APPROXIMATE INERTIAL MANIFOLDS FOR AN ADVECTION-DIFFUSION PROBLEM

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Abstract  In the framework of the infinite-dimensional dynamical systems theory, a family of approximate inertial manifolds (a.i.m.s) for a problem modelling the Fickian diffusion of a substance into a Newtonian fluid is constructed. Estimates of the distance between these manifolds and the exact solution of the problem are given, proving that, at large times, the solution is kept in some very narrow neighbourhoods of the a.i.m.s.

1. INTRODUCTION

The concept of approximate inertial manifold (a.i.m.) arose in the framework of the theory of inertial manifolds. First defined in [3], inertial manifolds are finite-dimensional (at least) Lipschitz invariant manifolds, that attract exponentially all trajectories of an evolution equation.

For many evolution equations the existence of an inertial manifold is not yet proved, since the proof requires the existence of a certain large between two successive eigenvalues of the linear part (the so-called spectral gap) and this requirement is not fulfilled [7]. Even if an inertial manifold exists, this inertial manifold is difficult to construct. In this situation, approximations of the inertial manifolds were constructed, the a.i.m.s.[3]. An a.i.m. is a finite dimensional (at least) Lipschitz manifold having the property that every trajectory of the problem enters in a narrow neighbourhood of the a.i.m. at a certain time and remains there for ever. This property inspired some new methods of approximating the solutions of an evolution equation namely the nonlinear Galerkin and post-processed Galerkin methods. Thus, appart for the property of the a.i.m. of keeping the attractor of the problem in some narrow neighbourhood, these methods bring a new and very concrete motivation to study the a.i.m.s.

Among several methods of constructing an a.i.m. we choosed the algorithm given in [8] for the Navier-Stokes equation. After presenting in Sections 2- 4 the problem and the results of existence of the solutions and dissipativity of the problem, in Sections 5, 6 we split the phase space in the direct sum of a finite-dimensional subspace and an infinite-dimensional one and give estimates of the norm of the projection of the solution on the infinite dimensional subspace.
We present the algorithm of [8] in Sections 7 and 8 and then use it in order to construct a family of a.i.m.s for our problem, in Sections 9 and 10. We also find the distance between the a.i.m.s and the exact solution.

2. THE PROBLEM, THE FUNCTIONAL FRAMEWORK

Consider the generalized problem governing the (Fick-ian) diffusion of a substance into a Newtonian incompressible fluid. It can be written as the following Cauchy problem for the evolution advection-diffusion equations

\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u = f(t), \quad (2.1)
\]

\[
\text{div} u = 0, \quad (2.2)
\]

\[
\frac{\partial c}{\partial t} - D \Delta c + u \cdot \nabla c = h(t), \quad (2.3)
\]

\[
u = u(0) = u_0, \quad (2.4)
\]

\[
\nu = c(0) = c_0, \quad (2.5)
\]

where \( u = u(t,x) \) is the fluid velocity, \( x \in (0,l) \times (0,l) = \Omega, \ u(.,x) : [0,\infty) \rightarrow \mathbb{R}^2, \ c(.,x) : [0,\infty) \rightarrow \mathbb{R}, \nu \) is the kinematic viscosity, \( D \) is the diffusion coefficient and \( c = c(t,x) \) is the concentration of the substance that is diffused into the fluid; \( f = f(t,x) \) represent the body forces and \( h = h(t,x) \) is a production function for \( c \). These are given functions and they are supposed to depend analytically on time. We assume that the function \( h \) it may describe either a production or a consumption of the diffused substance and, as a consequence, it may have either positive or negative values.

We consider periodic boundary conditions. Hence it is assumed that, for every fixed \( t \), \( u(t,\cdot) \) belongs to the space \( \mathcal{H}_1 \overset{\text{def}}{=} \{ v \in [L_{\text{per}}^2(\Omega)]^2 : \text{div} v = 0 \} \), with the scalar product \((u,v) = \int_\Omega (u_1v_1 + u_2v_2) \, dx \), (where \( u = (u_1,u_2), \ v = (v_1,v_2) \)) and \( \| \cdot \| \) the induced norm. For \( c \) we assume \( c(t,\cdot) \in \mathcal{H}_2 \overset{\text{def}}{=} L_{\text{per}}^2(\Omega) \), with \( L_{\text{per}}^2(\Omega) \) is endowed with the standard scalar product and the induced norm denoted also by \( \| \cdot \| \). From the context it will be understood whether we talk about the norm in \( \mathcal{H}_1 \) or that in \( \mathcal{H}_2 \).

We assume that for every \( t \), \( f(t) \in \mathcal{H}_1 \), and \( h(t) \in \mathcal{H}_2 \). As far as the dependence on \( t \) is concerned the most realistic hypothesis is that of periodicity in time for the functions \( f(\cdot) \) and \( h(\cdot) \). Anyhow, we suppose that these functions are bounded: there is a number \( M_f > 0 \) such that \( |f(t)| \leq M_f \), and there is a number \( M_h > 0 \) such that \( |h(t)| \leq M_h \) for every \( t > 0 \).

We also use the spaces \( \mathcal{V}_1 \overset{\text{def}}{=} \{ u \in [H_{\text{per}}^1(\Omega)]^2 : \text{div} u = 0 \} \), with the scalar product \((u,v) = \sum_{i,j=1}^2 \left( \frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right) \), and \( \mathcal{V}_2 \overset{\text{def}}{=} H_{\text{per}}^1(\Omega) \) with the
scalar product \((\langle c_1, c_2 \rangle) = \sum_{j=1}^{2} \left( \frac{\partial c_1}{\partial x_j}, \frac{\partial c_2}{\partial x_j} \right)\). The induced norms are denoted by \(\| \cdot \|\).

The operator \(A \overset{\text{def}}{=} -\Delta\) is defined on \(D(A) = \left\{ u \in [H^2_{\text{per}}(\Omega)]^2 : \text{div} u = 0 \right\}\) and it is self-adjoint.

For the bilinear form \(B(u, v) = (u \cdot \nabla) v\) the following inequalities [4], [6], [8]

\[
|B(u, v)| \leq c_1 |u|^\frac{3}{2} |\Delta u|^\frac{1}{2} |v|, \quad (\forall) \ u \in D(A), \ v \in V_1, \ (2.6)
\]

\[
|B(u, v)| \leq c_2 |u| |v|^\frac{1}{2} \left[ 1 + \ln \left( \frac{|\Delta u|^2}{|u|^2} \right) \right]^{\frac{1}{2}}, \quad (\forall) \ u \in D(A), \ v \in V(2.7)
\]

\[
|B(u, v)| \leq c_3 |u| |v|^\frac{1}{2} \left[ 1 + \ln \left( \frac{|\Delta v|^2}{|v|^2} \right) \right]^{\frac{1}{2}}, \quad (\forall) \ u \in V_1, \ v \in D(A)(2.8)
\]

hold [4], [6], [8]. We remind the following properties of the three-linear form \(b(u, v, w) = (B(u, v), w)\) (valid for periodic boundary conditions [5]):

\[
b(u, v, w) = -b(u, w, v), b(u, v, v) = 0,
\]

as well as the following inequalities [6], [5]

\[
|b(u, v, w)| \leq k |u|^\frac{1}{2} |v|^\frac{1}{2} |w|, \quad (\forall) \ u, v, w \in V_1, \ (2.9)
\]

\[
|b(u, v, w)| \leq k |u|^\frac{1}{2} |v|^\frac{1}{2} |w|^\frac{1}{2} |\Delta v|^\frac{1}{2} |w|, \quad (\forall) \ u \in V_1, \ v \in D(A), \ w \in \Omega(2.10)
\]

The operator \(A \overset{\text{def}}{=} -\Delta\) is defined on \(D(A) = [H^2_{\text{per}}(\Omega)]^2\) and it is self-adjoint.

We define the bilinear form \(B(u, v) = u \nabla v\). The inequalities (2.6)-(2.8) have the immediate consequences

\[
|B(u, c)| \leq c_1 |u|^\frac{3}{2} |\Delta u|^\frac{1}{2} |c|, \quad (\forall) \ u \in D(A), \ c \in V_2, \ (2.11)
\]

\[
|B(u, c)| \leq c_2 |u| |c| \left[ 1 + \ln \left( \frac{|\Delta u|^2}{|u|^2} \right) \right]^{\frac{1}{2}}, \quad (\forall) \ u \in D(A), \ c \in V(2.12)
\]

\[
|B(u, c)| \leq c_3 |u| |c| \left[ 1 + \ln \left( \frac{|\Delta c|^2}{|c|^2} \right) \right]^{\frac{1}{2}}, \quad (\forall) \ u \in V_1, \ c \in D(A)(2.13)
\]

For the periodic boundary conditions it is usual to consider the averages of the unknown functions on the periodicity cell, namely ([6] for \(u(t, x)\))

\[
u_m(t) = \frac{1}{l^2} \int_{\Omega} u(t, x) \, dx, \quad c_m(t) = \frac{1}{l^2} \int_{\Omega} c(t, x) \, dx.
\]
These are solutions of the equations
\[
\frac{d}{dt}u_m(t) = f_m(t), \quad (2.14)
\]
\[
\frac{d}{dt}c_m(t) = h_m(t), \quad (2.15)
\]
where $f_m, h_m$ are the averages of $f$ respectively $h$ over $\Omega$, therefore we can assume that $u_m$ and $c_m$ are given.

The functions $\tilde{u}(t, x) = u(t, x) - u_m(t)$, $\tilde{c}(t, x) = c(t, x) - c_m(t)$, are solutions of the equations
\[
\frac{\partial \tilde{u}}{\partial t} - \nu \Delta \tilde{u} + (\tilde{u} \cdot \nabla) \tilde{u} + (u_m \cdot \nabla) \tilde{u} = f - f_m, \quad (2.16)
\]
\[
\frac{\partial \tilde{c}}{\partial t} - D \Delta \tilde{c} + \tilde{u} \cdot \nabla \tilde{c} + u_m \cdot \nabla \tilde{c} = h - h_m. \quad (2.17)
\]

Since $u_m(t)$ is known, the study of the above equations is very similar to that of (2.1), (2.3). It is then usual [6] to study equations (2.1), (2.3) with the conditions $u_m(t) = 0$, $c_m(t) = 0$. In the following qualitative investigation we adopt these conditions, but we remind that, when a quantiative study is in view, the equations (2.16), (2.17) should be studied, together with (2.14), (2.15).

Thus, further, in the study of (2.1), (2.3), we use the spaces $\mathcal{H}_1 = \{ u \in \mathcal{H}_1, \ u_m = 0 \}$, $\mathcal{V}_1 = \{ u \in \mathcal{V}_1, \ u_m = 0 \}$, $\mathcal{H}_2 = \{ c \in \mathcal{H}_2, \ c_m = 0 \}$, $\mathcal{V}_2 = \{ c \in \mathcal{V}_2, \ c_m = 0 \}$.

The operator $A$ is positive-definite on $D(A) \cap \mathcal{H}_1$ and has a compact inverse [4]. Similarly, the operator $A$ is positive-definite on $D(A) \cap \mathcal{H}_2$ and has a compact inverse.

3. EXISTENCE OF THE SOLUTIONS

The flow of the incompressible viscous fluid in which the diffusion takes place is not affected by the substance that is diffused, whence the decoupling of the equations of the model (2.1)-(2.5).

Hence, for the problem (2.1), (2.4) we have the classical existence and uniqueness results for the equations Navier-Stokes in $\mathbb{R}^2$ with periodic boundary conditions.

**Theorem 1** [6]. a) If $u_0 \in \mathcal{H}_1$, $f$ is analytical in time and for every $t$, $f(t, \cdot) \in \mathcal{H}_1$, then the problem (2.1), (2.4) has an unique solution $u$, analytical in time and such that for every $t$, $u(t, \cdot) \in \mathcal{V}_1$. b) If, in addition to the hypotheses in a), $u_0 \in \mathcal{V}_1$, $f(t, \cdot) \in \mathcal{V}_1$, then $u(t, \cdot) \in D(A)$. 
By using the Galerkin-Faedo method we can easily prove the following theorem.

**Theorem 2.** a) In the conditions a) of Theorem 1 and if $h$ is analytical in time and for every $t$, $h(t, \cdot) \in \mathcal{H}_2$, $c_0 \in \mathcal{H}_2$, then there is a unique solution $c$ of the problem (2.3)-(2.5), analytical in time and such that $c(t, \cdot) \in \mathcal{H}_2$.

b) In the conditions b) of Theorem 1 and if $c_0 \in \mathcal{V}_2$, $h(t, \cdot) \in \mathcal{V}_2$ then $c(t, \cdot) \in D(A)$, $\forall t > 0$.

Since, by Theorems 1, 2 the existence and uniqueness of the solution $(u, c)$ is global with respect to time, it follows that the problem generates two semi-dynamical systems: $\{S_1(t)\}_{t \geq 0}$, $S_1(t) : \mathcal{H}_1 \to \mathcal{H}_1$, $S_1(t) u_0 = u(t, u_0)$, where $u(t, x, u_0)$ is the solution of (2.1), (2.2), (2.4), and $\{S(t)\}_{t \geq 0}$, $S(t) : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \times \mathcal{H}_2$, $S(t)(u_0, c_0) = (u(t, u_0, c_0), c(t, u_0, c_0))$, where $c(t, x, u_0, c_0)$ is the solution of (2.3), (2.5), with $u$ solution of (2.1), (2.2), (2.4).

4. DISSIPATIVITY OF THE PROBLEM

It was proved [1], [10], that the semi-dynamical system generated by (2.1), (2.2), (2.4) is dissipative, in the sense that in $\mathcal{H}_1$ there is an absorbing ball for it. More precisely, there is a $\rho_0 > 0$ such that for every $R > 0$, there is a $t_0(R) > 0$ with the property that for every $u_0 \in \mathcal{H}_1$ with $|u_0| \leq R$, we have $|S_1(t) u_0| \leq \rho_0$ for $t > t_0(R)$. In addition, there is an absorbing ball in $\mathcal{V}_1$ for $\{S_1(t)\}$, i.e. there is a $\rho_1 > 0$ such that for every $R > 0$, $|u_0| \leq R$ implies $|S_1(t) u_0| \leq \rho_1$ for $t > t_1(R)$.

The component $c$ of the solution $(u, c)$ is also dissipative both in $\mathcal{H}_2$ and $\mathcal{V}_2$. More precisely, the following result holds.

**Theorem 3.** a) There is a $\eta_0 > 0$ with the property that for every $R_c > 0$ there is a $t_{c0}(R_c) > 0$ such that for $|c_0| \leq R_c$

$$|c(t, u_0, c_0)| \leq \eta_0, \ t \geq t_{c0}(R_c).$$

b) There is a $\eta_1 > 0$ with the property that for every $R, R_c > 0$ there is a $t_{c1}(R, R_c) > 0$ such that

$$\|c(t, u_0, c_0)\| \leq \eta_1$$

for $|u_0| \leq R, \ |c_0| \leq R_c, \ t \geq t_{c1}(R, R_c)$.

The proof of this theorem is contained in the proof of Theorem 2 and we do not insist on it.
5. SPLITTING OF SPACES, PROJECTION OF EQUATIONS

The eigenvalues of $A = -\Delta$ in $H_1$ are

$$\lambda_{j_1,j_2} = \frac{4\pi^2}{l^2} (j_1^2 + j_2^2), \quad (j_1, j_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}. \quad (5.18)$$

For $H_1$ there is a total system $\mathcal{f}$ whose elements are eigenfunctions of $A = -\Delta$. They have the form $|l\rangle \frac{\sqrt{2}}{l} \sin \left(\frac{2\pi j_1}{l}\right), \frac{\sqrt{2}}{l} \cos \left(\frac{2\pi j_2}{l}\right)$, where $j = (j_1, j_2), j \neq (j_2, -j_1), |j| = (j_1^2 + j_2^2)^{1/2}, l^2 = \frac{j_1}{\pi l} + \frac{j_2}{\pi l}, (j_1, j_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

The eigenvalues of $A = -\Delta$ are also (5.18) and there is a total system for $H_2$ formed of eigenfunctions of $A = -\Delta$, namely those having the form $\frac{\sqrt{2}}{l} \sin \left(\frac{2\pi j_1}{l}\right), \frac{\sqrt{2}}{l} \cos \left(\frac{2\pi j_2}{l}\right)$.

Due to the symmetries of the functions $\sin$ and $\cos$ to every ordered pair of natural numbers $(j_1, j_2)$, for both operators $A$ and $A$, four eigenfunctions as above (with ”+” or ”-” between the two terms $\frac{j_1}{\pi l}$ and $\frac{j_2}{\pi l}$) correspond.

We fix a value $m \in \mathbb{N}$ and consider the set $\Gamma_m$ of eigenvalues $\lambda_{j_1,j_2}$ having $|j_1|, |j_2| \leq m$.

Denote $\lambda := \lambda_1 = \frac{4\pi^2}{l^2}, \Lambda := \frac{4\pi^2}{l^2} (m + 1)^2, \delta := \frac{\lambda}{\Lambda} = \frac{1}{(m + 1)^2}$.

Let $w_1, ..., w_{m_1}$ (resp. $w_1, ..., w_{m_1}$) be the eigenfunctions of $A$ (resp. $A$) corresponding to $\lambda_{j_1,j_2} \in \Gamma_m$.

Let $L (u, v, ...)$ stand for the linear space spanned by $u, v, ...$.

Denote by $P$ the projection operator on $L (w_1, ..., w_{m_1})$, with $Q$ the projection operator on $L (w_{m_1+1}, ...)$, by $P$ the projection operator on $L (w_1, ..., w_{m_1})$, and by $Q$ the projection operator on $L (w_{m_1+1}, ...)$.

For the solution $u$ of the Navier-Stokes equation, $p = Pu, q = Qu$, and, similarly, for the diffusion component $c, c = Pc, q = Qc$.

By projecting the equations (2.1), (2.3) on the above introduced spaces, we have,

$$\frac{dp}{dt} - \nu \Delta p + PB (p + q) = Pf, \quad (5.19)$$
$$\frac{dq}{dt} - \nu \Delta q + QB (p + q) = Qf, \quad (5.20)$$
$$\frac{dc_p}{dt} - D \Delta c_p + PB (u, c) = Ph, \quad (5.21)$$
$$\frac{dc_q}{dt} - D \Delta c_q + QB (u, c) = Qh. \quad (5.22)$$
6. ESTIMATES FOR THE SMALL COMPONENT OF THE SOLUTION

It is already known that for the Navier Sokes equations the component \( q(t) \) of the solution is small in the sense that it has the order of \( \delta \) for \( t \) large enough. In [8] it is proved that for every initial state \( u_0 \) with \( |u_0| \leq R \) there is a moment \( t_2(R) \) such that, for \( t \geq t_2(R) \),

\[
|q(t)| \leq K_0 L^2 \delta, \quad \|q(t)\| \leq K_1 L^{3/2} \delta^{1/2},
\]

where, for the chosen projection spaces, we have \( L = \ln(1 + 2m^2) \). Here and in the sequel the prime means time derivative.

Let us prove a similar result for \( c_q \). We assume in the following that for a fixed \( R \), we have \( |u_0| \leq R \).

**Theorem 4.** There is a moment \( t_{c2}(R) \) such that for every \( t \geq t_{c2}(R) \), the following inequalities

\[
|c_q(t)| \leq J_0 L^{3/2} \delta, \quad \|c_q(t)\| \leq J_1 L^{3/2} \delta^{1/2},
\]

\[
|c_q'(t)| \leq J_2 L^{3/2} \delta, \quad |\Delta c_q(t)| \leq J_3 L^2 \delta,
\]

hold, where \( J_0, J_1, J_2, J_3 \) are independent of \( m \).

**Proof.** In the sequel we frequently use the embedding inequalities

\[
\|c_q\| \geq \Lambda^{1/2} |c_q|, \quad |\Delta c_q| \geq \Lambda |c_q|,
\]

easy to derive by considering the Fourier series of \( c_q \) and the fact that \( \Lambda \) is the least eigenvalue of \( A \) on \( L^2(w_{m+1}^1, ...) \).

Taking the scalar product of (5.22) by \( c_q \) we have

\[
\left( \frac{dc_q}{dt}, c_q \right) - D(\Delta c_q, c_q) + (u \nabla c, c_q) = (Qh, c_q),
\]

such that, by using (2.12) and (2.13), we obtain

\[
\frac{1}{2} \frac{d}{dt} |c_q|^2 + D\Lambda |c_q|^2 \leq \frac{1}{2} |(Q (p \nabla c_p), c_q)| + |(Q (q \nabla c_p), c_q)| + |(Qh, c_q)|
\]

\[
\leq C L^{1/2} \|p\| \|c_p\| |c_q| + C L^{1/2} \|q\| \|c_p\| \|c_q| + |Qh| \|c_q|,
\]

\[
\leq \frac{C^2 L}{D\Lambda} \rho_1^2 \eta_1^2 + \frac{D\Lambda}{4} |c_q|^2 + \frac{C^2 L}{D\Lambda} \delta \eta_1^2 + \frac{D\Lambda}{4} |c_q|^2 +
\]

\[
\frac{M^2}{D\Lambda} + \frac{D\Lambda}{4} |c_q|^2
\]

and (since \( 1 < L \) for \( m > 1 \)) it follows
\[
\frac{d}{dt} |c_q|^2 + D\Lambda |c_q|^2 \leq \delta \left( \frac{C^2 L}{DX} \rho_1^2 \eta_1^2 + \frac{C^2 L}{DX} \delta \eta_1^2 + \frac{M_0^2}{D\lambda} \right)
\]

= \quad K_0 L \delta,

whence, by applying Gronwall Lemma, (6.23)\textsubscript{1}, where \( J_0 = \sqrt{K_0} \).

Taking the scalar product of (5.22) by \(-\Delta c_q\), we obtain

\[
\frac{1}{2} \frac{d}{dt} |c_q|^2 + D |\Delta c_q|^2 \leq \left| (u\nabla c, \Delta c_q) + |Qh, \Delta c_q| \right|
\]

\[
\leq \left| (p\nabla c_p, \Delta c_q) + |p\nabla c_q, \Delta c_q| + |q\nabla c_q, \Delta c_q| + \left| (Qh, \Delta c_q) \right| \right.
\]

\[
\leq CL \frac{1}{\lambda} (|p| |c_p| + |p| |c_q| + |q| |c_q|) |\Delta c_q| + C |q| \frac{1}{\lambda} |c_q| |\Delta c_q| + \left| Qh, \Delta c_q \right|
\]

The use of the inequality \(|c_q| \leq |c| \leq \eta_1\) further leads to

\[
\frac{d}{dt} |c_q|^2 + \Delta \Lambda |c_q|^2 \leq \frac{CL}{D} \left( \rho_1^2 + \delta \right) \eta_1^2 + \frac{1}{D} M_0^2,
\]

and

\[
\frac{d}{dt} \|c_q(t)\|^2 \leq \|c_q(0)\|^2 e^{-DA t} + \frac{1}{\Lambda} \left( \frac{CL}{D^2} \left( \rho_1^2 + \delta \right) \eta_1^2 + \frac{1}{D^2} M_0^2 \right),
\]

where \( C \) is a generic notation for a constant not depending on \( m \).

Since, usually, \( m \) is large, \( L \) is large too and we obtain (6.23)\textsubscript{2}, for \( t \) great enough.

Then, like for the Navier-Stokes equation [6], we can prove that the extension of the semigroup to the complex time variable is analytic in time in a neighborhood of the time axis, and by using the Cauchy formula, we obtain (6.24)\textsubscript{1}.

Finally, by using (5.22) and the above estimates, the inequality (6.24)\textsubscript{2} follows. \( \square \)

7. THE FAMILY OF A.I.M.S FOR THE NAVIER STOKES EQUATIONS

The family of a.i.m.s for the Navier Stokes problem, is defined in [8] as follows. The first one is the graph \( M_0 \) of the function \( \Phi_0 : P^1 \to \mathbb{P}^1 \), that satisfies the relation

\[-\nu \Delta \Phi_0 (X) + QB(X) = Qf, \text{ where } X \in P^1.\]
We remark that this relation is similar to the equation \(-\nu \Delta q + QB(p) = Qf\), obtained from (5.20) by neglecting \(\frac{dq}{dt}\) in the presence of \(\Delta q\) and \(q\) in the presence of \(p\). Therefore, \(\Phi_0(X)\) has the following form

\[
\Phi_0(X) = (-\nu \Delta)^{-1} (Qf - QB(X)).
\]

The next a.i.m. defined in [8] is \(M_1\), the graph of the function \(\Phi_1: P \cdot H^1 \rightarrow Q \cdot H^1\), given by the solution of the problem

\[
-\nu \Delta \Phi_1(X) + QB(X) + QB(X, \Phi_0(X)) + QB(\Phi_0(X), X) = Qf,
\]

that is

\[
\Phi_1(X) = - (\nu \Delta)^{-1} [Qf - QB(X) - QB(X, \Phi_0(X)) - QB(\Phi_0(X), X)].
\]

For \(j \geq 2\), the a.i.m. \(M_j\) is defined as the graph of \(\Phi_j: P \cdot H^1 \rightarrow Q \cdot H^1\), with \(\Phi_j(X)\) the solution of

\[
-\nu \Delta \Phi_j(X) + QB(X) + QB(X, \Phi_{j-1}(X)) + QB(\Phi_{j-1}(X), X) +
\]

\[
+ QB(\Phi_{j-2}(X)) + D \Phi_{j-2}(X) \Gamma_{u,j-2}(X) = Qf,
\]

where \(D \Phi_{j-2}(X)\) is the differential of \(\Phi_{j-2}(X)\), and

\[
\Gamma_{u,j-2}(X) = \nu \Delta X - PB(X + \Phi_{j-2}(X)) + Pf.
\]

Hence

\[
\Phi_j(X) = - (\nu \Delta)^{-1} [Qf - QB(X) - QB(X, \Phi_{j-1}(X)) - QB(\Phi_{j-1}(X), X) - QB(\Phi_{j-2}(X)) - D \Phi_{j-2}(X) \Gamma_{u,j-2}(X)]
\]

where \(D \Phi_{j-2}(X)\) is the differential of \(\Phi_{j-2}(X)\), and

\[
\Gamma_{u,j-2}(X) = \nu \Delta X - PB(X + \Phi_{j-2}(X)) + Pf.
\]

This construction is better understood with the definition of the so-called "induced trajectories".

8. **INDUCED TRAJECTORIES FOR THE NAVIER-STOKES PROBLEM**

In [8] a family of functions, \(\{k_{j,m}: j \in \mathbb{N}\}\), is defined by the equalities

\[
-\nu \Delta k_{0,m} + QB(p) = Qf, \quad (8.25)
\]

\[
-\nu \Delta k_{1,m} + QB(p, k_{0,m}) + QB(k_{0,m}, p) = 0,
\]
\[-\nu \Delta k_{j,m} + QB(p, k_{j-1,m}) + QB(k_{j-1,m}, p) + \]
\[+ \sum_{s=0}^{j-3} QB(k_{j-2,m}, k_{s,m}) + \sum_{r=0}^{j-3} QB(k_r,m, k_{j-2,m}) \]
\[+ QB(k_{j-2,m}, k_{j-2,m}) + k_{j-2,m} = 0, \quad j \geq 2, \]
where \( p(t) \) is, as before, the projection through \( P \) of the solution \( u(t) \).
Then the functions \( q_j \) are defined \([8]\) through the relations
\[q_j = k_{0,m} + \ldots + k_{j,m}, \]
and therefore, satisfy the equations
\[\begin{align*}
-\nu \Delta q_0 + QB(p) &= Qf, \\
-\nu \Delta q_1 + QB(p) + QB(p, q_0) + QB(q_0, p) &= Qf, \\
-\nu \Delta q_j + q_j' + QB(p) + QB(p, q_{j-1}) \\
+ QB(q_{j-1}, p) + QB(q_{j-2}, q_{j-2}) &= Qf, \quad j \geq 2.
\end{align*}\]

The following inequalities
\[|k_{jm}| \leq \kappa_j \delta^{1+j/2} L^{(j+1)/2}, \quad \|k_{jm}\| \leq \kappa_j \delta^{1+j/2} L^{(j+1)/2}, \quad (8.26)\]
\[|k'_{jm}| \leq \kappa_j \delta^{1+j/2} L^{(j+1)/2}, \quad (8.27)\]
\[|q_j| \leq \kappa_j \delta L^{1/2}, \quad \|q_j\| \leq \kappa_j \delta L^{1/2}, \quad |q'_j| \leq \kappa_j \delta L^{1/2}, \quad (8.28)\]
are proved in \([8]\), where \( \kappa_j \) are independent of \( m \), but depend on \( j, \nu, |f|, \lambda \).

They will help us to estimate the distance between the trajectories of the Navier-Stokes problems and the a.i.m.s.
Indeed, since \( u_j(t) \in M_j \),
\[dist_{\mathcal{M}_j} (u(t), M_j) \leq dist (u(t), u_j(t)) = |q(t) - q_j(t)|.\]

By estimating this last norm and the similar one for \( V_1 \), in \([8]\) and \([9]\) it is proved that
\[\begin{align*}
\text{dist}_{\mathcal{M}_j} (u(t), M_j) &\leq \tilde{\kappa}_j L^{\frac{j+1}{2}} \delta^{\frac{3+j}{2}}, \\
\text{dist}_{V_1} (u(t), M_j) &\leq \tilde{\kappa}'_j L^\frac{j}{2} \delta^{\frac{2+j}{2}}.
\end{align*}\]
9. "INDUCED TRAJECTORIES" FOR THE DIFFUSED SUBSTANCE

Following the above reasoning, we define the functions $h_{j,m} : \mathbb{R}^+ \to \mathbb{Q}^2$, $j \in \mathbb{N}$, as solutions of the equations

$$-D\Delta h_{0,m} + QB(p, c_p) = Qh, \quad (9.29)$$
$$-D\Delta h_{1,m} + QB(p, h_{0,m}) + QB(k_{0,m}, c_p) = 0, \quad (9.30)$$

$$h'_{j-2,m} - D\Delta h_{j,m} + QB(p, h_{j-1,m}) + QB(k_{j-1,m}, c_p) +$$
$$+ \sum_{r=0}^{j-3} QB(k_{r,m}, h_{j-2,m}) + \sum_{s=0}^{j-3} QB(k_{j-2,m}, h_{s,m}) + QB(k_{j-2,m}, h_{j-2,m}) = 0, \quad (9.31)$$

for $j \geq 2$.

In these definitions $p = p(t) = Pu(t)$ and $c_p = c_p(t) = Pc(t)$, where $(u(t), c(t))$ is the solution of the diffusion problem, and $k_{j,m}$ are given by (8.25).

We prove the following

**Theorem 5.** There are some constants $\kappa_j, \varsigma_j$ independent of $m$, but depending on $j$ and on $\nu$, $|f|$, $\lambda_1, D$, and there is a $t_{c2}(R)(\geq t_{c1}(R))$ such that for $m, j$, and every $t \geq t_{c2}(R)$, the estimates

$$|h_{j,m}| \leq \kappa_j L^{\frac{j+1}{2}} \delta^{\frac{j+1}{2}}, \quad \|h_{j,m}\| \leq \kappa_j L^{\frac{j+1}{2}} \delta^{\frac{j+1}{2}}, \quad (9.32)$$

$$|h'_{j,m}| \leq \kappa_j L^{\frac{j+1}{2}} \delta^{\frac{j+1}{2}}, \quad |Ah_{j,m}| \leq \kappa_j L^{\frac{j+1}{2}} \delta^{\frac{j+1}{2}}, \quad (9.33)$$

hold, $(\forall) j \geq 0$.

**Proof.** We look for an estimate for the norm of $h_{0,m}$. From (9.29), by using (2.12), we successively obtain

$$|D\Delta h_{0,m}| = |Q(p\nabla c_p)| + |Qh| \leq |p\nabla c_p| + |Qh| \leq cL^\frac{1}{2} \|p\| \|c_p\| + M_h,$$

whence

$$|h_{0,m}| \leq \left( \frac{c p \eta_1 + M_h D}{D\lambda} \right)^{\frac{1}{2}} = \frac{c p \eta_1 + M_h D}{D\lambda} L^{\frac{1}{2}} \delta, \quad \|h_{0,m}\| \leq \frac{c p \eta_1 + M_h D}{D\lambda^{\frac{1}{2}}} \delta^{\frac{1}{2}}.$$

With $\kappa_0 = \frac{c p \eta_1 + M_h D}{D\lambda^{\frac{1}{2}}}$, $\kappa'_0 = \frac{c p \eta_1 + M_h D}{D\lambda^{\frac{1}{2}}}$, the relations (9.32) are proved for $j = 0$. 
Since $u$ and $c$ are analytic in time, so are $p$ and $c_p$. Moreover, these last functions are the restrictions to the real axis of functions of a complex variable that are analytic in a neighbourhood of the real axis. It follows that $h_{0,m}$ has the same property. Then, by using the same reasoning as for the Navier-Stokes equations [6], we prove the first relation of (9.33), for $j = 0$.

From the definition of $h_{1,m}$, with (2.12), (2.13), and by taking into account the estimates for $h_{0,m}$, we successively deduce

$$|Dh_{1,m}| \leq c \|p\| \|h_{0,m}\| L_{\frac{3}{2}} + c \|\kappa_{0,m}\| \|c_p\| L_{\frac{3}{2}}$$

$$\leq c p_1 \kappa_{0}^{2} L_{\frac{3}{2}} + c \kappa_{0}^{1/2} ||h_{1,m}|| \leq (c p_1 \kappa_{0}^{2} + c \kappa_{0}^{1/2}) L_{\frac{3}{2}}^{1/2},$$

$$|h_{1,m}| \leq \left( \frac{c p_1 \kappa_{0}^{2} + c \kappa_{0}^{1/2}}{D} \right) L_{\frac{3}{2}}^{1/2},$$

$$||h_{1,m}|| \leq \left( \frac{c p_1 \kappa_{0}^{2} + c \kappa_{0}^{1/2}}{D} \right) L_{\frac{3}{2}}^{1/2}.$$ 

We find as for $h_{0,m}$, that $h_{1,m}$ is analytic in time and, then, $h_{1,m}'$ is majorized by an expression of the order of $L_{\delta}^{\frac{1}{2}}$.

For $k \geq 2$ we proceed by induction. We assume that the inequalities (9.32), (9.33) are true and that $h_{j,m}$ is the restriction to the real axis of an analytic function of $t$ for every $j < k - 1$. From the definition of $h_{k,m}$, we have

$$|Dh_{k,m}| \leq |h_{k-2,m}'| + |Q(p \nabla h_{k-1,m})| + |Q(k_{k-1,m} \nabla c_p)| + \sum_{r=0}^{k-3} |Q(k_{r,m} \nabla h_{k-2,m})| +$$

$$+ \sum_{s=0}^{k-3} |Q(k_{s,m} \nabla h_{k-2,m})| + |Q(k_{k-2,m} \nabla h_{k-2,m})|$$

$$\leq CL_{\frac{3}{2}}^{k-\frac{1}{2}} \kappa_{0}^{\frac{1}{2}} + CL_{\frac{3}{2}}^{k} ||p|| ||h_{k-1,m}|| + CL_{\frac{3}{2}}^{k} ||k_{k-1,m}|| ||c_p|| +$$

$$+ CL_{\frac{3}{2}}^{k} \sum_{r=0}^{k-3} ||k_{r,m}|| ||h_{k-2,m}|| + CL_{\frac{3}{2}}^{k} \sum_{s=0}^{k-3} ||k_{s,m}|| ||h_{s,m}|| +$$

$$+ CL_{\frac{3}{2}}^{k} ||k_{k-2,m}|| ||h_{k-2,m}||,$$

whence

$$|Dh_{k,m}| \leq CL_{\frac{3}{2}}^{k-\frac{1}{2}} \kappa_{0}^{\frac{1}{2}} + CL_{\frac{3}{2}}^{k} \kappa^{1/2} L_{1}^{k} \kappa_{1}^{1/2} L_{1}^{k} \kappa_{1}^{1/2} L_{1}^{k} \kappa_{1}^{1/2} L_{1}^{k} +$$

$$+ \kappa_{k-2} L_{1}^{k-\frac{1}{2}} \kappa_{1}^{k-\frac{1}{2}} \sum_{r=0}^{k-3} \kappa_{r}^{\frac{1}{2}} L_{1}^{\frac{k-1}{2}} L_{1}^{\frac{k-1}{2}} +$$

$$+ \kappa_{k-2} L_{1}^{k-\frac{1}{2}} \kappa_{1}^{k-\frac{1}{2}} \sum_{s=0}^{k-2} \kappa_{s}^{\frac{1}{2}} L_{1}^{k-\frac{1}{2}} L_{1}^{\frac{k-1}{2}} \kappa_{1}^{\frac{1}{2}} L_{1}^{k} \kappa_{1}^{\frac{1}{2}} L_{1}^{k} \kappa_{1}^{\frac{1}{2}} L_{1}^{k} \kappa_{1}^{\frac{1}{2}} L_{1}^{k}.$$
Consequently,

\[ |h_{k,m}| \leq \frac{C''}{D\lambda} L_{1+1} L_{1+1} \delta_{1+1}^{-1} = \kappa_{k} L_{1+1} \delta_{1+1}^{-1}, \]

where \( \kappa_{k} = \frac{C''}{D\lambda} \), hence, the conclusion. The other inequalities are proved as for \( j = 0 \) and \( j = 1 \).

We define now \( c_{q,j} : \mathbb{R}^+ \to \mathbb{R}^2, j \in \mathbb{N} \) by

\[ c_{q,j} = h_{0,m} + h_{1,m} + ... + h_{j,m}. \]

As a consequence of their definition, these new functions satisfy the relations

\[ -D\Delta c_{q,0} + QB(p, c_p) = Qh, \quad (9.34) \]
\[ -D\Delta c_{q,1} + QB(p, c_p) + QB(p, c_{q,0}) + QB(q_0, c_p) = Qh, \quad (9.35) \]
\[ c'_{q,j-2} - D\Delta c_{q,j} + QB(p, c_p) + QB(p, c_{q,j-1}) + QB(q_{j-1}, c_p) + QB(q_{j-2}, c_{q,j-2}) = Qh, \quad (9.36) \]

for \( j \geq 2 \). By summing the relations proved for \( h_{k,m} \) we obtain

\[ |c_{q,j}| \leq \varsigma_j L_{j+1} \delta_j, \quad \|c_{q,j}\| \leq \varsigma'_j L_{j+1} \delta_j^2, \quad |c'_{q,j}| \leq \varsigma''_j L_{j+1} \delta_j, \quad j \geq 0. \]

Now, along the lines of [8], we define the induced trajectories for \( c \)

\[ c_j(t) = c_p(t) + c_{q,j}(t) \]

and find the distance between these functions and \( c(t) \). We define \( \xi_j(t) = c_j(t) - c(t) = c_{q,j}(t) - c_q(t) \).

**Theorem 6.** There are some constants, \( \kappa_j \), independent of \( m \) but depending on \( j \) and \( \nu, \|f\|, \lambda_1, D \) and there is a moment \( t_3 \) such that for every \( m, j \) and for \( t \geq t_3 \) the inequalities

\[ |\xi_j| \leq \kappa_j L_{j+1} \delta_j^{j+3}, \quad (9.37) \]
\[ \|\xi_j\| \leq \kappa'_j L_{j+1} \delta_j^{j+1}, \quad (9.38) \]
\[ |\xi'_j| \leq \kappa''_j L_{j+1} \delta_j^{j+3}, \quad (9.39) \]
Hold for $\forall j \geq 0$.

**Proof.** We start with the definition of $c_{q,0}$ and from the equation for $c_q$, written in the form

$$\Delta c_{q,0} = -\frac{1}{D} [Qh - Q(p \nabla c_p)],$$

$$\Delta c_q = -\frac{1}{D} [Qh - Q(u \nabla c) - \frac{dc_q}{dt}].$$

By subtracting the two relations, we obtain

$$\Delta (c_{q,0} - c_q) = \frac{1}{D} \left[ Q(p \nabla c_p) - Q(u \nabla c) - \frac{dc_q}{dt} \right]$$

(9.40)

$$= -\frac{1}{D} \left[ Q(p \nabla c_q) + Q(q \nabla c_p) + Q(q \nabla c_q) + \frac{dc_q}{dt} \right],$$

whence, by using the same inequalities as above, we have

$$|\xi_0| = |c_{q,0} - c_q| \leq \frac{1}{D\Lambda} \left| Q(p \nabla c_q) + Q(q \nabla c_p) + Q(q \nabla c_q) + \frac{dc_q}{dt} \right|$$

$$\leq \frac{CL_1^2}{D\Lambda} \delta^\frac{3}{2} = \frac{CL_2^2}{D\Lambda} \delta^\frac{3}{2} = \kappa_0 L_1^2 \delta^\frac{3}{2},$$

i.e. the inequality (9.37) for $j = 0$. Similarly, we obtain (9.38) for $j = 0$.

As we can see from (9.40), $\xi_0$ is analytical in $t$ and can be extended to a function of a complex variable, analytical in a neighborhood of the $t$ axis. Then, by using the Cauchy formula, it follows that $|\xi'_0|$ is of the same order as $|\xi_0|$.

Now, for $|\xi_1|$ we have

$$\Delta (c_{q,1} - c_q) = \frac{1}{D} \left[ Q(p \nabla c_p) + Q(p \nabla c_{q,0}) + Q(q_0 \nabla c_p) - Q(u \nabla c) - \frac{dc_q}{dt} \right]$$

$$= \frac{1}{D} \left[ Q(p \nabla (c_{q,0} - c_q)) + Q((q_0 - q) \nabla c_p) - Q(q \nabla c_q) - \frac{dc_q}{dt} \right],$$

and, by taking the norm and using the previous results, we get (9.37) and (9.38) for $j = 1$.

The inequality for $|\xi'_1|$ is obtained as that for $|\xi'_0|$.

From this point on, we proceed by induction, assuming that (9.37), (9.38) and (9.39) hold for every $j \leq k - 1$. 


We have
\[\Delta (c_{q,k} - c_q) = \frac{1}{D} \left[ Q \left( p \nabla \left( c_{q,k-1} - c_q \right) \right) + Q \left( (q_{k-1} - q) \nabla c_p \right) + Q((q_{k-2} - q) \nabla c_{q,k-2}) + Q(q \nabla (c_{q,k-2} - c_q)) + c'_{q,k-2} - c'_q \right],\]

hence
\[|\xi_k| = |c_{q,k} - c_q| \leq \frac{1}{D} \delta \left[ CL^{1/2} \rho_1 L^{1/2} \delta^{1/4} + CL^{1/2} \eta_1 L^{1/2} \delta^{1/4} + CL^{1/2} \delta^{1/8} \ln \delta^{1/8} + CL^{1/2} \delta^{1/8} \right] \]
\[\|\xi_k\| \leq CL^{1/2} \delta^{1/4}.\]

By using the Cauchy formula for the extension of \(\xi_k(t)\) to a complex variable (analytical in a neighbourhood of the time axis), we obtain for \(|\xi_k'|\) the same type of estimate as for \(|\xi_k|\).

10. APPROXIMATE INERTIAL MANIFOLDS FOR OUR PROBLEM

Here we construct the a.i.m.s for the diffusion problem (2.1)-(2.5).

The first a.i.m. is the manifold \(\tilde{M}_0\) defined as the graph of \(\Psi_0 : P\tilde{\mathcal{H}}_1 \times P\tilde{\mathcal{H}}_2 \rightarrow Q\tilde{\mathcal{H}}_1 \times Q\tilde{\mathcal{H}}_2\),

\[\Psi_0(X, X) = \left( -(\nu \Delta)^{-1} (Qf - QB(X)) , - (D\Delta)^{-1} [Qh - QB(X,X)] \right).\]

Denote the components of \(\Psi_0(X, X)\) by \((\Psi_{u0}(X), \Psi_{c0}(X, X))\) (obviously \(\Psi_{u0}(X) = \Phi_0(X)\)).

The second a.i.m., denoted by \(\tilde{M}_1\), is the graph of \(\Psi_1 : P\tilde{\mathcal{H}}_1 \times P\tilde{\mathcal{H}}_2 \rightarrow Q\tilde{\mathcal{H}}_1 \times Q\tilde{\mathcal{H}}_2\), \(\Psi_1(X, X) = (\Psi_{u1}(X), \Psi_{c1}(X, X))\), where \(\Psi_{u1}(X) = \Phi_1(X)\) and

\[\Psi_{c1}(X, X) = \frac{-1}{D} \Delta^{-1} [Qh - QB(X, X) - QB(X, \Psi_{c0}(X, X)) - QB(\Psi_{u0}(X), X)].\]

For \(j \geq 2\) we define the function \(\Psi_j : P\tilde{\mathcal{H}}_1 \times P\tilde{\mathcal{H}}_2 \rightarrow Q\tilde{\mathcal{H}}_1 \times Q\tilde{\mathcal{H}}_2\), \(\Psi_j(X, X) = (\Psi_{u,j}(X), \Psi_{c,j}(X, X))\), where \(\Psi_{u,j}(X) = \Phi_j(X)\) and
\[ \Psi_{c,j}(X,X) = -(D\Delta)^{-1}\left[ Qh - QB(X,X) - QB(X,\Psi_{c,j-1}(X,X)) - QB(\Psi_{u,j-1}(X,X) - QB(\Psi_{u,j-2}(X,X),\Psi_{c,j-2}(X,X)) - \quad \right. \]
where \( D\Psi_{c,j-2}(X,X) \) is the differential of \( \Psi_{c,j-2}(X,X) \) and
\[ \Gamma_{c,j-2}(X,X) = D\Delta X - PB(X + \Psi_{u,j-2}(X,X) + \Psi_{c,j-2}(X,X)) + Ph. \]

The graph of this function is the a.i.m. \( \tilde{M}_j \) of the diffusion problem.

**Theorem 7.** For \( t \) large enough, the distance between the solution of the diffusion equations and the a.i.m. \( \tilde{M}_j \) is bounded by
\[ \text{dist}_{3\hat{u}_j 3\hat{u}_j} \left( (u(t),c(t)),\tilde{M}_j \right) \leq K_1 L^{1+j} 3^{-3+j}. \quad (10.41) \]

**Proof.** For \( j = 0 \) and \( j = 1 \), by definition, \( \Psi_{u,j}(p) = q_j(p) \), \( \Psi_{c,j}(p,p) = c_{q,j}(p,p) \); the inequality (10.41) follows directly from the result (10.2) of [8] and from (2.1), \( j = 0 \), resp. \( j = 1 \) of our Theorem 6. For \( j \geq 2 \), we need a special proof for (10.41). We take as initial step in our inductive reasoning the case \( j = 2 \). We have
\[ \Psi_{c,2}(p,c_p) - c_q = \Psi_{c,2}(p,c_p) - c_{q,2} + c_{q,2} - c_q = \Psi_{c,2}(p,c_p) - c_{q,2} + \xi_2, \]
such that the difference between \( \Psi_{c,2}(p,c_p) \) and \( c_{q,2} \) satisfies the identities
\[ D\Delta |\Psi_{c,2}(p,c_p) - c_{q,2}| = QB(p,\Psi_{c,1}(p,c_p) - c_{q,1}) + \quad \]
\[ + QB(p,\Psi_{u,1}(p) - q_1,c_p) + \quad \]
\[ + QB(p,\Psi_{c,0}(p,c_p)) - QB(q_0,c_{q,0}) + \quad \]
\[ + D\Psi_{c,0}(p,c_p)(\Gamma_{u,0}(p),\Gamma_{c,0}(p,c_p)) - c_{q,0}' \]
\[ = D\Psi_{c,0}(p,c_p)(\Gamma_{u,0}(p),\Gamma_{c,0}(p,c_p)) - c_{q,0}' , \]
whence
\[ |\Psi_{c,2}(p,c_p) - c_{q,2}| \leq \left| (D\Delta)^{-2}[Q(\Gamma_{u,0}(p)\nabla c_p) + Q(p\nabla\Gamma_{c,0}(p,c_p))] \right. \]
\[ - Q(p'\nabla c_p) - Q(p\nabla c_p')] \right| \leq \frac{D}{\lambda^2} \delta^2 \left| QB(\Gamma_{u,0}(p) - p',c_p) + QB(p,\Gamma_{c,0}(p,c_p) - c_{q,0}') \right| \]
\[ \leq \frac{D}{\lambda^2} \delta^2 CL^{1/2} \left( ||\Gamma_{u,0}(p) - p'|| ||c_p|| + ||p|| ||\Gamma_{c,0}(p,c_p) - c_{q,0}'|| \right) . \]
Now,
\[
\left\| \Gamma_{u,0}(p) - p' \right\| = \left\| \text{PB}(p + \Phi_0(p)) - \text{PB}(p + q) \right\|
\]
\[
= \left\| \text{PB}(p, \Phi_0(p) - q) + \text{PB}(\Phi_0(p) - q, p) + \right. \text{PB}(\Phi_0(p), \Phi_0(p)) - \text{PB}(q, q) \right\|
\]
\[
\leq \Lambda^2 CL^{\frac{1}{2}} (\|p\| \|\Phi_0(p) - q\| + \|\Phi_0(p) - q\| ||q|| + \|\text{PB}(\Phi_0(p), \Phi_0(p)) - \text{PB}(q, q)\|)
\]
\[
\leq CL\delta^{\frac{3}{2}},
\]
and
\[
\left\| \Gamma_{c,0}(p,c_p) - c'_p \right\| = \left\| \text{PB}(p + \Psi_{u,0}(p), c_p) + \Psi_{c,0}(p,c_p) - \text{PB}(u,c) \right\|
\]
\[
\leq \left\| \text{PB}(\Psi_{u,0}(p) - p, c_p) + \text{PB}(p, \Psi_{c,0}(p,c_p) - c_q) \right\|
\]
\[
+ \|\text{PB}(\Psi_{u,0}(p), \Psi_{c,0}(p,c_p)) - \text{PB}(c_p,c_p)\|\n\]
\[
\leq \Lambda^2 CL^{\frac{1}{2}} (\|\Psi_{u,0}(p) - p\| \|c_p\| + \|p\| \|\Psi_{c,0}(p,c_p) - c_q\| + \|\text{PB}(\Psi_{u,0}(p), \Psi_{c,0}(p,c_p)) - \text{PB}(c_p,c_p)\|)
\]
\[
\leq CL\delta^{\frac{3}{2}},
\]
whence
\[
|\Psi_{c,2}(p,c_p) - c_{q,2}(p,c_p)| \leq CL^{\frac{3}{2}} \delta^{\frac{3}{2}}.
\]
Assume that for every \( k \leq j - 1 \)
\[
|\Psi_{c,k}(p,c_p) - c_{q,k}(p,c_p)| \leq CL^{\frac{k+1}{2}} \delta^{\frac{k+3}{2}}.
\]
For \( j > 2 \), we have
\[
|D\Delta (\Psi_{c,j}(p,c_p) - c_{q,j}(p,c_p))| \leq
\]
\[
\leq CL^{\frac{j}{2}} \|p\| \|\Psi_{c,j-1}(p,c_p) - c_{q,j-1}\| + CL^{\frac{j}{2}} \|c_p\| \|\Psi_{u,j-1}(p) - q_{j-1}\| +
\]
\[
+ C \|\Psi_{u,j-2}(p) - q_{j-2}\|^\frac{1}{2} |\Delta (\Psi_{u,j-2}(p) - q_{j-2})|^\frac{1}{2} \|c_{q,j-2}\| +
\]
\[
+ C \|\Psi_{u,j-2}(p)\|^\frac{1}{2} |\Delta \Psi_{u,j-2}(p)|^\frac{1}{2} \|\Psi_{c,j-2}(p,c_p) - c_{q,j-2}\| +
\]
\[
+ |D\Psi_{c,j-2}(p,c_p)(\Gamma_{u,j-2}(p), \Gamma_{c,j-2}(p,c_p)) - c'_{q,j-2}| (10.42)
\]
\[
\leq CL^{\frac{j+1}{2}} \delta^{\frac{j+1}{2}} + \|D\Psi_{c,j-2}(p,c_p)(\Gamma_{u,j-2}(p), \Gamma_{c,j-2}(p,c_p)) - c'_{q,j-2}| (10.42)
\]
A careful analysis of the order of magnitude and the use of the induction hypothesis and of the inequalities (2.6)-(2.8), (2.11)-(2.13) leads us to the conclusion that the terms in the right-hand side of (10.42) have the same order of magnitude with respect to \( \delta \), as \( \delta \to 0. \)
Hence, finally,
\[ |\Psi_{c,j}(p,c_p) - c_q| = |\Psi_{c,j}(p,c_p) - c_{q,j} + c_{q,j} - c_q| \leq |\Psi_{c,j}(p,c_p) - c_{q,j}| + |\xi_j| \leq CL^{j+1}\delta^{-\frac{j+3}{2}}. \]

Since
\[ \text{dist} \left( (u(t), c(t)), \tilde{M}_j \right) \leq C \left( |\Phi_j(p) - q| + |\Psi_{c,j}(p,c_p) - c_q(p,c_p)| \right) , \]
the conclusion of the theorem follows. \( \square \)

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References

ERRATUM TO: APPROXIMATE INERTIAL MANIFOLDS FOR AN ADVECTION-DIFFUSION PROBLEM

[ROMAI Journal, 2, 1(2006), 91-108]; Anca-Veronica Ion

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The inequalities (7), (8) in the cited article should read:

\[
|B(u, v)| \leq c_2 \|u\| \|v\| \left[ 1 + \ln \left( \frac{|\Delta u|^2}{\lambda_1 \|u\|^2} \right) \right]^{1/2}, \forall \ u \in D(A) \setminus \{0\}, v \in V_1,
\]  

(1)

\[
|B(u, v)| \leq c_3 \|u\| \|v\| \left[ 1 + \ln \left( \frac{|\Delta v|^2}{\lambda_1 \|v\|^2} \right) \right]^{1/2}, \forall \ u \in V_1, v \in D(A) \setminus \{0\}.
\]  

(2)

It is also important to give some additional explanations concerning their proof, since from the text of the article it follows that they are presented in [1], [2] and [3]. The first one is indeed re-called in the three cited works, is often used in the literature dedicated to the Navier-Stokes equations and is a consequence of the Brézis-Gallouët inequality

\[
\|u\|_{L^\infty(\Omega)} \leq C \|u\| \left( 1 + \ln \frac{\|u\|^2}{\lambda_1^{1/2} \|u\|} \right)^{1/2}.
\]

The second one is not met in the three works cited. It is a direct consequence of the inequality

\[
|(B(u, v), Aw)| \leq c_3 \|u\| \|v\| \|Aw\| \left[ 1 + \ln \left( \frac{|\Delta v|^2}{\lambda_1 \|v\|^2} \right) \right]^{1/2}, \forall \ u \in V_1, v, w \in D(A)
\]  

(3)
enounced in [4], and proved in [5].

The inequalities (12) and (13) from our article should read

\[ |B(u, c)| \leq c_2 \|u\| \|c\| \left[ 1 + \ln \left( \frac{|\Delta u|^2}{\lambda_1 \|u\|^2} \right) \right]^{\frac{1}{2}}, \quad (\forall) \; u \in D(A) \setminus \{0\}, \; c \in V_2, \]

(4)

\[ |B(u, c)| \leq c_3 \|u\| \|c\| \left[ 1 + \ln \left( \frac{|\Delta c|^2}{\lambda_1 \|c\|^2} \right) \right]^{\frac{1}{2}}, \quad (\forall) \; u \in V_1, \; c \in D(A) \setminus \{0\}. \]

(5)

and are obtained as consequences of (1) and (2) above.

We apologize to the reader for these imprecisions in our paper.

References


