ON THE NUMBER OF THE EQUILIBRIA FOR A PENDULUM WITH NEO-HOOKEAN ROD

Marina Pandrea, Nicolae - Doru Stănescu  
Department of Applied Mechanics, University of Pitești

Abstract  In this paper we study the pendulum with a non-linear neo-Hookean type sustain rod. In this situation a linear relation is not suitable to connect the force in the rod and the deformation of the rod. In addition, the sustain point of the pendulum has a vertical displacement. This displacement is a function of the instantaneous rod elongation. On the basis of the obtained equation of motion of such a pendulum, we perform a study concerning the number of the equilibrium positions for the pendulum. In our paper, all the possible cases are considered. We find that, in dependence on the values of the parameter, defining the vertical forced oscillations of an end of the rod, there are zero, one or two equilibrium positions.

1. INTRODUCTION

The system proposed for study is drawn in fig. 1. It consists in by the rod $AB$ of length $l$ and negligible mass and the ball of mass $m$, the ball being situated at the point $B$.

![Mathematical model](image_url)

Fig. 1. Mathematical model.

The length of the rod $AB$ in undeformed state is $l_0$. Its static elongation under the action of the ball weight $mg$ is denoted by $z_{st}$. The joint from the end $A$ of the rod has a vertical motion under the law $Y = Y(t)$, which we assume as known. The elastic force in the rod is assumed to be defined by a potential $U$. 

which will be presented further. The system has two degrees of freedom: the length \( l \) of the rod \( AB \) and the angle \( \theta \) formed by the rod with the vertical descendent direction.

With the following notation
\[
l = l_0 + z_{st} + z; \quad H = \frac{A_0 G l_0}{m l_0}; \quad \omega^2 = \frac{g}{l^2}; \quad \ddot{Y} = \frac{Y}{l_0};
\]
where \( z \) is the rod elongation relative to the static equilibrium position, \( G \) is the shear modulus, and \( A_0 \) is the rod cross-sectional area, and assuming that \( U = -A_0 G l_0 \left( \frac{\lambda^2}{2} + \frac{1}{\lambda} \right) \) and \( \ddot{Y} = BH\lambda \), where \( B \) is a real constant, one obtains the equations of motion
\[
\ddot{\lambda} - \lambda \dot{\theta}^2 + H \left( \lambda - B \cos \theta \lambda - \frac{1}{\lambda} \right) = \omega^2 \cos \theta, \quad \lambda \ddot{\theta} + 2 \lambda \dot{\theta} + \left( \omega^2 + BH \lambda \right) \sin \theta = 0.
\]

2. EQUILIBRIA

Denoting \( \xi_1 = \lambda, \xi_2 = \theta, \xi_3 = \dot{\lambda}, \xi_4 = \dot{\theta} \), the system (2) is transformed in a system of four non-linear first order differential equations
\[
\frac{d\xi_1}{dt} = \xi_3; \quad \frac{d\xi_2}{dt} = \xi_4;
\]
\[
\frac{d\xi_3}{dt} = \xi_1 \xi_4^2 - H \left[ (1 - B \cos \xi_2) \xi_1 - \frac{1}{\xi_1^2} \right] + \omega^2 \cos \xi_2;
\]
\[
\frac{d\xi_4}{dt} = -\frac{2 \xi_3 \xi_4}{\xi_1} - \frac{\omega^2}{\xi_1} + BH \xi_1 \sin \xi_2.
\]

Its equilibria are obtained at the intersection of the nullclines
\[
\xi_3 = 0; \quad \xi_4 = 0;
\]
\[
\frac{\xi_1 \xi_4^2}{\xi_1} - H \left[ (1 - B \cos \xi_2) \xi_1 - \frac{1}{\xi_1^2} \right] + \omega^2 \cos \xi_2 = 0;
\]
\[
-\frac{2 \xi_3 \xi_4}{\xi_1} - \frac{\omega^2}{\xi_1} + BH \xi_1 \sin \xi_2 = 0.
\]

(3)

There are only two possibilities: \( \xi_2 = 0 \) or \( \xi_2 = \pi \).

3. CASE \( \xi_2 = 0 \)

From the third relation (3) we obtain
\[
(1 - B) \xi_1^3 - 1 - \frac{\omega^2}{H} \xi_1^2 = 0.
\]

(4)

If \( B = 1 \), from (4) it follows the equation \( -\frac{\omega^2}{H} \xi_1^2 = 1 \), which has no solution in \( \mathbb{R} \); therefore, for \( B = 1 \) there exists none equilibrium position.
If $B \neq 1$, then dividing the equation (4) by $(1 - B)$, performing the transformation
\[ \xi_1 = \zeta_1 + \frac{\omega_0^2}{3H(1 - B)}, \] (5)
and denoting
\[ a = -\frac{\omega_0^6}{3H^2(1 - B)^2}; \quad b = -\frac{2\omega_0^6}{27H^3(1 - B)^3} - \frac{1}{1 - B}, \] (6)
we obtain
\[ \zeta_1^3 + a\zeta_1 + b = 0. \] (7)

By the Hudde method, the number of the real roots of the equation (7) depends on the discriminant $\Delta = 4a^3 + 27b^2$. Thus, if $\Delta < 0$, then the equation (7) has three distinct real roots; if $\Delta = 0$, then the equation (7) has three real roots but two of them are equal, and if $\Delta > 0$, then the equation (7) has one and only one real root.

In our case, the discriminant reads $\Delta = \frac{4\omega_0^6}{H^3(1 - B)^4} + \frac{27}{(1 - B)^3}$. Denote by $f$ the function $f = \zeta_1^3 + a\zeta_1 + b$ and by $f'$ the derivative $f' = 3\zeta_1^2 + a$.

**Case $\Delta = 0$.** In this case $\Delta$ is a sum of two strictly positive terms. We have no equilibrium position.

**Case $\Delta < 0$.** For the same reasons as in the previous paragraph we obtain no equilibrium points.

**Case $\Delta > 0$.** In this case $\frac{4\omega_0^6}{H^3(1 - B)^4} + 27 > 0$. Equation (7) has one and only one real root. If we want to have an equilibrium position, we must impose the condition that the equation (4) has a positive real solution. Recalling the transformation (5) one obtains the relation $\zeta_1 > -\frac{\omega_0^2}{3H(1 - B)}$.

Remembering that the derivative $f'$ has two real roots $-\sqrt{-a/3}$, and $\sqrt{-a/3}$, because $a < 0$ (see the first relation (6)), the first root corresponding to a maxim and the second root to a minin of the function $f$ we must have the relation
\[ f\left(-\frac{\omega_0^2}{3H(1 - B)}\right) < 0, \]
implying $1 - B > 0$.

**Case $\zeta_2 = \pi$** From the third relation (3) it follows
\[ (1 + B)\xi_1^3 + \frac{\omega_0^2}{H}\xi_1^2 - 1 = 0. \] (8)

If $B = -1$ from (8) we have
\[ \frac{\omega_0^2}{H}\xi_1^2 = 1, \]
with the solution

\[ \xi_1 = \frac{\sqrt{H}}{\omega_0} \]

Thus we have an equilibrium position.

If \( B \neq -1 \), we divide the equation (8) by \((1 + B)\) to get

\[ \xi_1^3 + \frac{\omega_0^2}{H (1 + B) \xi_1^2} - \frac{1}{1 + B} = 0. \tag{9} \]

Performing the transformation

\[ \xi_1 = \eta_1 - \frac{\omega_0}{3H(1+B)^{1/2}}, \tag{10} \]

from (9) we obtain the equation

\[ \eta_1^3 - \frac{\omega_0^4}{3H^2 (1 + B)^2} \eta_1 + \frac{2\omega_0^6}{27H^3 (1 + B)^3} - \frac{1}{1 + B} = 0. \tag{11} \]

Denote

\[ a = -\frac{\omega_0^4}{3H^2 (1+B)^2}; \quad b = \frac{2\omega_0^6}{27H^3 (1+B)^3} - \frac{1}{1+B}, \tag{12} \]

such that the equation (11) take the form (7).

The comments relative to the number of the real roots of the equation (7) remain valid, with the remark that now the discriminant reads

\[ \Delta = -\frac{4\omega_0^6}{H^3 (1+B)^2} + \frac{27}{(1+B)^2}. \]

**Case** \( \Delta = 0 \). This case implies \( \frac{4\omega_0^6}{H^3 (1+B)^2} = \frac{27}{(1+B)^2} \), so \( B = -1 \pm \sqrt{\frac{4\omega_0^6}{27H^2}} \).

If \( B = -1 + \sqrt{\frac{4\omega_0^6}{27H^2}} \), then, from (12), we have \( a = -\frac{9H}{4\omega_0^2}; b = -\frac{1}{2\sqrt{\frac{4\omega_0^6}{27H^2}}} \). The solutions of the equation \( f'(\eta_1) = 0 \) are

\[ \eta_1^{(1)} = -\sqrt{-\frac{a}{3}} = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \eta_1^{(1)} = \sqrt{-\frac{a}{3}} = \sqrt{\frac{3H}{4\omega_0^2}}. \]

On the other hand \( f \left( \eta_1^{(1)} \right) = 0 \); therefore the double root of the equation \( f(\eta_1) = 0 \) is \( \eta_1^{(1)} \). Equation \( f(\eta_1) = 0 \) has the roots

\[ \eta_1^* = \eta_1^{(1)} = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \eta_1^{**} = -2\eta_1^{(1)} = 2\sqrt{\frac{3H}{4\omega_0^2}}. \]
the first of them being double.

With the transformation (10), the solutions of the equation (9) read
\[ \xi_1^* = -2\sqrt{\frac{3H}{4\omega_0^2}} < 0; \quad \xi_1^{**} = \sqrt{\frac{3H}{4\omega_0^2}} > 0. \]

Again, the root \( \xi_1^* \) is double.

We have one equilibrium position given by \( \xi_1^{**} \). If \( B = -1 - \sqrt{\frac{4\omega_0^6}{27H^3}} \), then from the relations (12) we have \( a = -\frac{9H}{4\omega_0^2}, b = \frac{1}{2\sqrt{\frac{4\omega_0^6}{27H^3}}} \). The solutions of the equation \( f'(\xi_1) = 0 \) are
\[ \xi_1^{(1)} = -\sqrt{-\frac{a}{3}} = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \xi_1^{(2)} = \sqrt{-\frac{a}{3}} = \sqrt{\frac{3H}{4\omega_0^2}} \]

We have \( f(\xi_1^{(1)}) = 0 \), therefore the double root of the equation \( f(\xi_1) = 0 \) is \( \xi_1^{(2)} \).

Equation \( f(\xi_1) = 0 \) has the roots
\[ \xi_1^* = -2\xi_1^{(2)} = -2\sqrt{\frac{3H}{4\omega_0^2}}; \quad \xi_1^{**} = \xi_1^{(2)} = \sqrt{\frac{3H}{4\omega_0^2}}, \]
the second of them being double.

By the transformation (10), the solutions of the equation (9) read
\[ \xi_1^* = -\sqrt{\frac{3H}{4\omega_0^2}}; \quad \xi_1^{**} = 2\sqrt{\frac{3H}{4\omega_0^2}}, \quad (13) \]
the second of them being double.

We have one equilibrium position given by \( \xi_1^{**} \).

**Case** \( \Delta > 0 \). In this case there exists only one root for the equation \( f(\xi_1) = 0 \). We obtain \(-\frac{4\omega_0^6}{H^3(1+B)} + \frac{27}{(1+B)^2} > 0\); therefore
\[ (1+B)^2 > \frac{4\omega_0^6}{27H^3}. \quad (14) \]
If follows that \( B \in \left( -\infty, -\frac{2\omega_0^3}{27H^2} - 1 \right) \cup \left( \frac{2\omega_0^3}{27H^2} - 1, \infty \right), b = \frac{1}{1+B} \left[ \frac{2\omega_0^6}{27H^3(1+B)^2} - 1 \right]. \)

If \( B < -\frac{2\omega_0^3}{\sqrt{27H^2}} - 1 \), we shall prove that \( b > 0 \). Indeed, in this case \( 1+B < 0 \) and the condition \( b > 0 \) reads \( \frac{2\omega_0^6}{27H^3(1+B)^2} - 1 < 0 \), an obvious relation from (14). Therefore \( b > 0 \).
The roots of the equation \( f'(\zeta_1) = 0 \) are given by \( \zeta^{(1)}_1 = -\sqrt{-a/3} \) and \( \zeta^{(2)}_1 = \sqrt{-a/3} \). In this case \( f \) has one negative real root less than \( \zeta^{(1)}_1 \).

On the other hand \( \sqrt{-a/3} = -\omega_0^3/3H(1+B) \). It follows that the transformation (10) reads \( \xi_1 = \zeta_1 + \sqrt{-a/3} \) and therefore the equation (8) has one negative real root; we have no equilibrium position.

If \( B > 2\omega_0^3/\sqrt{27H^3} - 1 \), then \( b < 0 \). Indeed, in this case \( 1+B > 0 \) and the condition \( b < 0 \) reads
\[
\frac{2\omega_0^3}{27H^3(1+B)^2} - 1 < 0,
\]
an obvious relation from (14). Therefore \( b < 0 \).

It follows that the equation \( f(\zeta_1) = 0 \) has exactly one positive real root greater than \( -\sqrt{-a/3} \).

Since \( \sqrt{-a/3} = \omega_0^3/3H(1+B) \) it follows that the transformation (10) reads \( \xi_1 = \zeta_1 - \sqrt{-a/3} \). Therefore the equation (8) has exactly one positive real root; we have one equilibrium position.

Case \( \Delta < 0 \). The equation \( f(\zeta_1) = 0 \) has now three distinct real roots.

From \( \Delta < 0 \) it follows \( -\frac{4\omega_0^3}{H^3(1+B)^3} + \frac{27}{(1+B)^3} < 0 \), so
\[
B \in \left( -\frac{2\omega_0^3}{27H^3} - 1, \frac{2\omega_0^3}{27H^3} - 1 \right).
\]

The expression of \( b \) from (12) reads
\[
b = \frac{2\omega_0^6 - 27H^3(1+B)^2}{27H^3(1+B)^3}.
\]

Denoting \( \gamma = 1 + B \) the expression (16) becomes \( b = \frac{2\omega_0^6 - 27H^3\gamma^2}{(27H^3\gamma^2)^3} \) and from (15) one obtains \( \gamma \in \left( -\frac{2\omega_0^3}{27H^3}, \frac{2\omega_0^3}{27H^3} \right) \). Remark that \( \sqrt{-\frac{\pi}{3}} = -\omega_0^3/3H|\gamma| \).

The expression of \( b \) becomes zero for \( \gamma_1 = -\sqrt{\frac{2\omega_0^6}{27H^3}} ; \gamma_2 = \sqrt{\frac{2\omega_0^6}{27H^3}} \).

We have the following six possibilities:
- if \( b > 0 \) and \( \gamma < 0 \), then the transformation (20) becomes
\[
\xi_1 = \zeta_1 + \sqrt{-a/3}.
\]

The function \( f \) has one negative real root less than \( -\sqrt{-a/3} \), one positive real root situated between 0 and \( \sqrt{-a/3} \), and one positive real root greater than \( \sqrt{-a/3} \). Recalling now the formula (17) we obtain that the equation (8) has one negative root and two positive real roots; therefore there exist two equilibrium positions;
- if \( b < 0 \) and \( \gamma < 0 \), then the transformation (10) has the same form (17). The equation \( f(\zeta_1) = 0 \) has one negative real root less than \( -\sqrt{-a/3} \),
one negative real root situated between $-\sqrt{-a/3}$ and 0 and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has one negative real root and two positive real roots; therefore there exist two equilibrium positions:
- if $b > 0$ and $\gamma > 0$, then the transformation (10) takes the form
  \[ \xi_1 = \zeta_1 - \sqrt{-a/3}. \] (18)

The equation $f(\zeta_1) = 0$ has one negative real root less than $-\sqrt{-a/3}$, one positive real root situated between 0 and $\sqrt{-a/3}$, and one positive real root greater than $\sqrt{-a/3}$. From (18) it follows that the equation (8) has two negative real roots and one positive real root; therefore there exists one equilibrium position;
- if $b < 0$ and $\gamma > 0$, then the transformation (10) reads again as (18). The equation $f(\zeta_1) = 0$ has two negative real roots and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has two negative real roots and one positive real root; therefore there exists one equilibrium position;
- if $\gamma = -\frac{\omega_0^2 \sqrt{2}}{\sqrt{27} H^3}$, then $b = 0$, $\gamma < 0$ and the transformation (10) reads again as (17). The equation $f(\zeta_1) = 0$ has one negative real root less than $-\sqrt{-a/3}$, one root equal to 0 and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has two positive real roots; therefore there exist two equilibrium positions;
- if $\gamma = -\frac{\omega_0^2 \sqrt{2}}{\sqrt{27} H^3}$, then $b = 0$, $\gamma > 0$ and the transformation (10) reads as (18). The equation $f(\zeta_1) = 0$ has one negative real root, one root equal to 0 and one positive real root greater than $\sqrt{-a/3}$. It follows that the equation (8) has two negative real roots and one positive real root; therefore there exists one equilibrium position.

4. CONCLUSIONS

In our paper we studied the pendulum with one neo-Hookean rod. We determined the equations of motion and we presented the number of equilibria as a function of the real parameter $B$. We obtain that this number can be 0, 1 or 2.

References
