ROMAI J., **4**, 1(2008), 149–168

PSEUDOPERTURBITERATION METHODS IN GENERALIZED EIGENVALUE PROBLEMS

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Abstract **Abstract** The review of the authors' results is given on pseudoperturbiteration methods allowing to refine the spectral characteristics, eigenvalues, eigen- and Jordan elements of the operator-functions depending on spectral parameter and adjoint to them. The main attention is payed to the development of iteration processes and their various applications.

Keywords: generalized eigenvalue problems; adjoint problem; refining of spectral characteristics; perturbation and bifurcation theory; iteration processes; applications in mathematical physics.

2000 MSC: 47A55, 47A75, 58E07, 65J99.

1. INTRODUCTION

The idea of pseudoperturbation method (PPM), i.e. perturbation operator construction such that the known approximations to spectral characteristics would became exact for the perturbed operator, is due to M.K. Gavurin [1] in his application to the refining of simple eigenvalues and eigenelements of self-adjoint operators in Hilbert spaces. Later on his PhD-student F. Kuhnert [2] has solved this problem for non-self-adjoint operators. The further development of PP-method was given in the articles by B. V. Loginov, D. G. Rakhimov and N. A. Sidorov (see review article [3]), where the refining problem was solved for multiple eigenvalues, eigenvectors and generalized Jordan chains (GJCh) of linear by spectral parameter operator-function in Banach spaces. The suggested there PP-operator did not make possible to use GJChs of adjoint operator-function, and in the general case, the problem remained

unsolved. In 2003 [4, 5] two forms of PP-operators were suggested which allow to refine multiple eigenvalues and GJChs of the eigenvalue problem linear by spectral parameter and adjoint to it in Banach spaces. They symmetrically used the known approximations to enumerating spectral characteristics with subsequent application of Newton-Kantorovich method to their refining. In the articles [6, 7] the development of PP-method to nonlinear with respect to spectral parameter operator-function was given based on its linearization possibilities.

Here four iteration processes for the determination of exact spectral characteristics are suggested together with the investigation of their convergence rates and stability with respect to small computation errors. The main attention is payed to various illustrations of PP-iteration method; some of them are reviewed in [8]. The PP-iteration (PPI) methods are considered in a group symmetry conditions. Also their connections with parameter continuation methods [9] are investigated. The theory of PPI-methods and the proof of their computational stability is based on bifurcation and perturbation theories [10].

The obtained results are supported by grant RFBR-Romanian Academy, N 07-01-91680-PA_a.

2. PSEUDOPERTURBATION OPERATOR CONSTRUCTION

In Banach spaces E_1 and E_2 the linear by spectral parameter eigenvalue problem with bounded for simplicity linear operators $B, A \in L(E_1, E_2)$

$$(B - tA)x = 0 \tag{1}$$

is considered. Let the unknown eigenvalue λ be the Fredholm point of the operator-function $B - tA$ with eigenelements $N(B-\lambda A) = span\{\varphi_1^{(1)}, \ldots, \varphi_n^{(1)}\}$, $N(B^* - \lambda A^*) = span\{\psi_1^{(1)}, \ldots, \psi_n^{(1)}\}$ and corresponding $A-$ and A^*- Jordan chains [10] of lengths $p_1 \le p_2 \le \ldots \le p_n$

$$\begin{aligned}
&(B - \lambda A)\varphi_i^{(s)} = A\varphi_i^{(s-1)},\ (B^* - \lambda A^*)\psi_i^{(s)} = A^*\psi_i^{(s-1)},\ s = \overline{2, p_i},\ i = \overline{1, n},\\
&k = \det[\langle A\varphi_i^{(p_i)}, \psi_j^{(1)}\rangle] \neq 0,\ L_{ij} = [\langle A\varphi_i^{(p_i+1-s)}, \psi_j^{(l)}\rangle]_{l=\overline{1,p_j},\, s=\overline{1,p_i}} \neq 0,\\
&L = \det[L_{ij}] \neq 0,
\end{aligned} \tag{2}$$

which can be chosen [10, 11] to satisfy the biorthogonality relations

$$\begin{array}{ll} \langle \varphi_i^{(j)}, \gamma_k^{(l)} \rangle = \delta_{ik}\delta_{jl}, & \langle z_i^{(j)}, \psi_k^{(l)} \rangle = \delta_{ik}\delta_{jl}, \quad j(l) = 1, \ldots, p_i(p_k) \\ \gamma_k^{(l)} = A^* \psi_k^{(p_k+1-l)}, & z_i^{(j)} = A\varphi_i^{(p_i+1-j)}, \quad i,k = 1, \ldots, n. \end{array} \tag{3}$$

Let some sufficient good approximations λ_0, $\varphi_{i0}^{(s)}$, $\psi_{i0}^{(s)}$ to unknown eigenvalue λ and GJChs be given, $|\lambda - \lambda_0| \le \varepsilon$, $\|\varphi_i^{(s)} - \varphi_{i0}^{(s)}\| \le \varepsilon$, $\|\psi_i^{(s)} - \psi_{i0}^{(s)}\| \le \varepsilon$, with close to unit relevant magnitudes k_0 and L_0 (2). The problem is posed: to construct pseudoperturbation operator D_0, for which the given approximations would be exact for the perturbed operator.

Lemma 1. *Passing to linear combinations the systems* $\{\gamma_{k0}^{(l)}\}_{k=\overline{1,n}}^{l=\overline{1,p_k}}$, $\gamma_{k0}^{(l)} = A^*\psi_{k0}^{(p_k+1-l)}$, $\{z_{i0}^{(j)}\}_{i=\overline{1,n}}^{j=\overline{1,p_i}}$, $z_{i0}^{(j)} = A\varphi_{i0}^{(p_i+1-j)}$ *satisfying the biorthogonality relations* (3) $\langle \varphi_{i0}^{(j)}, \gamma_{k0}^{(l)} \rangle = \delta_{ik}\delta_{jl}$, $\langle z_{i0}^{(j)}, \psi_{k0}^{(l)} \rangle = \delta_{ik}\delta_{jl}$ *can be determined.*

Proof. In fact, let some sufficiently good approximations $\{\varphi_{i0}^{(j)}\}_{i=\overline{1,n}}^{j=\overline{1,p_i}}$, $\{\widetilde{\psi}_{\mu 0}^{(\nu)}\}_{\mu=\overline{1,n}}^{\nu=\overline{1,p_\mu}}$ and $z_{i0}^{(j)} = A\varphi_{i0}^{(p_i+1-j)}$ be given. Setting $\widetilde{\gamma}_{\mu 0}^{(\nu)} = A^*\widetilde{\psi}_{\mu 0}^{(p_\mu+1-\nu)}$ form the linear combinations $\gamma_{k0}^{(l)} = \sum\limits_{\mu=1}^{n}\sum\limits_{\nu=1}^{p_\mu} K_{k\mu}^{l\nu}\widetilde{\gamma}_{\mu 0}^{(\nu)}$, subject to the conditions $\langle \varphi_{i0}^{(j)}, \gamma_{k0}^{(l)} \rangle = \delta_{ik}\delta_{jl}$, $i,k = 1, \ldots, n$, $j(l) = 1, \ldots, p_i(p_k)$. Then according to (2) the coefficients $K_{k\mu}^{l\nu}$ can be uniquely determined by the system of equations

$$\sum_{\mu=1}^{n}\sum_{\nu=1}^{p_\mu} K_{k\mu}^{l\nu}\langle A\varphi_{i0}^{(j)}, \widetilde{\psi}_{\mu 0}^{(p_\mu+1-\nu)} \rangle = \delta_{ik}\delta_{jl}, \; i,k = 1, \ldots, n, \; j(l) = 1, \ldots, p_i(p_k).$$

Setting now $\psi_{k0}^{(p_k+1-l)} = \sum\limits_{\mu=0}^{n}\sum\limits_{\nu=1}^{p_\mu} K_{k\mu}^{l\nu}\widetilde{\psi}_{\mu 0}^{(p_\mu+1-\nu)}$ from this system it follows $\langle z_{i0}^{(p_i+1-j)}, \psi_{k0}^{(p_k+1-l)} \rangle = \delta_{ik}\delta_{jl}$, $z_{i0}^{(p_i+1-j)} = A\varphi_{i0}^{(j)}$.

Computing the discrepancies

$$\begin{array}{ll} \sigma_{i0}^{(1)} = (B - \lambda_0 A)\varphi_{i0}^{(1)}, & \sigma_{i0}^{(j)} = (B - \lambda_0 A)\varphi_{i0}^{(j)} - A\varphi_{i0}^{(j-1)}, \\ \tau_{i0}^{(1)} = (B^* - \lambda_0 A^*)\psi_{i0}^{(1)}, & \tau_{i0}^{(j)} = (B^* - \lambda_0 A^*)\psi_{i0}^{(j)} - A^*\psi_{i0}^{(j-1)}, \end{array} \qquad j = 2, \ldots, p_i$$

determine [4] two forms of pseudoperturbation operator

$$D_0 x = \sum_{i=1}^{n}\sum_{j=1}^{p_i}\langle x, \gamma_{i0}^{(j)} \rangle \Big[\sigma_{i0}^{(j)} - \sum_{k=1}^{n}\sum_{s=1}^{p_k}\langle \sigma_{i0}^{(j)}, \psi_{k0}^{(s)} \rangle z_{k0}^{(s)}\Big] + \sum_{i=1}^{n}\sum_{j=1}^{p_i}\langle x, \tau_{i0}^{(j)} \rangle z_{i0}^{(j)} \tag{4}$$

$$D_0 x = \sum_{i=1}^{n}\sum_{j=1}^{p_i}\langle x, \gamma_{i0}^{(j)} \rangle \sigma_{i0}^{(j)} + \sum_{i=1}^{n}\sum_{j=1}^{p_i}\Big[\langle x, \tau_{i0}^{(j)} - \sum_{k=1}^{n}\sum_{s=1}^{p_k}\langle \varphi_{k0}^{(s)}, \tau_{i0}^{(j)} \rangle \gamma_{k0}^{(s)} \rangle\Big] z_{i0}^{(j)} \tag{5}$$

Theorem 1. [4] *Pseudoperturbation operators* (4) *and* (5) *have the following properties*

$$\begin{matrix} D_0\varphi_{i0}^{(s)} = \sigma_{i0}^{(s)} \\ D_0^*\psi_{i0}^{(s)} = \tau_{i0}^{(s)} \end{matrix} \Rightarrow \begin{matrix} (B-\lambda_0 A - D_0)\varphi_{i0}^{(1)} = 0, & (B-\lambda_0 A - D_0)\varphi_{i0}^{(s)} = A\varphi_{i0}^{(s-1)} \\ (B^*-\lambda_0 A^* - D_0^*)\psi_{i0}^{(1)} = 0, & (B^*-\lambda_0 A^* - D_0^*)\psi_{i0}^{(s)} = A^*\psi_{i0}^{(s-1)} \end{matrix} \tag{6}$$

3. ITERATIVE PROCESSES AND THEIR CONVERGENCE RATES

A. Newton-Kantorovich method

According to the generalized E. Schmidt lemma [10] and general perturbation theory at the sufficiently exact initial approximations ($\|D_0\|$ is small) there exists the bounded operator $\Gamma = \widetilde{B}^{-1} = \left[(A_0 - \lambda_0 A_1 - D_0) + \sum\limits_{i=1}^{n} \left\langle \cdot, \gamma_{i0}^{(1)} \right\rangle z_{i0}^{(1)}\right]^{-1}$. Consequently the Jordan elements for the exact eigenvalue are determined by formulae [4]

$$\varphi_i^{(s)} = (I-(\lambda-\lambda_0)\Gamma A + \Gamma D_0)^{-1}\left[\Gamma A (I-(\lambda-\lambda_0)\Gamma A + \Gamma D_0)^{-1}\right]^{s-1}\varphi_{i0}^{(1)}, \quad s = 1,2,\ldots \tag{7}$$

and it is not difficult to see [4] that the exact eigenvalue $t=\lambda$ is $K = \sum\limits_{i=1}^{n} p_i$-multiple root of the equation

$$f(t) = \det[l_{ij}(t-\lambda_0)] = 0,$$

$$l_{ij}(t-\lambda_0) = \left\langle \left((t-\lambda_0)A - D_0\right)\left[I - \Gamma(t-\lambda_0)A + \Gamma D_0\right]^{-1}\varphi_{i0}^{(1)}, \psi_{k0}^{(1)} \right\rangle$$

$$= -\left\langle \left\{ I - \left[I - \Gamma((t-\lambda_0)A_1 - D_0)\right]^{-1}\right\}\varphi_{i0}^{(1)}, \gamma_{k0}^{(1)} \right\rangle,$$

$$l_{ij,s}(t-\lambda_0) = \frac{1}{s!}\frac{d^s l_{ij}(t-\lambda_0)}{dt^s}$$

$$= \left\langle [I-(t-\lambda_0)\Gamma A + \Gamma D_0]^{-1}\left(\Gamma A\left[I-(t-\lambda_0^{)}\Gamma A + \Gamma D_0\right]^{-1}\right)^s \varphi_{i0}^{(1)}, \gamma_{j0}^{(1)} \right\rangle,$$

i.e. $\left.\dfrac{d^s f(t)}{dt^s}\right|_{t=\lambda} = 0$ for $s < K$ and $\left.\dfrac{d^K f(t)}{dt^K}\right|_{t=\lambda} = \det\left[\dfrac{d^{p_i} l_{ij}(t-\lambda_0)}{dt^{p_i}}\right]_{t=\lambda} \neq 0$ by virtue of the relation

$$\frac{1}{p_i!}\frac{d^{p_i} l_{ik}(0)}{dt^{p_i}} = \left\langle (I+\Gamma D_0)^{-1}\left(\Gamma A (I+\Gamma D_0)^{-1}\right)^{p_i}\varphi_{i0}^{(1)}, \gamma_{k0}^{(1)} \right\rangle$$

$$= \left\langle (\Gamma A)^{p_i}\varphi_{i0}^{(1)}, \gamma_{k0}^{(1)} \right\rangle -$$

$$- \left\langle \left[\Gamma D_0 (\Gamma A)^{p_i} + (\Gamma A)(\Gamma D_0)(\Gamma A)^{p_i-1} + \ldots + (\Gamma A)^{p_i}(\Gamma D_0)\right]\varphi_{i0}^{(1)}, \gamma_{k0}^{(1)} \right\rangle +$$

$$+ o(\|D_0\|) = \left\langle \varphi_{i0}^{(1)}, \gamma_{k0}^{(1)} \right\rangle + O(\|D_0\|) = \delta_{ik} + O(\|D_0\|). \tag{8}$$

In order to determine the exact λ of the equation $f^{(K-1)}(t)=0$, the modified or basic Newton-Kantorovich (N.-K.) method can be applied taking $t=\lambda_0$ as the initial approximation

$$\lambda_\nu=\lambda_{\nu-1}-\left[f^{(K)}(\lambda_0)\right]^{-1}f^{(K-1)}(\lambda_{\nu-1}),\nu=1,2,\dots \tag{9}$$

$$\lambda_\nu=\lambda_{\nu-1}-\left[f^{(K)}(\lambda_{\nu-1})\right]^{-1}f^{(K-1)}(\lambda_{\nu-1}),\nu=1,2,\dots \tag{10}$$

Theorem 2. *Newton-Kantorovich method applied to the equation* $f^{(K-1)}(t)=0$*, for sufficiently small* $\|D_0\|$*, determines its unique solution* λ *and has the square convergence rate.*

Proof. Indeed, by virtue of formulae [10,§31]

$$\varphi_{j0}^{(s)}=(\Gamma_0 A)^{s-1}\varphi_{j0}^{(1)}=\varphi_{j0}^{(s-\left[\frac{s}{p_j}\right]p_j)}\text{ and }\left\langle\varphi_{j0}^{(s)},\gamma_{k0}^{(1)}\right\rangle=\delta_{jk}\delta_{(s-\left[\frac{s}{p_j}\right]p_j),1} \tag{11}$$

and according to Hadamard inequality for determinant, there exists a constant C such that $\left|f^{(s)}(\lambda_0)\right|\le C\|D_0\|$, $s\neq K$. Also the relation (8) gives $f^{(K)}(\lambda_0)=p_1!\dots p_n!\det\left[\delta_{ik}+O(\|D_0\|)\right]$, whence by virtue of the continuity $f^{(K)}(t)$ there exists a constant such that $\left|f^{(K)}(\lambda_v)\right|^{-1}\le m(\rho)$ in some ρ-neighborhood $S_\rho(\lambda_0)$ of the point λ_0. The continuity of $f^{(K+1)}(\lambda)$ in $S_\rho(\lambda_0)$ gives the Lipschitz condition with some constant $l\quad\left|f^{(K)}(\lambda_1)-f^{(K)}(\lambda_2)\right|\le l\left|\lambda_1-\lambda_2\right|,\quad\lambda_1,\lambda_2\in S_\rho(\lambda_0)$. Consequently [12, §34.2] if $q=\frac{1}{2}m^2lC\|D_0\|<1$ and $\rho'=mC\|D_0\|\sum\limits_{k=0}^{\infty}q^{2^K-1}<\rho$, the equation $f^{(K-1)}(t)=0$ has in the ball $S'_\rho(\lambda_0)$the unique solution λ to which the iterations (10) are square convergent.

Corollary. *According to theorem [12, §34.3] , if* $2m^2lC\|D_0\|<1$ *and* $r'=\frac{1}{ml}(1-\sqrt{1-2m^2lC\|D_0\|})<\rho$ *the iterations* (9) *of the modified N.-K. method converge in the ball* $\overline{S'_r(\lambda_0)}$ *to the unique solution* λ*. The convergence rate of* $\{\lambda_\nu\}$ *to* λ *is given by the inequality* $|\lambda-\lambda_\nu|\le\frac{1-\sqrt{1-2m^2lC\|D_0\|}}{\sqrt{1-2m^2lC\|D_0\|}}mC\|D_0\|$.

After the determination of the exact eigenvalue λ the elements of GJChs can be determined directly from the following equations [4]

$$\begin{aligned}&(B-\lambda A)\varphi_j^{(1)}+\sum_{i=1}^{n}\left\langle\varphi_j^{(1)},\gamma_{i0}^{(1)}\right\rangle z_{i0}^{(1)}=z_{j0}^{(1)},\\&(B-\lambda A)\varphi_j^{(s)}+\sum_{i=1}^{n}\left\langle\varphi_j^{(s)},\gamma_{i0}^{(1)}\right\rangle z_{i0}^{(1)}=A\varphi_j^{(s-1)},\end{aligned}\quad s=2,\dots,p_j,j=1,\dots,n \tag{12}$$

$$\begin{aligned}
&(B^* - \lambda A^*)\psi_j^{(1)} + \sum_{i=1}^{n} \left\langle z_{i0}^{(1)}, \psi_j^{(1)} \right\rangle \gamma_{i0}^{(1)} = \gamma_{j0}^{(1)}, \\
&(B^* - \lambda A^*)\psi_j^{(s)} + \sum_{i=1}^{n} \left\langle z_{i0}^{(1)}, \psi_j^{(s)} \right\rangle \gamma_{i0}^{(1)} = A^*\psi_j^{(s-1)}.
\end{aligned} \qquad s = \overline{2, p_j}, j = \overline{1, n} \quad (13)$$

B. Newton-Kantorovich method with cubic convergence

In the articles [13, 14] one modification of the basic N.-K. method was suggested that has the cubic convergence. It is based on the change of the original nonlinear equation $g(t) = 0$ by the equation $F(t) = g(t)e^{-kt} = 0$, which has the same roots. This change gives the modification of the iteration process by the following one

$$t_{n+1} = t_n - g(t_n)\left[g'(t_n) - kg(t_n)\right]^{-1}, \quad (14)$$

where the arbitrary parameter k is chosed in the special form $k = \dfrac{g''(t_n)}{2g'(t_n)}$. In fact, the setting $\varepsilon_n = t^* - t_n$ gives for the iterations (14) the following relation $\varepsilon_{n+1} = \varepsilon_n + g(t_n)\left[g(t_n) - kg(t_n)\right]^{-1}$, whence by using of the Taylor expansions of $g(t_n)$ and $g'(t_n)$ in a neighbourhood of the exact solution t^*, the equality

$$\begin{aligned}
\varepsilon_{n+1} &= -\varepsilon_n^2 \frac{\dfrac{1}{2}g''(t^*) - kg'(t^*) - \varepsilon_n\left(\dfrac{1}{3}g'''(t^*) - \dfrac{k}{2}g''(t^*)\right) + o\left(|\varepsilon_n|\right)}{g'(t^*) - \varepsilon_n\left(g''(t^*) - kg'(t^*)\right) + o\left(|\varepsilon_n|\right)} \\
&= \varepsilon_n^3 \frac{\dfrac{1}{3}g'''(t^*) - \dfrac{k}{2}g''(t^*) + o\left(|\varepsilon_n|\right)}{g'(t^*) - \varepsilon_n\dfrac{1}{2}g''(t^*) + o\left(|\varepsilon_n|\right)}.
\end{aligned}$$

Under the theorem of [12, §34.2] assumptions about convergence rate of the basic Newton method in [14] its analog about cubic convergence of the suggested modification is proved.

Consequently the relevant iteration scheme for our problem has the form

$$t_n = t_{n-1} - \frac{2g(t_{n-1})g'(t_{n-1})}{2g'^2(t_{n-1}) - g''(t_{n-1})g(t_{n-1})} = \quad (15)$$

$$g = f^{(K-1)}(t)t_{n-1} - \frac{2f^{(K-1)}(t_{n-1})f^{(K)}(t_{n-1})}{2\left[f^{(K)}(t_{n-1})\right]^2 - f^{(K+1)}(t_{n-1})f^{(K-1)}(t_{n-1})},$$

which has the cubic convergence at sufficiently small $\|D_0\|$.

C. Eitken-Steffensen iteration process

In the monographs [15, 16, 17] it was investigated the Eitken-Steffensen iteration process which does not require the derivatives computation, however

possessing the square convergence rate

$$\lambda_n = \lambda_{n-1} - \frac{g(\lambda_{n-1})}{g(\lambda_{n-1}) - g\left(\psi(\lambda_{n-1})\right)} g(\lambda_{n-1}), \psi(t) = t - f(t). \qquad (16)$$

For our problem we must take $g(t) = f^{(K-1)}(t)$, and (16) takes the form

$$\lambda_n = \lambda_{n-1} - \frac{f^{(K-1)}(\lambda_{n-1})}{f^{(K-1)}(\lambda_{n-1}) - f^{(K-1)}\left(\psi(\lambda_{n-1})\right)} f^{(K-1)}(\lambda_{n-1}).$$

On every step of this process it is required to solve $4n^2(K+1)^2$ linear equations (12), (13) for $\lambda = \lambda_{n-1}$ and $\psi(\lambda_{n-1}) = \lambda_{n-1} - f^{(K-1)}(\lambda_{n-1})$, respectively.

Remark 1. The iteration processes **A** and **C** can be changed on the relevant processes for the computation of multiple roots [15], for our problem of K-multiple roots of the equation $f(t) = \det[l_{ij}(t - \lambda_0)] = 0$,

A. $t_n = t_{n-1} - K\dfrac{f(t_{n-1})}{f'(t_{n-1})}$, **C.** $\lambda_n = \lambda_{n-1} - K\dfrac{f(\lambda_{n-1})}{f(\lambda_{n-1}) - f(\psi(\lambda_{n-1}))} f(\lambda_{n-1})$, which have superlinear convergence rate.

D. Gavurin iteration process

By means of pseudoperturbation operator D_0 (4), (5) the eigenvalue problem (1) is rewritten in the form of the perturbation problem for the eigenvalue λ_0 with the small parameter D_0 [10, §32]

$$B_0\varphi \equiv (B - \lambda_0 A - D_0)\varphi = (\Delta\lambda)A\varphi - D_0\varphi, \quad \mu = \Delta\lambda = \lambda - \lambda_0, \qquad (17)$$

which can be reduced [10] to the branching equation (BEq)

$$L(\mu, \|D_0\|) = \det\left[\langle((\Delta\lambda)A - D_0)\left[I - \Gamma_0\left((\Delta\lambda)A - D_0\right)\right]^{-1}\varphi_{i0}^{(1)}, \psi_{k0}^{(1)}\rangle\right] = 0. \qquad (18)$$

The length of the decreasing part of the Newton diagram for (13) is equal to $K = \sum\limits_{i=1}^{n} p_i$, since $L(\mu, 0) = \sum\limits_{s=k}^{\infty} L_{s0}\mu^s$, $L_{s0} = 0$, $s < k$,

$L_{K0} = \det\left[\langle \mu A\left(I - \Gamma_0 A\right)^{-1}\varphi_{i0}^{(1)}, \psi_{k0}^{(1)}\rangle\right] = \det\left[\langle A\varphi_{i0}^{(p_i)}, \psi_{k0}^{(1)}\rangle\right] = 1.$

However, some of BEq coefficients $L_{s,K-s}$ may be nonzero. Let ν be the first number for which $L_{\nu,K-\nu} \neq 0$.

From formulae (7) it follows that as first approximations to the exact Jordan chain elements we must to take the following

$$\begin{aligned}
&\varphi_{j1}^{(1)} = (I + \Gamma D_0)^{-1}\varphi_{j0}^{(1)},\\
&\varphi_{j1}^{(2)} = (I + \Gamma D_0)^{-1}\Gamma A(I + \Gamma D_0)^{-1}\varphi_{j0}^{(1)} = (I + \Gamma D_0)^{-1}\Gamma A\varphi_{j1}^{(1)}, \ldots,\\
&\varphi_{j1}^{(p_i)} = (I + \Gamma D_0)^{-1}\left[\Gamma A(I + \Gamma D_0)^{-1}\right]^{p_j - 1}\varphi_{j0}^{(1)} = (I + \Gamma D_0)^{-1}\Gamma A\varphi_{j1}^{(p_i - 1)},
\end{aligned} \qquad (19)$$

$$\begin{aligned}
&\psi_{j1}^{(1)} = (I+\Gamma^* D_0^*)^{-1}\psi_{j0}^{(1)},\\
&\psi_{j1}^{(2)} = (I+\Gamma^* D_0^*)^{-1}\Gamma^* A^*(I+\Gamma^* D_0^*)^{-1}\psi_{j0}^{(1)} = (I+\Gamma^* D_0^*)^{-1}\Gamma^* A^*\psi_{j1}^{(1)},\dots,\\
&\psi_{j1}^{(p_i)} = (I+\Gamma^* D_0^*)^{-1}\left[\Gamma^* A^*(I+\Gamma^* D_0^*)^{-1}\right]^{p_j-1}\psi_{j0}^{(1)}\\
&\qquad = (I+\Gamma^* D_0^*)^{-1}\Gamma^* A^*\psi_{j1}^{(p_i-1)}.
\end{aligned} \tag{20}$$

These elements are the solutions of the equations

$$\begin{aligned}
&(B-\lambda_0 A)\varphi_{j1}^{(1)} + \sum_{k=1}^{n}\left\langle \varphi_{j1}^{(1)}, \gamma_{k0}^{(1)}\right\rangle z_{k0}^{(1)} = z_{j0}^{(1)},\\
&(B-\lambda_0 A)\varphi_{j1}^{(s)} + \sum_{k=1}^{n}\left\langle \varphi_{j1}^{(s)}, \gamma_{k0}^{(1)}\right\rangle z_{k0}^{(1)} = A\varphi_{j1}^{(s-1)},
\end{aligned} \qquad s=2,\dots,p_j, j=1,\dots,n$$

$$\begin{aligned}
&(B^*-\lambda_0 A^*)\psi_{j1}^{(1)} + \sum_{k=1}^{n}\left\langle z_{k0}^{(1)}, \psi_{j1}^{(1)}\right\rangle \gamma_{k0}^{(1)} = \gamma_{j0}^{(1)},\\
&(B^*-\lambda_0 A^*)\psi_{j1}^{(s)} + \sum_{k=1}^{n}\left\langle z_{k0}^{(1)}, \psi_{j1}^{(s)}\right\rangle \gamma_{k0}^{(1)} = A^*\psi_{j1}^{(s-1)},
\end{aligned} \qquad s=2,\dots p_j, j=1,\dots,n$$

respectively. The relevant first approximation λ_1 to the exact eigenvalue λ can be found by the Newton diagram method from the BEq (18). Now after Lemma 1 application, i.e. after the biorthogonalization of the first approximations, we repeat the iteration process: the pseudoperturbation operator D_1 is determinating and the second approximations to Jordan elements $\varphi_{j2}^{(s)}$, $\psi_{j2}^{(s)}$, $s=1,\dots,p_j$, $j=1,\dots,n$ with the relevant λ_2 determination and so on.

Theorem 3. *The M.K. Gavurin iteration processes is converging with the rate of the order* $1+\dfrac{1}{\nu}$*, i.e. it is superlinearly convergent.*

The technically difficult proof coincides with the proof of relevant theorems in [18, 19].

4. STABILITY OF ITERATION PROCESSES

In [20, 21] regularization by A.N. Tychonov questions in perturbation and bifurcation theories are investigated. The symbol $\sim$ denotes δ-approximations of the corresponding magnitudes, $\|\widetilde{A}x - Ax\| \le \delta(\|x\| + a\|Ax\|)$, $\forall x \in D(A)$. Here we give the result for linear equations, on the base of which the computational stability of suggested iteration processes can be satisfied by Theorem 4 [20, 21]. Let in the equation $Ax = f$, $A : E_1 \supset D(A) \to E_2$, $\overline{D(A)} = E_1$ the operator A be Fredholm; $N(A) = span\{\varphi_1,\dots,\varphi_n\}$, $N^*(A) = span\{\psi_1,\dots,\psi_n\}$ be its zero and defect subspaces; $\{\gamma_i\}_1^n \in E_1^*$ and $\{z_i\}_1^n \in E_2^*$ be corresponding

biorthogonal systems. Let $x^* = \Gamma f$, $\Gamma = \left(A + \sum_{i=1}^{n} \langle \cdot, \gamma_i \rangle z_i\right)^{-1}$ be the normal solution of the equation $Ax = f$. If $\delta < q[a + (1 + na)\|\Gamma\|]^{-1}$, $0 < q < 1$, then the unique solution of the regularized equation $\widetilde{A}x + \sum_{i=1}^{n} \langle x, \gamma_i \rangle z_i = \widetilde{f}$ is determined by the formula $\widetilde{x} = \left(\widetilde{A}x + \sum_{i=1}^{n} \langle x, \gamma_i \rangle z_i\right)^{-1} \widetilde{f}$ and satisfies the following estimate $\|\widetilde{x} - \Gamma f\| \leq (1 - q)^{-1} \|\Gamma\|(1 + a\|f\| + \|\Gamma f\|)\delta$.

5. APPLICATIONS OF PSEUDOPERTURBITERATION METHODS

A. Pseudoperturbiteration method for differential equations with displacements in boundary conditions

Here we consider three problems with displacements: 1. one-dimensional Bitsadze-Samarskii problem [22]; 2. four eigenvalue problems with two displacements [23]; 3. E. Schmidt eigenvalue problems with displacement [24, 25, 26].

1. In the space $C^2[(0, x_0) \cup (x_0, 1)] \cap C^1[0, 1]$ the linear eigenvalue problem $u'' + \lambda u = 0$, $u(0) = 0$, $u(x_0) = u(1)$ is considered. It has two sets of eigenvalues $\mu_m = \dfrac{2m\pi}{1 - x_0}$, $\mu_s = \dfrac{(2s + 1)\pi}{1 + x_0}$, to which the eigenfunctions $\varphi_m = \sin \mu_m x$, $\varphi_s = \sin \mu_s x$ correspond. The adjoint problem $v'' + \lambda v = 0$, $v(0) = 0$, $v(1) = 0$, $v'(x_0 + 0) - v'(x_0 - 0) = v'(1)$ in the space $C^2[(0, x_0) \cup (x_0, 1)] \cap C[0, 1]$ has the same eigenvalues with relevant eigenfunctions

$$\psi_m = \begin{cases} 0, & 0 \leq x \leq x_0, \\ \sin \mu_m (1 - x), & x_0 \leq x \leq 1; \end{cases}$$

$$\psi_s = \begin{cases} \dfrac{\sin \mu_s (1 - x_0) \cdot \sin \mu_s x}{\sin \mu_s x_0}, & 0 \leq x \leq x_0, \\ \sin \mu_s (1 - x), & x_0 \leq x \leq 1. \end{cases}$$

The Jordan chains of the length two exist in the condition $\mu_m = \mu_s = \mu_0 \Rightarrow x_0 = \dfrac{2s - 2m + 1}{2s + 2m + 1}$. They are computed in the form satisfying the biorthogonality conditions (3).

The realization of PPI-method is fulfilled for $m = s = 1$, $x_0 = \frac{1}{5}$, $\lambda = \mu_0^2 = \frac{25\pi^2}{4}$

$$\varphi^{(1)} = \sin\frac{5\pi}{2}x, \qquad \psi^{(1)} = \begin{cases} 0, & 0 \le x \le \frac{1}{5}, \\ \frac{125\pi}{6}\cos\frac{5\pi}{2}x, & \frac{1}{5} \le x \le 1; \end{cases}$$
$$\varphi^{(2)} = -\frac{1}{5\pi}x\cos\frac{5\pi}{2}x, \quad \psi^{(2)} = \begin{cases} -\frac{10}{3}\sin\frac{5\pi}{2}x, & 0 \le x \le \frac{1}{5}, \\ -\frac{25}{6}(1-x)\sin\frac{5\pi}{2}x, & \frac{1}{5} \le x \le 1. \end{cases} \tag{21}$$

As approximations $\widehat{\varphi}_0^{(1)}$, $\widehat{\psi}_0^{(1)}$, $\widehat{\varphi}_0^{(2)}$, $\widehat{\psi}_0^{(2)}$ to eigenfunctions $\varphi_0^{(1)}$, $\widetilde{\psi}_0^{(1)}$ and Jordan elements $\varphi_0^{(2)}$, $\widetilde{\psi}_0^{(2)}$ were selected relevant parts of Taylor series for (7), and the approximation to eigenvalue λ_0 was determined to within 1 from the equation $\left\langle \frac{d^2}{dx^2}\widehat{\varphi}_0^{(2)} + t\widehat{\varphi}_0^{(2)} + \widehat{\varphi}_0^{(1)}, \widehat{\psi}_0^{(1)} \right\rangle = 0$, by stopping the iteration process at the achievement of a sufficient accuracy of approximation to λ_0. Note that in the considered example the exact spectral characteristics were known.

The problems 2. are investigated in [23], the problems 3. will be described in part **C** 4.

B. The refining of critical spectral parameter at Poincaré-Andronov-Hopf bifurcation

This problem in the general form is considered in [25], separately for the DE of the first order $Ax' = Bx - R(x,\varepsilon)$, $R(0,0) = 0$, $R_x(0,0) = 0$ and DE of higher order of the form $A_s x^{(s)} + A_{s-1}x^{(s-1)} + \ldots + A_1 x' = Bx - R(x, x^{(1)}, \ldots, x^{(s-1)}, \varepsilon)$, $R(0,\ldots,0,0) = 0$, $R_{x^k}(0,\ldots,0,0) = 0$ with bounded, for simplicity, linear operators. Therefore we give here only concrete example in part **C**.3° as spatially one-dimensional dynamical problem with a displacement illustrating the results [25].

C. Generalized pseudoperturbiteration methods for computation of E. Schmidt eigenvalue problems

At the beginning of the past century, E. Schmidt has introduced [26] systems of eigenvalues $\{\lambda_k\}$ (counted with their multiplicity) and eigenelements $\{\varphi_k\}_1^\infty$, $\{\psi_k\}_1^\infty$ satisfying the relations $B\varphi_k = \lambda_k\psi_k$, $B^*\psi_k = \lambda_k\varphi_k$ for integral Fredholm operators and allowing to generalize Hilbert-Schmidt theory on non-self-adjoint completely continuous operators in abstract separable Hilbert

space [27]. As s-numbers these systems have found many applications in computational mathematics and ill-posed problems theory.

1. For a pair of linear, bounded for simplicity, operators $B, A \in L(H)$ the generalized E. Schmidt eigenvalues and eigenelements are introduced [28] by the following equalities: $B\varphi = \lambda A\psi$, $B^*\psi = \lambda A^*\varphi$. Here H is a Hilbert space, the eigenvalue λ can be chosen real. The above system can be rewritten in matrix form

$$(\mathcal{B} - \lambda\mathcal{A})\Phi = 0, \quad (\mathcal{B}^* - \lambda\mathcal{A}^*)\Psi = 0, \tag{22}$$

where $\mathcal{A} = \mathrm{diag}\{A^*, A\}$, $\mathcal{B} = \mathrm{secondarydiag}\{B^*, B\}$, which is non-selfadjoint. Therefore Schmidt eigenelements can have generalized Jordan chains $\Phi_k^{(j)} = \left(\varphi_k^{(j)}, \psi_k^{(j)}\right)$ and $\Psi_k^{(j)} = \left(\hat{\varphi}_k^{(j)}, \hat{\psi}_k^{(j)}\right)$, $j = \overline{1, p_k}$, $k = \overline{1, n}$. They can be chosen so that the following orthogonality relations be satisfied

$$\left(\Phi_i^{(j)}, \Gamma_k^{(l)}\right) = \left(\varphi_i^{(j)}, A\hat{\varphi}_k^{(p_k+1-l)}\right) + \left(A\psi_i^{(j)}, \hat{\psi}_k^{(p_k+1-l)}\right) = \delta_{ik}\delta_{jl},$$

$$\Gamma_k^{(l)} = \mathcal{A}^*\Psi_k^{(p_k+1-l)},$$

$$\left(Z_i^{(j)}, \Psi_k^{(l)}\right) = \left(\varphi_i^{(p_i+1-j)}, A\hat{\varphi}_k^{(l)}\right) + \left(A\psi_i^{(p_i+1-j)}, \hat{\psi}_k^{(l)}\right) = \delta_{ik}\delta_{jl},$$

$$Z_i^{(j)} = \mathcal{A}\Phi_i^{(p_i+1-j)}, \; i, k = \overline{1, n}, j(l) = \overline{1, p_i(p_k)}.$$

Let for n-multiple Schmidt eigenvalue λ and generalized Jordan chains (GJCh) some sufficiently good approximations λ_0, $\Phi_{k0}^{(j)}$, $\Psi_{k0}^{(j)}$, $j = \overline{1, p_k}$, $k = \overline{1, n}$, be given. The problem arises to construct the pseudoperturbation operator such that these approximations would be exact magnitudes for the perturbed problem and to give iteration procedure for their refining (as an eigenvalue and GJCh of the given operators) on the base of perturbation theory [28].

2. The second problem connected with the E. Schmidt spectrum arises in abstract Dirac type systems [29] at dynamical bifurcation

$$\mathcal{A}\frac{dX}{dt} \equiv \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & -B^* \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - R(X, \varepsilon) \equiv \mathcal{B}X - R(X, \varepsilon),$$

from pure imaginary n-multiple $\mathcal{A}$-eigenvalue $\pm i\alpha$ of the operator $\mathcal{B}$. Then according to [30]

$$\mathbf{B}(\alpha)\Phi_k^{(1)} \equiv \begin{pmatrix} \mathcal{B} & \alpha\mathcal{A} \\ -\alpha\mathcal{A} & \mathcal{B} \end{pmatrix} \begin{pmatrix} U_{1k}^{(1)} \\ U_{2k}^{(1)} \end{pmatrix} = 0,$$

$$\mathbf{B}^*(\alpha)\Psi_k^{(1)} \equiv \begin{pmatrix} \mathcal{B}^* & -\alpha\mathcal{A}^* \\ \alpha\mathcal{A}^* & \mathcal{B}^* \end{pmatrix} \begin{pmatrix} V_{1k}^{(1)} \\ V_{2k}^{(1)} \end{pmatrix} = 0,$$

$U_{sk}^{(1)} = \left(u_{s1k}^{(1)}, u_{s2k}^{(1)}\right)^T$, $V_{sk}^{(1)} = \left(v_{s1k}^{(1)}, v_{s2k}^{(1)}\right)^T \in H\dot{+}H$, $s = 1,2$, $k = \overline{1,n}$, which means that the numbers $\pm\alpha$ are n-multiple Schmidt A-eigenvalues of the operator B, resp. Schmidt A^*-eigenvalues of the operator B^*

$$Bu_{11k}^{(1)} = -\alpha A u_{22k}^{(1)},\ Bu_{21k}^{(1)} = \alpha A u_{12k}^{(1)};\ Bv_{22k}^{(1)} = \alpha A^* v_{11k}^{(1)},\ Bv_{12k}^{(1)} = -\alpha A^* v_{21k}^{(1)};$$

$$B^*u_{22k}^{(1)} = -\alpha A^* u_{11k}^{(1)},\ B^*u_{12k}^{(1)} = \alpha A^* u_{21k}^{(1)};\ B^*v_{11k}^{(1)} = \alpha A v_{22k}^{(1)},\ B^*v_{21k}^{(1)} = -\alpha A v_{12k}^{(1)}.$$

Again the biorthogonality properties for the conveniently chosen $\mathbf{A}$-Jordan chains of the operator $\mathbf{B}(\alpha) = \mathbf{B} - \alpha\mathbf{A}$, $\mathbf{B} = \mathrm{diag}(B, -B^*, B, -B^*)$, $\mathbf{A} =$ secondary $\mathrm{diag}(-A, -A^*, A, A^*)$, take place

$$\left(\mathbf{A}\Phi_{\mu k}^{(l_k+1-j)}, \Psi_{\nu s}^{(l)}\right) = \delta_{\mu\nu}\delta_{ks}\delta_{jl},\ \left(\Phi_{\mu k}^{(j)}, \mathbf{A}^*\Psi_{\nu s}^{(p_s+1-l)}\right) = \delta_{\mu\nu}\delta_{ks}\delta_{jl},\ \mu,\nu = 1,2,$$

$$j = \overline{1,p_k},\ l = \overline{1,p_s},\ k,s = \overline{1,n}, \Phi_{1k}^{(s)} = \left(U_{1k}^{(s)}, U_{2k}^{(s)}\right)^T,$$

$$\Phi_{2k}^{(s)} = \left(-U_{2k}^{(s)}, U_{1k}^{(s)}\right)^T, \Psi_{1k}^{(s)} = \left(V_{2k}^{(s)}, -V_{1k}^{(s)}\right)^T,\ \Psi_{2k}^{(s)} = \left(V_{1k}^{(s)}, V_{2k}^{(s)}\right)^T.$$

Thus the problem of refining the approximately given E. Schmidt n-multiple eigenvalue α and relevant eigenelements with Jordan chains $u_{ijk0}^{(s)}, v_{ijk0}^{(s)}$ arises as a generalization of the previous one.

The problem 2. can be presented in the same form (22) as the problem 1. by replacing $\mathcal{A}$ and $\mathcal{B}$ by $\mathbf{A}$ and $\mathbf{B}$, λ by α. In the following arguments the forms (22) will be used both for the problem 1. and 2.

For the approximations to n-multiple Schmidt eigenvalue of the problem 1. (analogously to the problem 2.) with relevant GJChs we introduce the following notations: λ_0, $\Phi_{i0}^{(j)}$, $\Psi_{k0}^{(j)}$, $\Gamma_{k0}^{(j)}$, $Z_{i0}^{(j)}$. Passing, if necessary, to linear combinations (lemma 1) they can be considered satisfying the biorthogonality relations. Computing the discrepancies

$$\sigma_{i0}^{(j)} = (\mathcal{B} - \lambda_0\mathcal{A})\Phi_{i0}^{(j)} + (1 - \delta_{1j})\mathcal{A}\Phi_{i0}^{(j-1)},$$

$$\tau_{i0}^{(j)} = (\mathcal{B}^* - \lambda_0 \mathcal{A}^*)\Psi_{i0}^{(j)} + (1 - \delta_{1j})\mathcal{A}^*\Psi_{i0}^{(j-1)},$$

introduce two following forms of pseudoperturbation operators D_0 (see (4), (5)), such that $D_0\Phi_{i0}^{(s)} = \sigma_{i0}^{(s)}$, $D_0^*\Psi_{i0}^{(s)} = \tau_{i0}^{(s)}$, so the approximations to eigenvalue and GJChs are exact for the perturbed operator $\mathcal{B} - \lambda_0\mathcal{A} - D_0$.

Now one of iteration procedure of 3. can be applied.

Remark 2. To generalized E. Schmidt eigenvalue problem with s operators

$$\begin{array}{llll} B_1\varphi_1^{(1)} = \lambda A\psi_s^{(1)}, & B_2\varphi_2^{(1)} = \lambda A\psi_{s-1}^{(1)}, & \ldots, & B_s\varphi_s^{(1)} = \lambda A\psi_1^{(1)}, \\ B_1^*\psi_s^{(1)} = \lambda A^*\varphi_1^{(1)}, & B_2^*\psi_{s-1}^{(1)} = \lambda A^*\varphi_2^{(1)}, & \ldots, & B_s^*\psi_1^{(1)} = \lambda A^*\varphi_s^{(1)}, \end{array}$$

and problems with polynomial or analytic operator-function of spectral parameter

$$B\varphi = A(\lambda)\psi \equiv \sum_{k=1}^{s} \lambda^k A_k\psi, \quad B^*\psi = A^*(\lambda)\varphi \equiv \sum_{k=1}^{s} \lambda^k A_k^*\varphi,$$

as generally in generalized eigenvalue problems with polynomial (analytic) dependence on spectral parameter their linearization by means of matrix operators [6, 7] is applied [28].

3. Consider here the example of one dimensional dynamical problem, arising at the determination of spectral parameter critical value at Poicaré-Andronov-Hopf bifurcation [25] for the equation

$$\mathcal{A}\frac{dX}{dt} = \mathcal{B}X + R(X,\varepsilon),\ R(0,\varepsilon) = 0, \quad \mathcal{A} = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix},\ \mathcal{B} = \begin{pmatrix} B & 0 \\ 0 & -B^* \end{pmatrix},$$

$$A = A^* = I,\ B = B^* = \frac{d^2x}{dt^2} + I$$

which can be written in the form of the periodical solutions determination for the boundary value problem

$$\begin{array}{l} \dfrac{\partial v}{\partial t} = \dfrac{d^2u}{dx^2} + u + R_1(u,v,\varepsilon), \qquad R_1(0,0,\varepsilon) = 0, \quad u(0,t) = 0, \quad u(x_0,t) = u(1,t), \\ \dfrac{\partial u}{\partial t} = -\dfrac{d^2v}{dx^2} - v + R_2(u,v,\varepsilon), \quad R_2(0,0,\varepsilon) = 0, \quad v(1,t) = 0, \quad v(x_0,t) = v(0,t). \end{array}$$

The linearized problem has the pure imaginary critical eigenvalues
$u, v \in C^2([0,x_0) \cup (x_0,1]) \cap C^1[0,1]$, $0 < x_0 < 1$,

$$u'' + u = i\alpha v, \quad v'' + v = -i\alpha u, \tag{23}$$

$$u(0) = 0, \quad u(x_0) = u(1), \quad v(1) = 0, \quad v(x_0) = v(0),$$

where it is sufficient to find positive values α and relevant eigenfunctions $u(x)$, $v(x)$. It is established that nontrivial solutions exist only for $\alpha > 1$. The adjoint problem in the space $C^2([0, x_0) \cup (x_0, 1]) \cap C[0, 1]$, has the form

$$\begin{aligned}
&\tilde{u}'' + \tilde{u} = -i\alpha\tilde{v}, \quad \tilde{v}'' + \tilde{v} = i\alpha\tilde{u}, \quad 0 < x_0 < 1,\\
&\tilde{u}(0) = 0,\ \tilde{u}(1) = 0,\ \tilde{u}(x_0 + 0) = \tilde{u}(x_0 - 0),\\
&\tilde{u}'(x_0 + 0) - \tilde{u}'(x_0 - 0) = \tilde{u}'(1),\\
&\tilde{v}(0) = 0, \tilde{v}(1) = 0, \tilde{v}(x_0 + 0) = \tilde{v}(x_0 - 0),\\
&\tilde{v}'(x_0 + 0) - \tilde{v}'(x_0 - 0) = -\tilde{v}'(0).
\end{aligned} \tag{24}$$

The equation

$$\begin{aligned}
\Delta \equiv [\sin\nu - \sin\nu(1 - x_0)]\,(\sin h\mu - \sin h\mu x_0) +\\
+ (\sin\nu - \sin\nu x_0)\,[\sinh\mu - \sinh\mu(1 - x_0)] = 0,
\end{aligned}$$

$\mu = \sqrt{\alpha - 1}$, $\nu = \sqrt{\alpha + 1}$, determines, for every $0 < x_0 < 1$, the eigenvalue $\alpha = \alpha(x_0)$ of the problem (23). The system $\Delta = 0$, $\Delta'_\alpha = 0$ is inconsistent and Jordan elements are absent.

The PPI-method is illustrated here for the case $x_0 = 0,5$. Here $\Delta = 32 \sinh\frac{\mu}{4}\cosh\frac{3\mu}{4}\sin\frac{\nu}{4}\cos\frac{\nu}{4}(1 - 4\sin^2\frac{\nu}{4}) \Rightarrow \alpha = 16\pi^2 n^2 - 1,\ n = 1, 2, \ldots$; $\alpha = 4\pi^2(1 + 2m)^2 - 1,\ m = 0, 1, 2, \ldots$; $\alpha = 16(\pi s \pm \frac{\pi}{6})^2 - 1,\ s = 0, 1, 2, \ldots$

The smallest eigenvalue is $\alpha = \dfrac{4\pi^2}{9} - 1$ at $s = 0$, for which the explicit formulae for eigenfunctions of the problems are obtained. For this eigenvalue the computational experiment is made with approximation $\widetilde{\alpha} = 3.4375$, convergent on the fifth step up to 10^{-16}. Schmidt's eigenfunctions are determined then after refining $\widetilde{\alpha}$ on the relevant formulae.

4. In [24] in the functional class $C^2([0, x_0) \cup (x_0, 1]) \cap C^1[0; 1]$, $0 < x_0 < 1$ the E. Schmidt boundary eigenvalue problem

$$u'' + \lambda v = 0,\ v'' + \lambda u = 0, \quad u(0) = 0,\ u(x_0) = u(1), \quad v(1) = 0,\ v(x_0) = v(0)$$

is considered. The adjoint problem is stated. The determinant of the boundary conditions is calculated

$$\Delta = \big(\sinh\mu - \sinh\mu x_0\big)\big[\sin\mu - \sin\mu(1-x_0)\big] + \\ + \big(\sin\mu - \sin\mu x_0\big)\big[\sinh\mu - \sinh\mu(1-x_0)\big],$$

$$\lambda = \mu^2.$$

Analysis of the system $\Delta = 0$, $\Delta'_\mu = 0$ shows that Jordan elements are absent. The eigenfunctions for direct and adjoint problems are determined. The computational experiment is made again for symmetric particular case $x_0 = 0,5$, where $\Delta = -8\sin\dfrac{\mu}{2}\big[\sinh\mu - \sinh\dfrac{\mu}{2}\big]\big[2\cos\dfrac{\mu}{2} - 1\big] \Rightarrow \mu_1 = 2\pi n$ or $\mu_2 = \pm\dfrac{2\pi}{3} + 4\pi s$; $n, s = 0, 1, 2, \ldots$; $\mu > 0$. For μ_1 the eigenfunctions are absent, for μ_2 $\sin\mu = \sin\mu x_0 = \pm\dfrac{\sqrt{3}}{2} \neq 0$ and for the direct problem

$$u(x) = \pm 2\sin(\mu x \pm \frac{\pi}{6}) + (\sinh\frac{\mu}{2})^{-1}\sinh(\mu x - \frac{\mu}{2}),$$
$$v(x) = \pm 2\sin(\mu x \pm \frac{\pi}{6}) - (\sinh\frac{\mu}{2})^{-1}\sinh(\mu x - \frac{\mu}{2}),$$

for the adjoint one

$$u(x) = v(x) = \begin{cases} \sin\mu x, & 0 \le x \le \dfrac{1}{2}, \\ \sin\mu(1-x), & \dfrac{1}{2} \le x \le 1. \end{cases}$$

The computational experiment is fulfilled for the smallest eigenvalue μ_0^2, $\mu_0 = \dfrac{2\pi}{3}$, with approximation $\widetilde{\mu}_0 = 2,0938$ to within of the order 10^{-21}.

One case $x_0 \neq 0,5$ is considered, where the approximation $\mu_0 = 2.0938$ was determined graphically. This is the *unique* example substantive of PPI-method application, when we can't indicate the exact solution.

5.° Model problems of electromagnetic oscillations in lossless resonators.

Let be given the closed domain $V \subset R^3$ with piecewise smooth boundary $S = \partial V$. The eigenvalue oscillations of lossless resonator are called solutions of the boundary eigenvalue problems for the homogeneous Maxwell system

$$\begin{array}{ll} \mathrm{rotE} = i\omega\mu\mathrm{H}, & \mathrm{rotH} = -i\omega\varepsilon\mathrm{E}; \\ \mathrm{divE} = 0, & \mathrm{divH} = 0; \end{array} \qquad \big[\mathrm{n},\mathrm{E}\big]\big|_S = 0, \quad \big(\mathrm{n},\mathrm{H}\big)\big|_S = 0 \qquad (25)$$

$\mathbf{n}$ is the unit outer normal to S, ε and μ are dielectric and magnetic permeabilities of the medium. (25) is the typical E. Schmidt eigenvalue problem, ω

is the Schmidt's eigenvalue parameter. In the article [31] the adjoint to (25) system

$$\begin{array}{ll} \mathrm{rot}\mathcal{E} = i\omega\varepsilon\mathcal{H}, & \mathrm{rot}\mathcal{H} = -i\omega\mu\mathcal{E}; \\ \mathrm{div}\mathcal{E} = 0, & \mathrm{div}\mathcal{H} = 0; \end{array} \qquad \big[\mathrm{n}, \mathcal{H}\big]\big|_S = 0, \quad \big(\mathrm{n}, \mathcal{E}\big)\big|_S = 0$$

is constructed and on the base of I.S. Arzânykh basic formulae of the field theory for problems related to the Helmholtz operator [32, 33] and potential theory the system of integral equations equivalent to eigenvalue problem (25) is derived. For this the Fredholm property of the system (25) is proved.

In the monographs [34]-[38] and articles [39, 40] exact solutions of (25) for rectangular, cylindrical and spherical resonators are found obtained by using variables separation method and symmetry properties of the domain V. The definite symmetries of the domain generates the development of PPI-methods in conditions of group symmetry.

D. Solution of Algebraic Equations.

The method of replacing an algebraic equation by the characteristic equation of the corresponding Frobenius matrix, which, as known, is not rational, makes possible to apply the pseudoperturbation construction to refine roots of polynomials [41]. Although here we can not give the explicit formula for the E. Schmidt operator, this approach leads to interesting relations and summation formulae.

6. PSEUDOPERTURBITERATION METHOD UNDER GROUP SYMMETRY CONDITIONS

Here we give only the general scheme of pseudoperturbiteration methods. For the case of discrete group symmetry the eigenelements and generalized Jordan chains for direct and adjoint eigenvalue problems (1) and considered generalizations can be restored by the group action on relevant basis magnitudes in generating trajectories spaces. Analogously in the case of continuous symmetry these magnitudes can be restored by the action of infinitesimal operators of Lie algebra on the basis of magnitudes in generating trajectories subspace [42]. The simplest example is given by the rotation operator $s\frac{\partial}{\partial t} - t\frac{\partial}{\partial s}$ on the eigenspace $N(B) = \{s^2, st, t^2\}$, where the generating sub-

spaces is formed by the basis elements s^2 and t^2. In the monograph [42] another examples in capillary-gravity surface waves or phase transitions in statistical crystal theory can be found. Such examples for critical imaginary eigenvalues at Poincarè-Andronov-Hopf bifurcation are contained in [43].

Therefore in order to apply PPI-methods it is sufficient to know approximations to basic elements of generating trajectories subspaces. Reproducing the other approximations by the group action in the case of discrete symmetry and the relevant infinitesimal operators in the case of continuous symmetry, we can obtain the complete approximate basis of eigenelements and generalized Jordan chains corresponding to multiple eigenvalue. Further we can construct the PP-operator having the relevant group symmetry with subsequent application of one iterational process of n. 3.

However we can not indicate substantial examples for illustration of PPI-method. The considered in [34, 35, 37, 40, 42, 43] examples of eigenvalue problems are solved exactly. We hope to find such examples on the base of the article [39].

Remark 3. Connections of PPI-methods with parameter continuation methods are remained for now on the level of the article [3].

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