{n+2} f(z)}{D_{\lambda}^{n+1} f(z)} - \frac{\mu}{\lambda} \frac{D_{\lambda}^{n+1} f(z)}{D_{\lambda}^{n} f(z)} + \frac{\mu - 1}{\lambda} + 1 < 1 + \beta z, \quad z \in U,
\]
where \( \beta = \frac{3\alpha - 1}{2\alpha} \), then \( f \in \mathcal{B}(n, \mu, \alpha, \lambda) \).

**Proof.** Consider \( p(z) = \frac{D_{\lambda}^{n+1} f(z)}{z} \left( \frac{z}{D_{\lambda}^{n} f(z)} \right)^{\mu} \). Then \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \). A simple differentiation yields
\[
\frac{zp'(z)}{p(z)} = \frac{1}{\lambda} \frac{D_{\lambda}^{n+2} f(z)}{D_{\lambda}^{n+1} f(z)} - \frac{\mu}{\lambda} \frac{D_{\lambda}^{n+1} f(z)}{D_{\lambda}^{n} f(z)} + \frac{\mu - 1}{\lambda}.
\]
Using (1) we get \( \text{Re} \left( 1 + \frac{zp'(z)}{p(z)} \right) > \frac{3\alpha - 1}{2\alpha} \). Thus, by Lemma 1.1, we deduce that
\[
\text{Re} \left\{ \frac{D_{\lambda}^{n+1} f(z)}{z} \left( \frac{z}{D_{\lambda}^{n} f(z)} \right)^{\mu} \right\} > \alpha.
\]
Therefore, \( f \in \mathcal{B}(n, \mu, \alpha, \lambda) \), by Definition 1.

As consequences of the above theorem we have the following corollaries.

**Corollary 1.** If \( f \in \mathcal{A} \) and \( \text{Re} \left\{ \frac{2zf''(z) + zf'''(z) - zf''(z)}{f'(z)} \right\} > -\frac{1}{2}, \quad z \in U \), then \( f \in \mathcal{B}(1, 1, 1/2, 1) \), hence \( \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in U \). That is, \( f \) is convex of order \( \frac{1}{2} \).

**Corollary 2.** If \( f \in \mathcal{A} \) and \( \text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{1}{2}, \quad z \in U \) then \( f'(z) > \frac{1}{2}, \quad z \in U \). In other words, if the function \( f \) is convex of order \( \frac{1}{2} \), then \( f \in \mathcal{B}(0, 0, \frac{1}{2}, 1) \equiv \mathbb{R} \left( \frac{1}{2} \right) \).

**Corollary 3.** If \( f \in \mathcal{A} \) and \( \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in U \). That is, \( f \) is a starlike function.

**Corollary 4.** If \( f \in \mathcal{A} \) and \( \text{Re} \left\{ \frac{zf'(z) + zf''(z) + zf'''(z)}{f'(z) + zf''(z)} \right\} > -\frac{1}{2}, \quad z \in U \), then \( \text{Re} \left\{ \frac{zf'(z)}{z} + f'(z) - 2 \right\} > 2, \quad z \in U \).
References


