SURVEY ON COMPUTER REALIZATION OF BRANCHING EQUATION CONSTRUCTION ON ALLOWED GROUP SYMMETRY AND SUBGROUP INVARIANT SOLUTIONS

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Abstract In order to study bifurcation problems, the nonlinear equations of stationary and dynamical branching are written in operator form in Banach spaces. The Lyapoumov-Schmidt asymptotical method reduces their solving to the construction and investigation of finite-dimensional nonlinear algebraic systems. Multiple degeneracy of linearized operator frequently stipulated by the group of symmetries of the nonlinear problem entails technical difficulties at the construction of the branching equation (BEq). Herein the computer programs are suggested for the construction of the general form of the Lyapoumov-Schmidt branching equation both for stationary and Andronov-Hopf bifurcation on the allowed group of symmetries with its subsequent investigation. Special attention is paid to planar and spatial crystallographic groups. For a discrete group of symmetries the program for the determination of subgroup structure and dual to it structure of branching systems is presented with applications to nonlinearly perturbed Helmholtz equation.

1. INTRODUCTION

Many applied problems of critical phenomena can be investigated as bifurcational symmetry breaking problems. At multivariate branching the nonlinear problem often has one or several families of solutions depending on free parameters. The existence of such solutions as a rule is related to the presence of the original problem symmetry. In bifurcational symmetry breaking problems the original nonlinear equation is invariant with respect to the Euclidean space $\mathbb{R}^s$ ($s \geq 1$) motion group and its solution invariant with respect to this group is the rest state or uniform motion. At the loss of stability there arise solutions with cellular structure, i.e. periodical solutions with crystallographic group of symmetries (semi-direct product $G = G_1 \rtimes \tilde{G}^1$ of $s$-parametrical continuous shift group $G_1(\alpha_1, \ldots, \alpha_s)$ and the group $\tilde{G}^1$ of the elementary cell of periodicity), which are mutually transformed by the action of the group $\tilde{G}^1$. These problems can be solved by the construction and investigation of the relevant branching equation (BEq).
The general scheme for the construction of sufficiently smooth (analytical) BEQ of a general form on the allowed group of symmetries both for stationary and dynamic situations was developed in a series of B. V. Loginov’s papers [2, 9] on the base of group analysis methods for differential equations [8]. However, at the realization of this scheme to concrete applications serious difficulties have arisen. Usually, because of the high order of degeneracy, undergone by the linear operator at the bifurcation point, the investigator can not account for all possible factors and it is impossible to perform this scheme in manual. Whence the need for a computer realization of the indicated scheme. Such a realization is considered here. Computer programs for BEQ construction and investigation are realized for BEQ with simple cubic lattice symmetry. The applications to statistical crystal theory are known [2, 3, 9].

In various applications the problem of subgroup invariant solutions arises. Computer program is proposed by which all subgroups of discrete symmetries with known composition law can be found. Here it is applied to the concrete problem of the nonlinearly perturbed Helmholtz equation for square lattice symmetry.

The base consists of the group symmetry inheritance theorem for branching equation and the methods of group analysis of differential equations. Partially, the description of the proposed programs of the construction of general BEQ is presented in [5]. In the following the terminology, designation of indicated works and results of program application are used without any additional explanation. For the sake of simplicity of presentation here only dynamic (Andronov-Hopf) bifurcation is considered.

Let $E_1$ and $E_2$ be two Banach spaces. The problem about periodical solutions of differential equation with a small parameter $\varepsilon$ reads

$$A \frac{dx}{dt} = Bx - R(x, \varepsilon), \quad R(0, \varepsilon) \equiv 0, \quad R_x(0, 0) = 0. \tag{1}$$

Here $A$ and $B$ are densely defined closed linear Fredholm operators $A : D_A \subset E_1 \to E_2$, $B : D_B \subset E_1 \to E_2$. Moreover $D_A \supset D_B$ and $A$ is subordinated to $B$ (i.e. $\|Ax\|_{E_2} \leq \|Bx\|_{E_2} + \|x\|_{E_1}$ on $D_B$) or $D_B \supset D_A$ and $B$ is subordinated to $A$ (i.e. $\|Bx\|_{E_2} \leq \|Ax\|_{E_2} + \|x\|_{E_1}$ on $D_A$). $R(x, \varepsilon)$ is nonlinear operator sufficiently smooth in a neighborhood of $(0, 0) \subset E_1 + R^1$.

The theorem about symmetry inheritance by relevant Lyapounov-Schmidt BEQ $0 = f(\xi, \bar{\xi}, \mu, \varepsilon) = \{f_j(\xi, \bar{\xi}, \mu, \varepsilon)\}_j^n : \Xi^n \to \Xi^n$ at Andronov-Hopf bifurcation is expressed by the equality

$$f(Ag \xi, \bar{Ag} \xi, \mu, \varepsilon) = Bg f(\xi, \bar{\xi}, \mu, \varepsilon). \tag{2}$$

Here $Ag$ and $Bg$ are $n$-dimensional representations of the group $G$, which is allowed by the original nonlinear equation, in the zero-subspace $N$ and defect subspace $N^*$ of the operator linearized at the bifurcation point.
The equality (2) means that the manifold $\mathcal{F} : f - f(\xi, \bar{\xi}) = 0$ is invariant to the transformation group $\tilde{\xi} = A_g \xi$, $\tilde{f} = B_g f$. Consequently, for the construction of the general BEq on the allowed group of symmetries the Lie-Ovsyannikov invariants and invariant manifolds techniques [8] may be applied.

If the left-hand side of the branching system is assumed to be continuous on $\xi$-variables, then for the construction of the general BEq it is sufficient to have only a complete system of functional independent invariants. In the analytic case, at the BEq expansion on homogeneous forms not all invariants can be expressed via powers of basic ones. But the use of additional invariants leads to the repetition of BEq summands. Therefore, when using additional invariants, the expansion of the BEq by invariant monomials should be factorized subject to constraints between the used invariants. This factorization with respect to the expression inside the brackets will be designated by the symbol $\{ \cdots \}^{out}$.

Thus, according to the group-analysis scheme of the BEq, for the construction of its general form one must: 

a) find the complete system of functionally independent invariants, allowed by BEq continuous group, 

b) introduce the additional invariants, 

c) establish the constraints between the used invariants and 

d) make the factorization of the BEq expansion by these constraints, i.e. construct the symbol $\{ \cdots \}^{out}$. All tasks of this logic-combinatorial problem are realized by means of computer programs for examples of construction of BEqs, which admit the simple cubic lattice symmetry.

The obtained results are entered into RFBR- and INTAS-06 applications.

2. LYAPOUNOV-SCHMIDT BEQ CONSTRUCTION WITH SIMPLE CUBIC LATTICE SYMMETRY

Here the construction of BEq for symmetry breaking problems of Andronov-Hopf bifurcation is presented when spatial cell of periodicity is an octahedron. Let $\dim N = 12$ for the choice of its basis of the form $\{ \varphi_j = e^{i(l_j \cdot q)+\alpha_0} \}^6_{j=1}$ $q = (x_1, x_2, x_3)$ with inverse lattice vectors $l_{2k-1} = 2\pi e_k$, $l_{2k} = -l_{2k-1}$, $k = 1, 2, 3$. The BEq is invariant with respect to the reflection-rotation octahedron group, which is generated by

$$
C_4^{(1)} \cong (1)(2)(3546), \quad C_4^{(2)} \cong (1625)(3)(4), \\
C_4^{(3)} \cong (1324)(5)(6), \quad J \cong (12)(34)(56).
$$

The BEq continuous symmetry group has the form

$$
\mathcal{A}_{g(\alpha)} = \text{diag} \left\{ e^{i(2\pi \alpha_1 + \alpha_0)}, e^{-i(2\pi \alpha_1 + \alpha_0)}, e^{i(2\pi \alpha_1 + \alpha_0)}, e^{-i(2\pi \alpha_1 + \alpha_0)}, e^{i(2\pi \alpha_2 + \alpha_0)}, e^{-i(2\pi \alpha_2 + \alpha_0)}, e^{i(2\pi \alpha_2 + \alpha_0)}, e^{-i(2\pi \alpha_2 + \alpha_0)}, e^{i(2\pi \alpha_3 + \alpha_0)}, e^{-i(2\pi \alpha_3 + \alpha_0)}, e^{i(2\pi \alpha_3 + \alpha_0)}, e^{-i(2\pi \alpha_3 + \alpha_0)} \right\}.
$$
By differentiating on $\alpha_0, \ldots, \alpha_3$ and reducing on common multiplier we receive the Lie algebra basis that corresponds to rotations in the coordinate planes $(\xi_{2k-1}, \xi_{2k})$, $k = 1, 2, 3$

\[ X_0 = (\xi_1, -\bar{\xi}_1, \xi_2, -\bar{\xi}_2, \xi_3, -\bar{\xi}_3, \xi_4, -\bar{\xi}_4, \xi_5, -\bar{\xi}_5, \xi_6, -\bar{\xi}_6), \]

\[ X_1 = (\xi_1, -\bar{\xi}_1, -\xi_2, \xi_2, 0, 0, 0, 0, 0, 0, 0, 0), \]

\[ X_2 = (0, 0, 0, 0, \xi_3, -\bar{\xi}_3, -\xi_4, \bar{\xi}_4, 0, 0, 0, 0), \]

\[ X_3 = (0, 0, 0, 0, 0, 0, 0, 0, \xi_5, -\bar{\xi}_5, -\xi_6, \bar{\xi}_6). \]

Then the basic invariants are defined by the next system of first order partial differential equations

\[ X_i^\nu \frac{\partial I}{\partial \xi_i} + F_j^\nu \frac{\partial I}{\partial f_j} = 0, \quad \nu = 0, 1, 2, 3. \]  \( (3) \)

Hereinafter in every monomial expression the symbol $\xi$ will be omitted, i.e. for example, the notation $\xi_1\xi_2\bar{\xi}_3\bar{\xi}_4 = 1\ 2\ 3\ 4$ is used. The computer program selects six invariants of second order $\xi_k\bar{\xi}_k$, $k = 1, \ldots, 6$ and next six invariants of fourth order

\[ 1\ 2\ 3\ 4; \quad 1\ 2\ 5\ 6; \quad 1\ 2\ 3\ 4; \]

\[ 1\ 2\ 5\ 6; \quad 3\ 4\ 5\ 6; \quad 3\ 4\ 5\ 6. \]
Three standard constraints between fourth order invariants

\[ 1 2 \bar{3} 4 \times \bar{1} 2 5 6 = 1 \bar{1} \times 2 2 \times 3 \bar{4} 5 \bar{6}, \]
\[ 1 2 3 \bar{4} \times 3 4 5 \bar{6} = 3 \bar{3} \times 4 4 \times 1 2 5 \bar{6}, \]
\[ 1 2 5 6 \times 3 4 5 \bar{6} = 5 5 \times 6 6 \times 1 2 3 4 \]

are separating the system of three fourth order invariants that are used for the BEq general construction

\[ 1 2 3 \bar{4}, \quad \bar{1} 2 5 6, \quad 3 4 5 \bar{6}. \]

This system is subordinated to one nonstandard constraint

\[ 1 2 3 \bar{4} \times \bar{1} 2 5 6 \times 3 4 5 \bar{6} = 1 \bar{1} \times 2 2 \times 3 \bar{3} \times 4 \bar{4} \times 5 \bar{5} \times 6 6. \]

Thus the BEQ corresponding to adopted symmetry takes the form

\[
\begin{align*}
& f_1(\xi, \bar{\xi}, \mu, \varepsilon) \\
& \equiv \sum_{p_{\alpha};q_{\beta}} a_{p_{\alpha}q_{\beta}} (\xi \bar{\xi})^p_1 \cdots (\bar{\xi} \xi)^p_6 [\xi_1 (\xi_1 \bar{\xi}_3 \bar{\xi}_4) q_1 (\bar{\xi}_3 \bar{\xi}_2 \xi_5 \xi_6) q_2 (\xi_3 \xi_4 \bar{\xi}_5 \bar{\xi}_6) q_3] \text{out} = 0.
\end{align*}
\]

The symbol \([\cdots]\text{out}\) occurs in the following way

\[
\begin{align*}
& f_1(\xi, \bar{\xi}, \mu, \varepsilon) \equiv \sum_{p_{\alpha};q_{\beta};k_1;k_2} (\xi_1 \bar{\xi}_1)^p_1 \cdots (\bar{\xi}_6 \xi_6)^p_6 [a_{p_{0}q_1} \xi_1 + a_{p_{0}k_1} \xi_1 (\xi_3 \xi_4 \bar{\xi}_5 \bar{\xi}_6)^{k_1} + \\
& + b_{p_{k_1}} \xi_1 (\bar{\xi}_3 \xi_4 \xi_5 \xi_6, \bar{\xi}_1)^{k_1} + c_{p_{k_1}} \xi_1 (\bar{\xi}_2 \xi_5 \xi_6)^{k_1} + d_{p_{k_1}} \xi_1 (\xi_2 \bar{\xi}_5 \bar{\xi}_6)^{k_1} + \\
& + e_{p_{k_1}} \xi_1 (\xi_2 \bar{\xi}_3 \bar{\xi}_4)^{k_1} + f_{p_{k_1}} \xi_1 (\bar{\xi}_2 \xi_3 \xi_4)^{k_1} + g_{p_{k_1},k_2} \xi_1 (\xi_2 \bar{\xi}_3 \bar{\xi}_4)^{k_1} + h_{p_{k_1},k_2} \xi_1 (\bar{\xi}_2 \xi_3 \xi_4)^{k_1} + \\
& + i_{p_{k_1},k_2} \xi_1 (\xi_2 \bar{\xi}_3 \bar{\xi}_4)^{k_1} + j_{p_{k_1},k_2} \xi_1 (\bar{\xi}_2 \xi_3 \xi_4)^{k_1} + k_{p_{k_1},k_2} \xi_1 (\xi_2 \bar{\xi}_3 \bar{\xi}_4)^{k_1} + l_{p_{k_1},k_2} \xi_1 (\bar{\xi}_2 \xi_3 \xi_4)^{k_1} + \\
& + m_{p_{k_1},k_2} \xi_1 (\xi_2 \bar{\xi}_3 \bar{\xi}_4)^{k_1} + n_{p_{k_1},k_2} \xi_1 (\bar{\xi}_2 \xi_3 \xi_4)^{k_1} + o_{p_{k_1},k_2} \xi_1 (\xi_2 \bar{\xi}_3 \bar{\xi}_4)^{k_1} + p_{p_{k_1},k_2} \xi_1 (\bar{\xi}_2 \xi_3 \xi_4)^{k_1} = 0, \\
& f_{2j}(\xi, \bar{\xi}, \mu, \varepsilon) \equiv J f_{2j-1}(\xi, \bar{\xi}, \mu, \varepsilon) = 0, \quad j = 1, 2, 3 \quad (4)
\end{align*}
\]

where \(p_j\) consecutively takes the values \(C_4^{(3)}\), \(C_4^{(2)}\). The first equation of the written branching system is invariant with respect to the octahedron subgroup generated by substitutions \(e, C_4^{(1)}, C_4^{(1)}\), \(C_4^{(1)}\), \(J \circ C_4^{(2)}\), \(J \circ C_4^{(2)}\), \(J \circ C_4^{(2)}\), \(J \circ u_{13} = J \circ C_4^{(1)} \circ C_4^{(2)}\), \(J \circ u_{58} = J \circ C_4^{(1)} \circ C_4^{(3)}\) that preserve the number 1 that gives the symmetry relation between its coefficients.
3. **BEQ INVESTIGATION**

At the Andronov-Hopf bifurcation investigation (i.e., in dynamical branching) difficulties arise too. They are generated by the fact that the branching system includes, except for \( n \) complex variables, the unknown small additional contribution \( \mu \) to the oscillation frequency that defines the limit cycle onset. All these variables should be calculated from the branching system. In the V. I. Yudovich articles (for example see [7]) the asymptotic approach for the determination of periodical branching solution was suggested. It is based on subspaces invariant techniques relatively to the BEq left-hand side. After the solutions determination in a certain subspace as a consequence of group symmetry we have the solutions as trajectories of this subspace. Hence there arise the need of computer program for the construction of the subspace system. For the description of this program we refer to [5].

In the example of BEq (2.1) with simple cubic lattice symmetry in case of the octahedron cell the computer program gives 1795 invariant subspaces. However their complete system consists of the following 46 subspaces

\[
\Xi_1 = (\xi_1, 0, 0, 0, 0, 0), \quad \Xi_2 = (\xi_1, \xi_1, 0, 0, 0, 0), \quad \Xi_3 = (\xi_0, 0, \xi_1, 0, 0, 0), \quad \Xi_4 = (\xi_1, \xi_2, 0, 0, 0, 0), \quad \Xi_5 = (\xi_1, 0, \xi_2, 0, 0, 0), \quad \Xi_6 = (\xi_1, \xi_0, 0, \xi_1, 0, 0), \quad \Xi_7 = (\xi_1, 0, 0, \xi_2, 0, 0), \\
\Xi_8 = (\xi_0, \xi_1, 0, \xi_1, 0, 0), \quad \Xi_9 = (\xi_0, 0, \xi_1, 0, 0), \quad \Xi_{10} = (\xi_1, \xi_2, 0, 0, 0, 0), \quad \Xi_{11} = (\xi_1, 0, 0, 0, \xi_1, 0), \quad \Xi_{12} = (\xi_1, 0, \xi_2, 0, 0, 0), \quad \Xi_{13} = (\xi_1, 0, 0, 0, 0, \xi_1), \quad \Xi_{14} = (\xi_1, \xi_0, 0, 0, 0, \xi_1), \quad \Xi_{15} = (\xi_1, \xi_1, 0, 0, 0, 0), \\
\Xi_{16} = (\xi_0, \xi_1, 0, \xi_1, 0, 0), \quad \Xi_{17} = (\xi_1, 0, 0, 0, 0, \xi_1), \quad \Xi_{18} = (\xi_1, \xi_2, 0, 0, 0, 0), \quad \Xi_{19} = (\xi_1, 0, 0, \xi_2, 0, 0), \quad \Xi_{20} = (\xi_0, 0, \xi_1, 0, \xi_1, 0), \quad \Xi_{21} = (\xi_0, 0, 0, \xi_1, 0, \xi_1), \quad \Xi_{22} = (\xi_0, 0, 0, 0, \xi_1, \xi_1), \\
\Xi_{23} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{24} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{25} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{26} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{27} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{28} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{29} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{30} = (\xi_0, 0, 0, 0, 0, \xi_1), \\
\Xi_{31} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{32} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{33} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{34} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{35} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{36} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{37} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{38} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{39} = (\xi_0, 0, 0, 0, 0, \xi_1), \\
\Xi_{40} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{41} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{42} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{43} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{44} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{45} = (\xi_0, 0, 0, 0, 0, \xi_1), \quad \Xi_{46} = (\xi_0, 0, 0, 0, 0, \xi_1).
\]

The main part of the BEq (2.1) has the form

\[
i\mu \xi_1 + a_1 \xi_1 \epsilon + b \xi_1 |\xi_1|^2 + (c + d) \xi_1 |\xi_1|^2 + 2d \xi_1 |\xi_2|^2 + c \xi_1 \xi_2 \epsilon + e \xi_1 \xi_2 \epsilon = 0;
\]

As an example of invariant subspaces techniques consider this branching system in the subspace \( \Xi_{21} = (\xi_1, \xi_0, \xi_2, 0, 0) \)

\[
i\mu \xi_1 + a_1 \xi_1 \epsilon + (b + c) \xi_1 |\xi_1|^2 + 2d \xi_1 |\xi_2|^2 + e \xi_1 \xi_2 \epsilon = 0.
\]
Assume that $\xi_1 \neq 0$, $\xi_2 \neq 0$, and let us search for the solutions in the form $\xi_1 = r_1 \varepsilon^{1/2} \exp(i\theta_1)$, $\xi_2 = r_2 \varepsilon^{1/2} \exp(i\theta_2)$, $\mu = \nu \varepsilon$. Dividing the first equation by $\varepsilon^{3/2}r_1 \exp(i\theta_1)$, the second one by $\varepsilon^{3/2}r_2 \exp(i\theta_2)$, we get the system

\[
(b + c)r_1^2 + (2d + e \exp(ia))r_2 + = -i\nu - a,
\]
\[
(2d + e \exp(-ia))r_1^2 + (b + c)r_2 + = -i\nu - a,
\]
\[
\alpha = 2(\theta_2 - \theta_1),
\]
whence

\[
r_1^2 = \frac{\Delta_1}{\Delta} = \frac{\Delta \Delta}{|\Delta|^2}, \quad r_2^2 = \frac{\Delta_2}{\Delta} = \frac{\Delta_2 \Delta}{|\Delta|^2},
\]

where

\[
\Delta = (b + c)^2 - (2d - b - c + e \exp(i\alpha))(2d - b - c + e \exp(-i\alpha)),
\]
\[
\Delta_1 = (i\nu + a)(2d - b - c + e \exp(ia)),
\]
\[
\Delta_2 = (i\nu + a)(2d - b - c + e \exp(-i\alpha)).
\]

Since $r_1^2$ and $r_2^2$ are real, it follows the next system of two equations for the determination of $\nu$ and $\alpha$: $\text{Im} (\Delta_1 \Delta) = \text{Im} (\Delta_2 \Delta) = 0$, $\text{Im} (\Delta_1 \Delta) = \text{Im} \{((b + c)^2 - 4d^2 - c^2 - 4de \cos \alpha) \cdot (i\nu + a)(2d - b - c + e \cos \alpha + i \sin \alpha)\} = 0$, $\text{Im} (\Delta_2 \Delta) = \text{Im} \{((b + c)^2 - 4d^2 - c^2 - 4de \cos \alpha) \cdot (i\nu + a)(2d - b - c + e (\cos \alpha - i \sin \alpha))\} = 0$.

This system reads, equivalently, in the form

\[
puv + (u + sv) \cos \alpha + (v + ru) \sin \alpha + q = 0,
\]
\[
puv + (u + sv) \cos \alpha - (v + ru) \sin \alpha + q = 0,
\]

where

\[
p = \text{Re}(2d - b - c), \quad u = \text{Im}(a \cos \alpha), \quad s = \text{Re} e,
\]
\[
q = \text{Im}(a(2d - b - c)), \quad v = \text{Re} (a e), \quad r = -\text{Im} e.
\]

After the determination of $\nu$ from every equation of the system, it follows the system for the determination of $\alpha$, i.e.

\[
\nu = -\frac{q + u \cos \alpha + v \sin \alpha}{p + s \cos \alpha + r \sin \alpha} = -\frac{q + u \cos \alpha - v \sin \alpha}{p + s \cos \alpha - r \sin \alpha}.
\]

Hereinafter by using the scheme [1] in Maple 6.0 one gets

\[
\alpha = 0, \quad \pi, \quad \pm \text{Arg} \left( \frac{\sqrt{2} q r v u - v^2 p^2 + u^2 r^2 - q^2 r^2 + v^2 s^2 - 2 u r v s}{u r - v s} - i \frac{q r - v p}{u r - v s} \right).
\]
Finally, $|\xi_1| = \frac{\Delta_1}{\Delta}$, $|\xi_2| = \frac{\Delta_2}{\Delta}$.

The construction of the asymptotic periodical branching solution in other subspaces is made as indicated above and it is not given here in view of its awkwardness.

4. SOLUTIONS INVARIANT TO SUBGROUPS OF DISCRETE SYMMETRIES

The general theory for finding the subgroup invariant solutions of ordinary differential equations was suggested in [10]. Then this theory was developed for bifurcation problems and it was realized in concrete problems for the construction of solutions invariant to normal divisors of discrete symmetry [2].

Here this theory is realized in the form of computer program for the construction of the subgroup structure for the discrete symmetry of nonlinear equation with subsequent formation of the relevant branching systems of solutions invariant to arbitrary subgroups. As a simple illustration of this abstract theory we consider here the equations

\begin{align*}
\Delta u + \lambda^2 \sinh u &= 0, \quad (6) \\
\Delta u + \lambda^2 \sin u &= 0 \quad (7)
\end{align*}

and search for periodic solutions with square lattice of periodicity.

Applications of these equations to low temperature plasma theory are given in [6].

At the choice of the base $N(B) = \{\varphi_j\}_{1}^{4} = \{\frac{1}{2\pi} e^{i\lambda_0(x+y)}, \frac{1}{2\pi} e^{-i\lambda_0(x+y)}\}$ the four dimensional BEq inheriting the square symmetry group $D_4$ has the form [9]

\begin{align*}
f_1(\xi, \varepsilon) &= a_0(\varepsilon)\xi_1 + \sum_{|p|\geq 2} a_{p_1 p_2}(\varepsilon)\xi_1(\xi_1\xi_2)^{p_1}(\xi_3\xi_4)^{p_2} = 0, \\
f_2(\xi, \varepsilon) &= r^2 \circ f_1(\xi, \varepsilon) = f_1(r^2 \circ \xi, \varepsilon) = 0, \\
f_3(\xi, \varepsilon) &= r \circ f_1(\xi, \varepsilon) = f_1(r \circ \xi, \varepsilon) = 0, \\
f_4(\xi, \varepsilon) &= r^3 \circ f_1(\xi, \varepsilon) = f_1(r^3 \circ \xi, \varepsilon) = 0; \quad r = (1324), s = (13)(24). \quad (8)
\end{align*}

The leading terms of the branching system (4.0) are the following

\begin{align*}
\xi_1 \varepsilon + A \xi_1^2 \xi_2 + B \xi_1 \xi_3 \xi_4 + \ldots &= 0, \\
\xi_2 \varepsilon + A \xi_2^2 \xi_1 + B \xi_2 \xi_3 \xi_4 + \ldots &= 0, \\
\xi_3 \varepsilon + A \xi_3^2 \xi_1 + B \xi_1 \xi_2 \xi_3 + \ldots &= 0, \\
\xi_4 \varepsilon + A \xi_4^2 \xi_3 + B \xi_1 \xi_2 \xi_4 + \ldots &= 0, \quad (9)
\end{align*}

where $A = \pm \frac{\lambda_0^2}{4}$, $B = \pm \frac{\lambda_0^2}{2}$, $\lambda_0^2 = \pi^2/a^2$, $a$ is the lattice width (the upper sign is related to equation (6) and the lower sign to (7))
The origin is the discrete square-group $D_4 = \{e, r, s, r^2, r^3, sr, sr^2, sr^3\}$, $(r^ks = sr^{4-k}, k = 1, 2, 3)$ generated by the permutations of basic element indices and the structure $L(D_4)$ of all its subgroups. If $H_0 = D_4 \supset H_1 \supset H_2 \supset \ldots \supset H_{\alpha} = \{H_k\}_1^\alpha$ is some chain of subgroups of the length $\alpha$ then there exists the basis $R_{\alpha}$ in $N$, in which the representation $A_g$ for every subgroup $H_i$ is splitting into irreducible ones. The set of all B-Eqs for $H$–invariant solutions forms the dual to inclusion structure $L'$ to $L(D_4)$: B-Eq of solutions which are invariant with respect to the more narrow subgroup contains the B-Eq of solutions which are invariant with respect to a more wide subgroup. For two chains $A = \{H_k\}_1^\alpha$ and $g^{-1}A_g = \{g^{-1}H_kg\}_1^\alpha$ of similar subgroups the connection between the $H_k$–invariant element subspaces and B-Eqs of $H_k$–invariant solutions respectively is realized by the element $g$.

The basis application of this abstract theoretical result application is the computer program for the subgroup structure of discrete symmetry construction. By means of this program, for every subgroups chain from the complete branching system the subsystem determining solutions invariant with respect to every of its representatives is selected.

Consider its algorithm. The program for the subgroups construction of finite discrete symmetry uses the Kelly table. The proposed program begins the subgroups structure construction with the bottom upwards from trivial subgroup $\{e\}$. Furthermore the ability for subgroups verification on normal divisor property is realized. Let describe the algorithm steps.

1. Let some subgroup $H_k = \{e, g_{r_1}, \ldots, g_{r_k}\}$ be found. Add to this subset some element $g$ that does not enter the set $I_k$ (in the first step $I_k = H_k$).
2. Form the set that consists of all kinds of products of elements in $H_k+1 = H_k \cup g$. As the result of the formation of such products one gets some set $H_{k+s} = \{e, g_{r_1}, \ldots, g_{r_k}, g, g_{r_k+1}, \ldots, g_{r_{k+s}}\}$. The order of this set is the initial group order or its divisor.
3. In case of equality the set $H_{k+s}$ coincides with the investigated group. Let us come back to the point 1 and add to the set $I_k$ the element $g$.
4. Otherwise we have find the subgroup. Add to the set $I_k$ elements from the subgroup $H_{k+s}$. For the found subgroup $H_{k+s}$ repeat all actions from the point 1.

Thus the suggested algorithm contains the recursion, leading to the determination of the subgroup structure.
In the structure $L(D_4)$ the suggested program selects the following subgroup chains

$$
A_1 : D_4 = A_{1,0} \supset A_{1,1} = \{e, rs, sr, r^2\} \supset A_{1,2} = \{e, rs\},
$$

$$
s^{-1}A_1 s = sA_1 : D_4 \supset sA_{1,1}s = \{e, rs, sr, r^2\} \supset sA_{1,2}s = \{e, sr\},
$$
conjugate with $A_1$ meaning the reflection $s$

$$
A_2 : D_4 = A_{2,0} \supset A_{2,1} = A_{1,1} = \{e, rs, sr, r^2\} \supset A_{2,2} = \{e, r^2\};
$$

$$
A_3 : D_4 = A_{3,0} \supset A_{3,1} = \{e, s, r^2, r^2s\} \supset A_{3,2} = \{e, r^2s\},
$$

$$
r^{-1}A_3r = r^3A_3r : D_4 \supset r^3A_{3,1}r = \{e, s, r^2, r^2s\} \supset r^3A_{1,2}r = \{e, s\},
$$
conjugate with $A_3$ meaning the counterclockwise rotation $r$

$$
A_4 : D_4 = A_{4,0} \supset A_{4,1} = A_{3,1} = \{e, s, r, r^2s\} \supset A_{4,2} = \{e, r^2\};
$$

$$
A_5 : D_4 = A_{5,0} \supset A_{5,1} = \{e, r, r^2, r^3\} \supset A_{5,2} = A_{2,2} = \{e, r^2\}.
$$

The subgroups $\{e, rs, sr, r^2\}$, $\{e, s, r^2, r^2s\}$, $\{e, r, r^2, r^3\}$, $\{e, r^2\}$ are normal divisors (on the figure they are shown by semiboldface lines).

Let $H$ be a subgroup of $D_4$. The subspace of $N(B)$ consisting of elements which are invariant to $H$ is determined by means of the projection operator [4]

$$
P(H) = \frac{1}{|H|} \sum_{g \in H} T(g),
$$

where $T(g)$ is the representation of $D_4$ by the matrix group generated by

$$
T(s) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
$$

and $T(r) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}$.

For every chain of subgroups the projection operators and hence the subgroup-invariant elements in $N(B)$ are determined. On the base of the formulas for
Let us consider the chain $A_1$. The projection operator $P(A_{1,1})$ transforms the $N(B)$ basis into the subspace of $A_{1,1}$-invariant elements $\text{span}\{\varphi_1^\times = \frac{\varphi_1 + \varphi_2}{\sqrt{2}}, \varphi_2^\times = \frac{\varphi_1 + \varphi_4}{\sqrt{2}}\}$. Adding to the base of this subspace the elements $\varphi_3^\times = \frac{\varphi_1 - \varphi_2}{\sqrt{2}}, \varphi_4^\times = \frac{-\varphi_1 + \varphi_4}{\sqrt{2}}$ the BEq in the new base

\[ a_0\eta_1 + \sum_{|p| \geq 1} \frac{a_{p_1 p_2}}{2^{p_1 + p_2}} \eta_1 (\eta_1^2 + \eta_3^2) (\eta_2^2 + \eta_4^2)^p = 0, \]

\[ a_0\eta_2 + \sum_{|p| \geq 1} \frac{a_{p_1 p_2}}{2^{p_1 + p_2}} \eta_2 (\eta_2^2 + \eta_3^2) (\eta_1^2 + \eta_4^2)^p = 0, \]

\[ a_0\eta_3 + \sum_{|p| \geq 1} \frac{a_{p_1 p_2}}{2^{p_1 + p_2}} \eta_3 (\eta_1^2 + \eta_3^2) (\eta_2^2 + \eta_4^2)^p = 0, \]

\[ a_0\eta_4 + \sum_{|p| \geq 1} \frac{a_{p_1 p_2}}{2^{p_1 + p_2}} \eta_4 (\eta_2^2 + \eta_3^2) (\eta_1^2 + \eta_4^2)^p = 0 \]

is obtained. Then the BEq for $A_{1,1}$-invariant solutions follows from (4.0) by setting $\eta_3 = 0, \eta_4 = 0$ in accordance with the form of the projector $P(A_{1,1})$

\[ a_0\eta_1 + \sum_{|p| \geq 1} \frac{a_{p_1 p_2}}{2^{p_1 + p_2}} \eta_1 (\eta_1^2 + \eta_3^2) = 0, \]

\[ a_0\eta_3 + \sum_{|p| \geq 1} \frac{a_{p_1 p_2}}{2^{p_1 + p_2}} \eta_3 (\eta_1^2 + \eta_3^2) = 0 \]

The leading part of (4.0) is

\[ \eta_1 \varepsilon + \frac{A}{2} \eta_3^2 + \frac{B}{2} \eta_1 \eta_2^2 = 0 \]

\[ \eta_2 \varepsilon + \frac{A}{2} \eta_3^2 + \frac{B}{2} \eta_1 \eta_2^2 = 0 \]

with the solutions

1) $\eta_2 = 0$, $\eta_1 = \pm \sqrt{-\frac{2\varepsilon}{A}}$, $\text{sign} \varepsilon = -\text{sign} A$; $u = \eta_1 \varphi_1^\times = \pm \sqrt{-\frac{\varepsilon}{A}} \cos \lambda_0 (x + y) + O(|\varepsilon|)$,

2) $\eta_1 = 0$, $\eta_2 = \pm \sqrt{-\frac{2\varepsilon}{A}}$, $\text{sign} \varepsilon = -\text{sign} A$; $u = \eta_2 \varphi_2^\times = \pm \sqrt{-\frac{\varepsilon}{A}} \cos \lambda_0 (y - x) + O(|\varepsilon|)$,
3) \( \eta_1^2 = \eta_2^2 = -\frac{2\varepsilon}{A+B}, \) sign \( \varepsilon = -\text{sign} (A+B); \) \( u = \pm \frac{2}{\pi} \sqrt{-\frac{\varepsilon}{A+B}} \cos \lambda_0 x \cos \lambda_0 y + O(|\varepsilon|). \)

For the chain \( sA_1s \) of conjugate subgroups the transformation \( s \) transforms the branching equations for solutions invariant with respect to subgroups of the first chain into the BEqs for solutions which are invariant relatively to subgroups of the second chain.

The subspace of \( N(B) \) elements invariant to the subgroup \( A_{1,2} \) \( \text{span}\{\varphi_1^x = \varphi_1 + \varphi_2, \varphi_2^x = \varphi_3, \varphi_3^x = \varphi_4\} \) is transformed by \( s \) in the subspace \( \text{span}\{\varphi_1^x, \varphi_2^x = \varphi_1^x, \varphi_3^x = \varphi_1 + \varphi_2\} \) of \( N(B) \) elements invariant with respect to the subgroup \( sA_{1,2}s \). The BEq for solutions invariant to \( A_{1,2} \)

\[
\begin{align*}
 a_0\eta_1 &+ \sum_{|p|\geq1} \frac{a_{p1p2}}{2p1} \eta_1^{2p1+1} \eta_2^{p2} \eta_3 \eta_4^{p1} = 0 \\
 a_0\eta_3 &+ \sum_{|p|\geq1} \frac{a_{p1p2}}{2p2} \eta_1^{p2} \eta_2^{p1+1} \eta_3 \eta_4^{p1} = 0 \\
 a_0\eta_4 &+ \sum_{|p|\geq1} \frac{a_{p1p2}}{2p2} \eta_1^{p2} \eta_2^{p1} \eta_3 \eta_4^{p1+1} = 0
\end{align*}
\] (14)

is transformed by \( s \) in the BEq for solutions invariant to \( sA_{1,2}s \)

\[
\begin{align*}
 a_0\eta_1 &+ \sum_{|p|\geq1} \frac{a_{p1p2}}{2p2} \eta_1^{p1+1} \eta_2^{p1} \eta_3 \eta_4^{2p2} = 0, \\
 a_0\eta_2 &+ \sum_{|p|\geq1} \frac{a_{p1p2}}{2p2} \eta_1^{p1} \eta_2^{p1+1} \eta_3 \eta_4^{2p2} = 0, \\
 a_0\eta_3 &+ \sum_{|p|\geq1} \frac{a_{p1p2}}{2p2} \eta_1^{p2} \eta_2 \eta_3 \eta_4^{2p1+1} = 0.
\end{align*}
\] (15)
Moreover the branching systems form the structure dual to the structure of subgroups of $D_4$: BEq of solutions which are invariant to more narrow subgroup contains the BEq of solutions which are invariant to a more wide subgroup. In fact, the BEq of $A_{11}$-invariant solutions ($sA_{11}s$-invariant solutions) setting in (4.0) $\eta_3 = \eta_4 = \eta_3 = \eta_4 = \eta_1 = \eta_2 = \frac{\eta_2}{\sqrt{2}}$ (correspondingly in (4.0) $\eta_1 = \eta_2 = \frac{\eta_2}{\sqrt{2}}$ or

$$
\eta_1 (\varepsilon + \frac{A}{2} \eta_1^2 + B \eta_3 \eta_1) = 0,
\eta_3 (\varepsilon + A \eta_3 \eta_1 + \frac{B}{2} \eta_1^2) = 0,
\eta_4 (\varepsilon + A \eta_3 \eta_1 + \frac{B}{2} \eta_1^2) = 0
$$

Consequently, the $A_{11}$–invariant solutions of the equations (6), (7) are represented by the formula $\eta_1 \varphi_1^{x} + \eta_3 \varphi_3^{x} + \eta_4 \varphi_4^{x}$, where the vector $(\eta_1, \eta_3, \eta_4)$ passes the solution set of the previous system. The $sA_{11}s$–invariant solutions are, $s(\eta_1 \varphi_1^{x} + \eta_3 \varphi_3^{x} + \eta_4 \varphi_4^{x}) = \eta_1^* \varphi_1^{x} + \eta_3^* \varphi_3^{x} + \eta_4^* \varphi_4^{x} = (\eta_1^* \varphi_1^{x} + \eta_3^* \varphi_2^{x} + \eta_4^* \varphi_4^{x})$ respectively. Among the set of the solutions of the system (4.0) should be mentioned

1) $\eta_3 = \eta_4 = 0$, $\eta_1 = \pm \sqrt{-\frac{2\varepsilon}{A}}$, sign $\varepsilon = -\text{sign } A$,

2) $\eta_3 = \eta_4 = \pm \sqrt{-\frac{\varepsilon(A+B-1)}{A(A+B)}}$, $\eta_1 = \pm \sqrt{-\frac{\varepsilon}{A+B}}$, sign $\varepsilon = -\text{sign } (A + B)$, sign $(A + B - 1) = \text{sign } A$,

3) $\eta_1 = \pm \sqrt{-\frac{\varepsilon}{A+B}}$, $\eta_3 \eta_4 = -\frac{\eta_2}{\sqrt{2}}$, sign $\varepsilon = -\text{sign } (A + B)$.

References


