SINGULAR PERTURBATION FOR A PROBLEM FROM HYDRAULICS

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Abstract Of concern is the comparison of the solution \((u, v)\) to the nonlinear b.v.p. \((P)\) below with the solution \((u^\varepsilon, v^\varepsilon)\) of its elliptic regularization \((P_\varepsilon)\). An asymptotic approximation for \((u^\varepsilon, v^\varepsilon)\) is constructed using the boundary layer method of Vishik-Lyusternik. The order of this approximation is found.

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1. INTRODUCTION

Consider problem \((P)\) (arising in hydraulics) given by
\[
\begin{align*}
&\begin{cases}
u_t + v_x + \alpha u(x,t) = f(x,t), \quad (x,t) \in Q = (0,1) \times (0,T) \\
v_t + u_x + \beta v(x,t) = g(x,t), \quad (x,t) \in Q = (0,1) \times (0,T),
\end{cases} \\
&u(0,t) = u(1,t) = 0, \quad 0 < t < T, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 < x < 1
\end{align*}
\]
and its elliptic regularization \((P_\varepsilon)\)
\[
\begin{align*}
&\begin{cases}
\varepsilon u_{tt}^\varepsilon - u_t^\varepsilon = v_x^\varepsilon + \alpha u^\varepsilon(x,t) - f(x,t), \quad (x,t) \in Q \\
\varepsilon v_{tt}^\varepsilon - v_t^\varepsilon = u_x^\varepsilon + \beta v^\varepsilon(x,t) - g(x,t), \quad (x,t) \in Q,
\end{cases} \\
u^\varepsilon(0,t) = u^\varepsilon(1,t) = 0, \quad 0 < t < T, \\
u^\varepsilon(x,0) = u_0^\varepsilon(x), \quad v^\varepsilon(x,0) = v_0^\varepsilon(x), \quad 0 < x < 1 \\
u^\varepsilon(x,T) = u_1^\varepsilon(x), \quad v^\varepsilon(x,T) = v_1^\varepsilon(x), \quad 0 < x < 1
\end{align*}
\]

Problem \((P)\) governs the unsteady fluid flow (water-hammer) with nonlinear pipe friction. It is also a model for the fluid flow through a tree-structured system of transmission pipelines [4]. In problem \((P)\), \(u\) denotes the instantaneous discharge at a point and \(v\) is the elevation of hydraulic gradeline.

We construct an asymptotic approximation \((Y^\varepsilon, Z^\varepsilon)\) of \((u^\varepsilon, v^\varepsilon)\) with the aid of the solution of problem \((P)\) and find the order of this approximation in the spaces \(L^2(Q)\) and \(C([0,T]; L^2(0,1))\).
This approximation of problem \((P)\) with its elliptic regularization is stronger than those from [3] and from other related papers. In [2], an elliptic regularization is also used to approximate the solution of the heat equation.

2. THE ASYMPTOTIC APPROXIMATION

Denote \(K = L^2 (0, 1)\) and \(H = L^2 (0, 1)^2\). It is known that, under some specific hypotheses, problems \((P)\) and \((P_\varepsilon)\) have unique strong solutions \((u, v)\) in \(W^{1,\infty} (0, T; H)\) ([4]) and \((u^\varepsilon, v^\varepsilon) \in W^{2,2} (0, T; H)\), respectively [1].

Our purpose is to investigate the behavior of the solution \((u^\varepsilon, v^\varepsilon)\) of \((P_\varepsilon)\) as \(\varepsilon \to 0\). Remark that \((u, v)\) does not generally satisfy the boundary condition \((C_\varepsilon)\), therefore at least in some neighborhood \((\rho, T)\) of the final point \(t = T\), functions \((u^\varepsilon, v^\varepsilon)\) and \((u, v)\) are not close to each other. Thus \((P_\varepsilon)\) is a singularly perturbed problem and the set \(L = (0, 1) \times (\rho, T)\) is a boundary layer. We shall prove that \(u^\varepsilon - u\) and \(v^\varepsilon - v\) converge to zero in \(C ([0, \rho]; K)\) and construct an asymptotic approximation for \((u^\varepsilon, v^\varepsilon)\) which is valid in the whole domain \(Q\). Using the boundary function method of Vishik -Lyusternik [5], [6], we are looking for an asymptotic approximation \((Y^\varepsilon, Z^\varepsilon)\) of \((u^\varepsilon, v^\varepsilon)\) of the form

\[
Y^\varepsilon (x, t) = u (x, t) + \Pi (x, \tau), \quad Z^\varepsilon (x, t) = v (x, t) + \Phi (x, \tau),
\]

where \(\tau = (T - t) / \varepsilon\) is the boundary layer variable and \(\Pi, \Phi\) are boundary layer functions. They satisfy \(\Pi (x, \infty) = \Phi (x, \infty) = 0, 0 < x < 1\). We find

\[
\Pi (x, \tau) = e^{-\tau} [u_1 (x) - u (x, T)], \quad \Phi (x, \tau) = e^{-\tau} [v_1 (x) - v (x, T)].
\]

Since in the equations which define the remainder, the second derivatives \(u_{tt}\), \(v_{tt}\) arise, we need high order regularity results for \((P)\). Suppose that:

\(\begin{align*}
(H1) \quad & \alpha, \beta \geq 0; \\
(H2) \quad & f, g \in W^{2,\infty} (0, T; K); \\
(H3) \quad & f (., 0), g (., 0) \in H^1 (0, 1), \quad u_0, v_0 \in H^2 (0, 1), \quad f (0, 0) = f (1, 0) = 0, \\
& u_0 (0) = u_0 (1) = 0, \quad v_0 (0) = v_0 (1) = 0.
\end{align*}\)

**Theorem 2.1.** If \((H1) - (H3)\) hold, then \((u, v) \in W^{2,\infty} (0, T; H)\) and \(u_x, v_x \in L^\infty (0, T; K)\).

**Proof.** One differentiates formally problem \((P)\) with respect to \(t\). Denoting \(U = u_t, V = v_t\), one obtains the equation

\[
\begin{cases}
U_t + V_x + \alpha U (x, t) = f_t (x, t), \quad (x, t) \in Q = (0, 1) \times (0, T) \\
V_t + U_x + \beta V (x, t) = g_t (x, t), \quad (x, t) \in Q = (0, 1) \times (0, T),
\end{cases}
\]

with the boundary conditions

\[
U (0, t) = U (1, t) = 0, \quad 0 < t < T,
\]
and 

\[ U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad 0 < x < 1, \]

where \( U_0(x) = f(x,0) - \alpha u_0(x) - v_0'(x) \) and \( V_0(x) = g(x,0) - \beta v_0(x) - u'_0(x) \), for \( x \in (0,1) \).

One shows that this problem admits a unique strong solution \((U,V) \in W^{1,\infty}(0,T;H)\). Let \( S(t), t \geq 0 \) be the nonlinear semigroup generated by the linear operator \( A : D(A) \subseteq H \times H \to H \times H \),

\[ D(A) = \{(U,V) \in H, \quad U(0,t) = U(1,t) = 0, \quad 0 < t < T\}, \]

\[ A(U,V) = (\alpha U, \beta V). \]

With the aid of this semigroup and of the constant variation formula, we express \((U,V)\) and show that it coincides with the derivatives \((u_t,v_t)\) of the solution of \((P)\). Thus we conclude the proof.

Now we are interested to find the order of \( u^\varepsilon - Y^\varepsilon, \quad v^\varepsilon - Z^\varepsilon \), where \((Y^\varepsilon, Z^\varepsilon)\) is given in \((1)\). The main result is the following.

**Theorem 2.2.** If \( u_1 \in H^1_0(0,1), \quad v_1 \in H^1(0,1) \), then under assumptions \((H1) - (H3)\), for sufficiently small \( \varepsilon > 0 \), the pair \((Y^\varepsilon, Z^\varepsilon)\) is an asymptotic approximation of \((u^\varepsilon, v^\varepsilon)\) in the entire domain \( Q \) and we get the estimates:

\[
|u^\varepsilon - u|_{L^2(Q)} = o\left(\varepsilon^{1/2}\right), \quad |v^\varepsilon - v|_{L^2(Q)} = o\left(\varepsilon^{1/2}\right), \quad (3)
\]

\[
|u^\varepsilon - u - \Pi|_{C([0,T];K)} = o\left(\varepsilon^{1/4}\right), \quad |v^\varepsilon - v - \Phi|_{C([0,T];K)} = o\left(\varepsilon^{1/4}\right). \quad (4)
\]

**Conclusions.** a) If \( u(.,T) = u_1, \quad v(.,T) = v_1 \), then \((u,v)\) is an asymptotic approximation of \((u^\varepsilon, v^\varepsilon)\) in the entire domain \( Q \). Problem \((P_\varepsilon)\) is regularly perturbed and estimates \((4)\) hold with \( \Pi = \Phi = 0 \).

b) Otherwise, for every \( \rho \in (0,T) \), \((u,v)\) is an asymptotic approximation of \((u^\varepsilon, v^\varepsilon)\) in \( Q_1 = (0,1) \times (0,\rho) \), \((P_\varepsilon)\) is singularly perturbed and we have

\[
|u^\varepsilon - u|_{C([0,\rho];K)} = o\left(\varepsilon^{1/4}\right), \quad |v^\varepsilon - v|_{C([0,\rho];K)} = o\left(\varepsilon^{1/4}\right). \quad (5)
\]

**References**


