A NOTE ON SUBCLASSES OF UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

Adriana Cătaş

University of Oradea
acatas@uoradea.ro

Abstract The object of this paper is to derive some inclusion relations regarding a new class denoted by $S^n_{m}(\lambda, \alpha)$ using the generalized Sălăgean operator.

Keywords: univalent, Sălăgean operator, differential subordination.

2000 MSC: 30C45.

1. INTRODUCTION

Let $A_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad n \in \mathbb{N}^* = \{1, 2, \ldots \},$$

(1)

which are analytic and univalent in the unit disc of the complex plane

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

(2)

with $A_1 = A$.

F.M. Al-Oboudi in [1] defined, for a function in $A_n$, the following differential operator

$$D^0 f(z) = f(z)$$

(3)

$$D^1 f(z) = D_\lambda f(z) = (1 - \lambda) f(z) + \lambda z f'(z)$$

(4)

$$D^m f(z) = D_\lambda (D^m_{\lambda-1} f(z)), \quad \lambda > 0.$$  

(5)

When $\lambda = 1$, we get the Sălăgean operator [6].
If \( f \) and \( g \) are analytic functions in \( U \), then we say that \( f \) is subordinate to \( g \), written \( f \prec g \), or \( f(z) \prec g(z) \), if there is a function \( w \) analytic in \( U \) with \( w(0) = 0 \), \( |w(z)| < 1 \), for all \( z \in U \) such that \( f(z) = g[w(z)] \) for \( z \in U \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

We shall use the following lemmas to prove our results.

**Lemma 1.1.** [3] Let \( h \) be a convex function with \( h(0) = a \) and let \( \gamma \in \mathbb{C}^* \) be a complex number with \( \text{Re} \ \gamma > 0 \). If \( p \in \mathcal{H}(a, n) \) and

\[
p(z) + \frac{1}{\gamma}zp'(z) \prec h(z),
\]

then

\[
p(z) \prec q(z) \prec h(z),
\]

where

\[
q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1} \, dt.
\]

The function \( q \) is convex and is the best \((a, n)\)-dominant.

**Lemma 1.2.** [4] Let \( q \) be a convex function in \( U \) and let

\[
h(z) = q(z) + n\alpha zq'(z),
\]

where \( \alpha > 0 \) and \( n \) is a positive integer. If

\[
p(z) = q(0) + p_n z^n + \cdots \in \mathcal{H}(q(0), n)
\]

and

\[
p(z) + \alpha zq'(z) \prec h(z),
\]

then

\[
p(z) \prec q(z)
\]

and this result is sharp.
2. MAIN RESULTS

Definition 2.1. Let \( f \in A_n, \ n \in \mathbb{N}^* \). We say that the function \( f \) is in the class \( S_m^n(\lambda, \alpha) \), \( \lambda > 0, \alpha \in [0, 1) \), \( m \in \mathbb{N} \), if \( f \) satisfies the condition

\[
\text{Re } [D_{\lambda}^m f(z)]' > \alpha, \quad z \in U. \tag{6}
\]

Remark 2.1. The class \( S_1^m(\lambda, \alpha) \equiv S^m(\lambda, \alpha) \) was studied in [2] and the class \( S_1^m(1, \alpha) \) was studied in [5].

Theorem 2.1. If \( \alpha \in [0, 1), \ m \in \mathbb{N} \) and \( n \in \mathbb{N}^* \) then

\[
S_{n}^{m+1}(\lambda, \alpha) \subset S_{n}^{m}(\lambda, \delta), \tag{7}
\]

where

\[
\delta = \delta(\lambda, \alpha, n) = 2\alpha - 1 + 2(1 - \alpha) \frac{1}{n\lambda} \beta \left( \frac{1}{\lambda n} \right), \tag{8}
\]

\[
\beta(x) = \int_0^1 \frac{t^{x-1}}{t+1} dt \tag{9}
\]

is the Beta function.

Proof. Let \( f \in S_{n}^{m+1}(\lambda, \alpha) \). By using the properties of the operator \( D_{\lambda}^m \), we get

\[
D_{\lambda}^{m+1} f(z) = (1 - \lambda) D_{\lambda}^m f(z) + \lambda z(D_{\lambda}^m f(z))' \tag{10}
\]

If we denote

\[
p(z) = (D_{\lambda}^m f(z))', \tag{11}
\]

where \( p(z) = 1 + p_n z^n + \ldots \), \( p(z) \in \mathcal{K}[1, n] \), then after a short computation we get

\[
(D_{\lambda}^{m+1} f(z))' = p(z) + \lambda z p'(z), \quad z \in U. \tag{12}
\]

Since \( f \in S_{n}^{m+1}(\lambda, \alpha) \), from Definition 2.1 one obtains

\[
\text{Re } (D_{\lambda}^{m+1} f(z))' > \alpha, \quad z \in U.
\]
Using (12) we get
\[ \text{Re} \left( p(z) + \lambda z p'(z) \right) > \alpha, \]
which is equivalent to
\[ p(z) + \lambda z p'(z) \prec \frac{1 + (2\alpha - 1)z}{1 + z} \equiv h(z). \tag{13} \]

Making use of Lemma 1.1 we have
\[ p(z) \prec q(z) \prec h(z), \]
where
\[ q(z) = \frac{1}{n\lambda z^{1/\lambda n}} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} \left( \frac{1}{t + 1} \right)^{1/\lambda n} dt. \]

The function \( q \) is convex and is the best \((1,n)\)-dominant.

Since
\[ (D^m_{\lambda} f(z))' \prec 2\alpha - 1 + 2(1-\alpha) \frac{1}{n\lambda} \frac{1}{z^{1/\lambda n}} \int_0^z \left( \frac{1}{t + 1} \right)^{1/\lambda n} dt \]
it follows that
\[ \text{Re} \left( D^m_{\lambda} f(z) \right)' > q(1) = \delta, \tag{14} \]
where
\[ \delta = \delta(\lambda, \alpha, n) = 2\alpha - 1 + 2(1-\alpha) \frac{1}{n\lambda} \beta \left( \frac{1}{\lambda n} \right), \tag{15} \]
\[ \beta \left( \frac{1}{\lambda n} \right) = \int_0^1 \left( \frac{1}{t + 1} \right)^{1/\lambda n} dt. \tag{16} \]

From (14) we deduce that \( f \in S^m_n(\lambda, \alpha, \delta) \) and the proof of the theorem is complete. \( \blacksquare \)

Making use of Lemma 1.2 we now prove the following theorems.

**Theorem 2.2.** Let \( q(z) \) be a convex function, \( q(0) = 1 \) and let \( h \) be a function such that
\[ h(z) = q(z) + n\lambda z q'(z), \quad \lambda > 0. \tag{17} \]
If $f \in A_n$ and satisfies the differential subordination

$$(D_{\lambda}^{m+1} f(z))' < h(z), \quad (18)$$

then

$$(D_{\lambda}^m f(z))' < q(z) \quad (19)$$

and the result is sharp.

**Proof.** From (12) and (18) one obtains

$$p(z) + \lambda z p'(z) < q(z) + n \lambda z q'(z) \equiv h(z).$$

Then, by Lemma 1.2, we get

$$p(z) < q(z),$$

or

$$(D_{\lambda}^m f(z))' < q(z), \quad z \in U$$

and this result is sharp.  

**Theorem 2.3.** Let $q$ be a convex function with $q(0) = 1$ and let $h$ be a function of the form

$$h(z) = q(z) + nz q'(z), \quad \lambda > 0, \ z \in U. \quad (20)$$

If $f \in A_n$ satisfies the differential subordination

$$(D_{\lambda}^m f(z))' < h(z), \quad z \in U, \quad (21)$$

then

$$\frac{D_{\lambda}^m f(z)}{z} < q(z) \quad (22)$$

and this result is sharp.

**Proof.** If we let

$$p(z) = \frac{D_{\lambda}^m f(z)}{z}, \quad z \in U,$$
then we obtain
\[
(D_m^n f(z))' = p(z) + zp'(z), \quad z \in U.
\]

The subordination (21) becomes
\[
p(z) + zp'(z) \prec q(z) + nzq'(z)
\]
and, by Lemma 1.2, we have (22). The result is sharp. ■

References


