A NOTE ON THE REACHABILITY SET OF PETRI NETS

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Abstract This paper treats the notion of reachability in Petri nets. More precisely, a method to compute the residue of the reachability set of a Petri net is present. Moreover, as an application, it is shown how this residue set can be used to compute the concurrency-degrees of Petri nets.

Keywords: parallel and distributed systems, Petri nets, reachability, computability.

1. INTRODUCTION

A Petri net is a mathematical model used for the specification and the analysis of parallel and distributed systems. An introduction about Petri nets can be found in [6].

Petri nets are a powerful language for system modelling and validation. They are now in widespread use for many different practical and theoretical purposes in various fields of software and hardware development.

The reachability set of a Petri net is the set of all the states which are reachable from the initial state of the system. In this paper we present a method to compute the residue (i.e. the set of minimal elements) of the reachability set of a Petri net, by applying a more general algorithm from [8].

It appears to be useful to have a measure of concurrency for parallel and distributed systems. What is the meaning of the fact that in the system $S_1$ the concurrency is greater than in the system $S_2$? The number of transitions which can fire simultaneously in a Petri net, which models a given real system, can be used as an intuitive measure of the concurrency of that system.

As an application of this result, we show how we can compute the inferior concurrency-degree of a Petri net, using the residue of its reachability set.

The remainder of this paper is organized as follows. Section 2 presents the basic terminology, notation and results concerning Petri nets. In Section 3 we present a method to compute the residue of the reachability set of a marked P/T-net, and, in Section 4, we show how we can compute the inferior
concurrency-degree of a marked P/T-net. Section 5 concludes this paper and formulates some open problems.

2. PRELIMINARIES

Assume as known the basic terminology and notation about sets, relations and functions, vectors, multisets and formal languages.

This section establishes the basic terminology, notation, and results concerning Petri nets in order to give the reader the necessary prerequisites to understand this paper (for details the reader is referred to [1], [3], [6]). Mainly, it follows [3].

2.1. MULTISETS AND INTEGER VECTOR SETS

The set of integers is denoted by \( \mathbb{Z} \), and the set of nonnegative integers by \( \mathbb{N} \).

First, let us just briefly remind that a multiset \( m \), over a non-empty set \( S \), is a function \( m : S \to \mathbb{N} \), usually represented as a formal sum: \( \sum_{s \in S} m(s)s \). Sometimes it will be identified with a \( |S| \)-dimensional vector. The operations and relations on multisets are defined component-wise. \( S_{MS} \) denotes the set of all multisets over \( S \). The empty multiset \( \sum_{s \in S} 0s \) is denoted by \( \emptyset \). The size of the multiset \( m \) is defined as \( |m| = \sum_{s \in S} m(s) \). The multiset \( m \) is called infinite iff \( |m| = \infty \).

In order to represent the infinity (“+∞”), a new symbol, denoted usually by \( \omega \), is added to the set of non-negative integers \( \mathbb{N} \), giving \( \mathbb{N}_\omega = \mathbb{N} \cup \{ \omega \} \). The symbol \( \omega \) plays the role of “+∞”, the binary operations +, −, ·, min and max, and the relation < been extended to \( \mathbb{N}_\omega \) in the obvious way:

a) \( n + \omega = \omega + n = \omega + \omega = \omega \),

b) \( \omega - n = \omega \),

c) \( (n + 1) \cdot \omega = \omega \cdot (n + 1) = \omega \) and \( 0 \cdot \omega = \omega \cdot 0 = 0 \),

d) \( \min(n, \omega) = \min(\omega, n) = n \) and \( \max(n, \omega) = \max(\omega, n) = \omega \),

e) \( n < \omega \), for all \( n \in \mathbb{N} \).

\( \mathbb{N}_k^k \) (with \( k \geq 1 \)) denotes the set of \( k \)-dimensional nonnegative integer vectors, and \( \mathbb{N}_\omega^k \) denotes the set of \( k \)-dimensional vectors with components from the set \( \mathbb{N}_\omega \). The relations =, ≥, ≤ for vectors (from \( \mathbb{N}_k^k \) or \( \mathbb{N}_\omega^k \)) are understood componentwise and \( x < y \) is a shorthand for \( (x \leq y \text{ and } x \neq y) \). The binary operations +, −, min, and max are evaluated componentwise too. This means, for instance, that

\[
\min\left((x_1, \ldots, x_k), (y_1, \ldots, y_k)\right) = \left(\min(x_1, y_1), \ldots, \min(x_k, y_k)\right).
\]
Definition 2.1. The residue of a set \( X \subseteq \mathbb{N}^k \), abbreviated \( \text{res}(X) \), is the set of minimal elements of \( X \) (w.r.t. the partial order \( \leq \) defined on \( \mathbb{N}^k \))

\[
\text{res}(X) = \text{minimal}(X) = \{ x \in X \mid \forall y \in X - \{ x \} : y \not\leq x \}.
\]

Remark 2.1. By Dickson’s lemma [2], any subset of \( \mathbb{N}^k \) contains only finitely many incomparable vectors. Since, by the above definition, the elements of the residue of any subset of \( \mathbb{N}^k \) are incomparable (w.r.t. the partial order \( \leq \)), it follows that the residue of any subset of \( \mathbb{N}^k \) is a finite set.

Petri nets

A Place/Transition net, shortly P/T-net, (finite, with infinite capacities), abbreviated PTN, is a 4-tuple \( \Sigma = (S, T, F, W) \), where \( S \) and \( T \) are two finite non-empty sets (of places and transitions, resp.), \( S \cap T = \emptyset \), \( F \subseteq (S \times T) \cup (T \times S) \) is the flow relation and \( W : (S \times T) \cup (T \times S) \rightarrow \mathbb{N} \) is the weight function of \( \Sigma \) satisfying \( W(x, y) = 0 \) iff \( (x, y) \notin F \).

A marking of a PTN \( \Sigma \) is a function \( m : S \rightarrow \mathbb{N} \), i.e. a multiset over \( S \); sometimes it will be identified with a \(|S|\)-dimensional vector. The operations and relations on vectors are defined component-wise. \( \mathbb{N}^S \) denotes the set of all markings of \( \Sigma \).

A marked PTN, abbreviated \( mPTN \), is a pair \( \gamma = (\Sigma, m_0) \), where \( \Sigma \) is a PTN and \( m_0 \), called the initial marking of \( \gamma \), is a marking of \( \Sigma \).

In the sequel we often use the term “Petri net” (PN) or “net” whenever we refer to a PTN (\( mPTN \)) and it is not necessary to specify its type (i.e. marked or unmarked).

Let \( \Sigma \) be a Petri net, \( t \in T \) and \( w \in T^* \). The functions \( t^- : S \rightarrow \mathbb{N} \), \( t^+ : S \rightarrow \mathbb{N} \), \( \Delta t, \Delta w : S \rightarrow \mathbb{Z} \) are defined by: \( t^-(s) = W(s, t) \), \( t^+(s) = W(t, s) \), \( \Delta t(s) = t^+(s) - t^-(s) \), and

\[
\Delta w(s) = \begin{cases} 
0, & \text{if } w = \lambda \\
\sum_{i=1}^{n} \Delta t_i(s), & \text{if } w = t_1 t_2 \ldots t_n (n \geq 1)
\end{cases}, \forall s \in S.
\]

The sequential behaviour of a net \( \Sigma \) is given by the firing rule, which consists of

- the enabling rule: a transition \( t \) is enabled at a marking \( m \) in \( \Sigma \) (or \( t \) is fireable from \( m \)), abbreviated \( m[t]_{\Sigma} \), iff \( t^- \leq m \);
- the computing rule: if \( m[t]_{\Sigma} \), then \( t \) may occur yielding a new marking \( m' \), abbreviated \( m[t]_{\Sigma} m' \), defined by \( m' = m + \Delta t \).

In fact, for any transition \( t \) of \( \Sigma \) we have a binary relation on \( \mathbb{N}^S \), denoted by \( [t]_{\Sigma} \) and given by: \( m[t]_{\Sigma} m' \) iff \( t^- \leq m \) and \( m' = m + \Delta t \). If \( t_1, t_2, \ldots, t_n, n \geq 1 \), are transitions of \( \Sigma \), \( [t_1 t_2 \ldots t_n]_{\Sigma} \) will denote the classical product of the
relations \([t_1]_\Sigma, \ldots, [t_n]_\Sigma\). Moreover, we also consider the relation \([\lambda]_\Sigma\) given by \([\lambda]_\Sigma = \{(m, m) \mid m \in \mathbb{N}^S\}\).

Let \(\Sigma\) be a P/T-net, and \(m \in \mathbb{N}^S\). The word \(w \in T^*\) is called a transition sequence from \(m\) in \(\Sigma\) if there exists a marking \(m'\) of \(\Sigma\) such that \(m[w]_\Sigma m'\). Moreover, \(m'\) is called reachable from \(m\) in \(\Sigma\). We denote by \(TS(\Sigma, m) = \{w \in T^* \mid m[w]_\Sigma\}\) the set of all transition sequences from \(m\) in \(\Sigma\), and by \(|m]_\Sigma = RS(\Sigma, m) = \{m' \in \mathbb{N}^S \mid \exists w \in TS(\Sigma, m) : m[w]_\Sigma m'\}\) the set of all reachable markings from \(m\) in \(\Sigma\).

If \(\gamma = (\Sigma, m_0)\) is a mPTN, for the case \(m = m_0\), the set \(TS(\Sigma, m_0)\) is abbreviated by \(TS(\gamma)\) and sometimes it is called the language of \(\gamma\), and the set \(RS(\Sigma, m_0)\) is abbreviated by \(RS(\gamma)\) (or \([m_0]_\gamma\)) and it is called the reachability set of the net \(\gamma\).

Functions \(m : S \rightarrow \mathbb{N}_\omega\) are called pseudo-markings; sometimes they are identified with \(|S|\)-dimensional vectors. \(\mathbb{N}_\omega^S\) denotes the set of all pseudo-markings. If \(m(s) = \omega\), then the component \(s\) of \(m\) is called an \(\omega\)-component; \(\Omega(m)\) denotes the set of all \(\omega\)-components of \(m\), i.e. \(\Omega(m) = \{s \in S \mid m(s) = \omega\}\). Obviously, any marking is a pseudo-marking. The firing rule is extended to pseudo-markings in the straightforward way: (ER) \(m[t]_\Sigma\) iff \(t^- \leq m\); (CR) \(m[t]_\Sigma m'\) iff \(m[t]_\Sigma\) and \(m' = m + \Delta t\). The other notions from markings (i.e. transition sequence, reachable marking etc.) are extended similarly to pseudo-markings.

3. THE RESIDUE OF THE REACHABILITY SET

The residue of the reachability set has some practical importance for P/T-nets, for instance it can be used to compute the concurrency-degree of a Petri net (as we see in the next section).

As we already know, for any marked P/T-net \(\gamma = (\Sigma, M_0)\), the residue of its reachability set, i.e. \(\text{res}([m_0]_\gamma)\), is a finite set (see Remark 2.1), but the problem is how to compute it.

In this section we show how we can compute the residue of the reachability set of a P/T-net, using an algorithm from [8].

**Definition 3.1.** For each \(m \in \mathbb{N}_\omega^k\), let \(\text{reg}(m) = \{x \in \mathbb{N}^k \mid x \leq m\}\) be the region specified by \(m\).

**Definition 3.2.** For each set of integer vectors \(X \subseteq \mathbb{N}^k\) we define the predicate \(p_X : \mathbb{N}_\omega^k \rightarrow \{\text{true}, \text{false}\}\) by \(p_X(m) = (\text{reg}(m) \cap X \neq \emptyset)\), for all \(m \in \mathbb{N}_\omega^k\).

A set \(X \subseteq \mathbb{N}^k\) is said to have property RES iff the predicate \(p_X(m)\) is decidable for each \(m \in \mathbb{N}_\omega^k\).

**Definition 3.3.** A set \(X \subseteq \mathbb{N}^k\) is called right-closed iff \(\forall x \in X, \forall y \in \mathbb{N}^k : x \leq y \Rightarrow y \in X\).
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Valk & Jantzen [8] proved the following result about the computability of the residue of vector sets:

**Theorem 3.1.** Let $X \subseteq \mathbb{N}^k$ be a right-closed set. Then $\text{res}(X)$ can be effectively constructed iff $X$ has property RES.

More exactly, the inverse implication was proved by giving an algorithm which compute the residue of a set $X \subseteq \mathbb{N}^k$ which has property RES (remark: the hypothesis $X$ being a right-closed set was not used in the proof of the inverse implication).

The algorithm and its correctness proof can be found in [8]. Also, that article presents some applications to decidability problems in Petri nets of this result.

Thus, to compute the residue of the reachability set of a P/T-net, it suffices to show that the reachability set of any P/T-net has the property RES.

**Definition 3.4.** The Reachability Problem (RP) : Given a mPTN and $m$ a marking of $\gamma$, is $m$ reachable in $\gamma$? The predicate associated with this problem is: $\text{RP}(\gamma, m) = \text{true}$ iff $m \in [m_0]_{\gamma}$.

**Remark 3.1.** It is well-known [5], [4] that the problem (RP) is decidable (i.e. the predicate RP is recursively decidable).

**Propozitia 3.1.** Let $\gamma = (\Sigma, m_0)$ be a mPTN. Then its reachability set $[m_0]_{\gamma}$ has the property RES.

**Proof.** Let $\gamma = (\Sigma, m_0)$ be a marked P/T-net, with $\Sigma = (S, T, F, W)$, and $[m_0]_{\gamma}$ its reachability set. We have to prove that the question $\text{reg}(m) \cap [m_0]_{\gamma} \neq \emptyset$ is decidable for any pseudo-marking $m \in \mathbb{N}_S^\omega$.

Let $m \in \mathbb{N}_S^\omega$. We distinguish two cases:

i) $\Omega(m) = \emptyset$, i.e. $m \in \mathbb{N}_S^\omega$. In this case, the set $\text{reg}(m)$ is finite and we have that

$$p_{[m_0]_{\gamma}}(m) = \text{true} \iff \exists m' \in \text{reg}(m) : \text{RP}(m') = \text{true}. \quad (1)$$

Therefore, in this case the predicate $p_{[m_0]_{\gamma}}(m)$ is decidable;

ii) $\Omega(m) \neq \emptyset$, i.e. $m \in \mathbb{N}_S^\omega - \mathbb{N}_S^\omega$. In this case relation (1) still holds, but the set $\text{reg}(m)$ is infinite.

If $\Omega(m) = S$, then $m = (\omega, \ldots, \omega)$ and $\text{reg}(m) = \mathbb{N}_S^\omega$. Thus, the predicate $p_{[m_0]_{\gamma}}(m)$ in this case is decidable, because $m_0 \in \mathbb{N}_S^\omega \cap [m_0]_{\gamma}$.

Therefore, let us consider that $\emptyset \subset \Omega(m) \subset S$. Without loss of generality, we may assume that the set of locations of the net $\gamma$, $S = \{s_1, s_2, \ldots, s_k\}$, is indexed in such an order that $\Omega(m) = \{s_{j+1}, \ldots, s_k\}$, with $1 \leq j < k$. Thus, $\{s_1, s_2, \ldots, s_j\}$ is the set of finite components of the marking $m$, i.e.

$$m(s_1) = n_1 \in \mathbb{N}, \ldots, m(s_j) = n_j \in \mathbb{N},$$
\( m(s_{j+1}) = \omega, \ldots, m(s_k) = \omega. \)

Therefore, \( \text{reg}(m) = \{ x \in \mathbb{N}^k \mid x_i \leq n_i, \forall 1 \leq i \leq j \} \) and in this case the question \( \text{reg}(m) \cap [m_0]_\gamma \neq \emptyset ? \) becomes the following decision problem:

\((*)\) Given any \( mPTN \gamma \) (with \( k \) locations) and the integers \( n_1, \ldots, n_j \in \mathbb{N} \), with \( 1 \leq j < k \), there exists a reachable marking \( m' \in [m_0]_\gamma \) such that \( m'(s_i) \leq n_i \), for all \( 1 \leq i \leq j \)?

But the decision problem \((*)\) is reducible to the following decision problem:

\((**)\) Given any \( mPTN \gamma \) (with \( k \) locations) and the integers \( n_1, \ldots, n_j \in \mathbb{N} \), with \( 1 \leq j < k \), there exists a reachable marking \( m' \in [m_0]_\gamma \) such that \( m'(s_i) = n_i \), for all \( 1 \leq i \leq j \)?

Indeed, the problem \((*)\) is equivalent to a finite conjunction of problems \((**)*\).

Now, we prove that the problem \((**)*\) is decidable.

So, let \( \gamma = (\Sigma, m_0) \) be a \( mPTN \), with \( \Sigma = (S,T,F,W) \), and the integers \( n_1, \ldots, n_j \in \mathbb{N} \), with \( 1 \leq j < k \), where \( k = |\Sigma| \) (\( S = \{ s_1, \ldots, s_k \} \)).

We construct a new \( mPTN \gamma' = (\Sigma', m'_0) \), with \( \Sigma' = (S', T', F', W') \), defined in the following way:

(i) \( S' = S \cup \{ \bar{s}, \bar{s} \} \), with two new locations \( \bar{s} \) and \( \bar{s} \);  

(ii) \( T' = T \cup \{ t'_j+1, \ldots, t'_k \} \cup \{ \bar{t}, \bar{t} \} \), with \( k - j + 2 \) new transitions \( t'_j+1, \ldots, t'_k \), \( \bar{t} \) and \( \bar{t} \); 

(iii) the flow relation \( F' \subseteq (S' \times T') \cup (T' \times S') \) is given by

\[
F' = F \cup \{(\bar{s}, t), (t, \bar{s}) \mid t \in T\} \cup \{(\bar{s}, \bar{t}), (\bar{t}, \bar{s}) \} \cup \{(s_i, \bar{t}) \mid 1 \leq i \leq j \} \cup \{(s_i, t'_i), (\bar{s}, t'_i), (t'_i, \bar{s}) \mid j < i \leq k \};
\]

(iv) the weight function \( W' : (S' \times T') \cup (T' \times S') \rightarrow \mathbb{N} \) is given by

\[
W'(x, y) = \begin{cases} 
W(x, y) & \text{if } (x, y) \in F \\
1 & \text{if } (x, y) \in \{(\bar{s}, t), (t, \bar{s}) \mid t \in T\} \\
1 & \text{if } (x, y) \in \{(\bar{s}, \bar{t}), (\bar{t}, \bar{s}) \} \\
1 & \text{if } (x, y) = (s_i, \bar{t}) \text{, } 1 \leq i \leq j \\
1 & \text{if } (x, y) \in \{(s_i, t'_i), (\bar{s}, t'_i), (t'_i, \bar{s}) \text{, } j < i \leq k \} \\
0 & \text{if } (x, y) \notin F' 
\end{cases}
\]

(v) the initial marking \( m'_0 \in \mathbb{N}^{S'} \) is given by

\[
m'_0(s) = \begin{cases} 
m_0(s) & \text{if } s \in S \\
1 & \text{if } s = \bar{s} \\
0 & \text{if } s = \bar{s} 
\end{cases}
\]
By the way $\gamma'$ is constructed, it is easy to see that to each transition sequence in $\gamma$ of the form
\begin{equation}
m_0 [w]_\Sigma m, \tag{5}
\end{equation}
such that
\begin{equation}
w \in T^* , m \in [m_0]_\gamma \text{ and } m(s_i) = n_i, \forall 1 \leq i \leq j
\end{equation}
it corresponds a transition sequence in $\gamma'$ of the form
\begin{equation}
m'_0 [w]_\Sigma m'[\tilde{t}]_\Sigma m'' [w']_\Sigma 01 [\tilde{t}]_\Sigma 0, \tag{7}
\end{equation}
where the markings $m', m'', 01, 0 \in N^{S'}$ are defined by
\begin{equation}
m'(s) = \begin{cases}
    m(s), & \text{if } s \in S \\
    1, & \text{if } s = \bar{s} \\
    0, & \text{if } s = s_i
\end{cases}, \quad m''(s) = \begin{cases}
    m(s), & \text{if } s \in \{s_{j+1}, \ldots, s_k\} \\
    0, & \text{if } s \in \{\bar{s}\} \cup \{s_1, \ldots, s_j\} \\
    1, & \text{if } s = \bar{s}
\end{cases}, \tag{8}
\end{equation}
and $0(s) = 0, \forall s \in S'$, $01(s) = 0, \forall s \in S' \setminus \{\bar{s}\}$ and $01(\bar{s}) = 1$,
and the word $w' \in T'^*$ is given by
\begin{equation}
w' = t'_{j+1}^{m(s_{j+1})} \cdots t'_k^{m(s_k)} \tag{9}
\end{equation}
(the order of the transitions $t'_{j+1}, \ldots, t'_k$ in the sequence $w'$ is not important, i.e. more generally $w'$ is an arbitrary sequence $w' \in \{t'_{j+1}, \ldots, t'_k\}^*$ such that $t'_i$ occurs exactly $m(s_i)$ times in the word $w'$, for each $j < i \leq k$).

Moreover, the marking $0$ is reachable in $\gamma'$ only by transition sequences of the form (7).

From these facts, it follows easily that there exists a reachable marking $m \in [m_0]_\gamma$ such that $m(s_i) = n_i$, for all $1 \leq i \leq j$, iff the marking $0$ is reachable in the net $\gamma'$. Thus, we can conclude that the problem (**) is decidable.

Therefore, the predicate $p_{[m_0]_\gamma}(m)$ is decidable in this case, too. ■

As a consequence, we have:

**Corollary 3.1.** The residue of the reachability set of any mPTN is computable.

**Proof** This affirmation follows from Theorem 3.1 and Proposition 3.1 (the residue set $\text{res}([m_0]_\gamma)$ is constructed by the algorithm given in [8]). ■

4. APPLICATION: COMPUTING THE CONCURRENCE-DEGREES FOR P/T-NETS

We recall the notion of concurrency-degree for Petri nets, which was defined in [9] in a more general way than the original definition introduced in [7],
by taking into consideration also the transitions concurrently enabled with themselves.

Let us briefly remind those definitions from [9].

**Definition 4.1.** A step $Y$ of a P/T-net $\Sigma$ is any non-empty and finite multiset over the set of transitions of $\Sigma$. The set of all steps of $\Sigma$ is denoted by $\mathcal{Y}(\Sigma)$.

**Definition 4.2.** Let $\Sigma$ be a P/T-net and $m$ an arbitrary marking of $\Sigma$.

i) The step-type concurrent behaviour of the net $\Sigma$ is given by the step firing rule, which consist of

- the step enabling rule: a step $Y$ is enabled at the marking $m$ in $\Sigma$ (or $Y$ is fireable from $m$), and we say also that $Y$ is a multiset of transitions concurrently enabled at $m$, abbreviated $m\{Y\}_\Sigma$, iff $\sum_{t \in T} Y(t) \cdot t^- \leq m$;

- the step computing rule: if the step $Y$ is enabled at $m$ in $\Sigma$, then $Y$ may occur at $m$ yielding a new marking $m'$, abbreviated $m\{Y\}_\Sigma m'$, defined by $m' = (m - \sum_{t \in T} Y(t) \cdot t^-) + \sum_{t \in T} Y(t) \cdot t^+$.

ii) A step $Y$ is called a maximal step enabled at the marking $m$ in $\Sigma$, if $Y$ is enabled at $m$ in $\Sigma$ and there exists no step $Y'$ enabled at $m$ in $\Sigma$ with $Y' > Y$.

**Definition 4.3.** Let $\Sigma$ be a P/T-net and $m$ an arbitrary marking of $\Sigma$. The concurrency-degree at the marking $m$ of the net $\Sigma$ is defined by

$$d(\Sigma, m) = \sup \{ |Y| ; Y \in \mathcal{Y}(\Sigma) \land m\{Y\}_\Sigma \} . \quad (10)$$

**Remark 4.1.** Intuitively, the notion of concurrency-degree at a marking $m$ of a Petri net $\Sigma$ represents the maximum (i.e. the supremum) number of transitions concurrently enabled at the marking $m$.

**Definition 4.4.** Let $\gamma = (\Sigma, m_0)$ be a marked P/T-net.

i) The inferior concurrency-degree of the net $\gamma$ is defined by

$$d^- (\gamma) = \min \{ d(\Sigma, m) \mid m \in [m_0]_\gamma \} . \quad (11)$$

ii) The superior concurrency-degree of the net $\gamma$ is defined by

$$d^+ (\gamma) = \sup \{ d(\Sigma, m) \mid m \in [m_0]_\gamma \} . \quad (12)$$

iii) If $d^- (\gamma) = d^+ (\gamma)$, then this number is called the concurrency-degree of $\gamma$ and it is denoted by $d(\gamma)$.

**Remark 4.2.** Directly from definitions we have

i) $0 \leq d^- (\gamma) \leq d^+ (\gamma) \leq \infty$.

ii) The inferior concurrency-degree of the net $\gamma$, $d^- (\gamma)$, represents the minimum number of transitions maximal concurrently enabled at any reachable marking of $\gamma$. In other words, at any reachable marking $m$ of $\gamma$ there exist
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d−(γ) transitions concurrently enabled at m.

iii) The superior concurrency-degree of the net γ, \( d^+(\gamma) \), represents the supremum number of transitions maximal concurrently enabled at any reachable marking of γ. In other words, at any reachable marking \( m \) of γ there exist at most \( d^+(\gamma) \) transitions concurrently enabled at \( m \).

iv) The concurrency-degree of γ means that at any reachable marking \( m \) of γ there exist \( d(\gamma) \) transitions concurrently enabled at \( m \), and there is no reachable marking \( m' \) of γ with more than \( d(\gamma) \) transitions concurrently enabled at \( m' \).

What about the computability of these concurrency-degrees for P/T-nets?

In [9], we proved that the concurrency-degree at a marking, \( d(\Sigma, m) \), is computable for any \( \Sigma \), and for any marking \( m \), by showing that the computation of the degree \( d(\Sigma, m) \) is reducible to solving the following integer linear programming problem

\[
\begin{align*}
(P_{\Sigma,m}) & \quad \max \sum_{1 \leq i \leq n} x_i \\
& \quad \left\{ \begin{array}{l}
\sum_{1 \leq i \leq n} W(s_j, t_i) \cdot x_i \leq m(s_j) \quad \forall 1 \leq j \leq k \\
1 \leq x_1, x_2, \ldots, x_n \in \mathbb{N}
\end{array} \right.
\end{align*}
\] (13)

where \( x_i \) are the variables of the problem and the non-negative integers \( W(s_j, t_i) \) \( (= t_i^-(s_j)) \), representing the weights of the net \( \Sigma \), are the coefficients of the variables in the linear constraints (where \( S = \{s_1, \ldots, s_k\} \) and \( T = \{t_1, \ldots, t_n\} \)).

In [9], we also proved that the superior concurrency-degree, \( d^+(\gamma) \), is computable for any \( mPTN \gamma = (\Sigma, m_0) \)

\[
\begin{align*}
d^+(\gamma) &= \max \{ d(\Sigma, m) \mid m \in MCS(\gamma) \}.
\end{align*}
\] (14)

where \( MCS(\gamma) \) is the minimal coverability set of γ.

Now, we have the following result about the inferior concurrency-degree:

**Theorem 4.1.** The inferior concurrency-degree, \( d^-(\gamma) \), is computable for any \( mPTN \gamma = (\Sigma, m_0) \):

\[
\begin{align*}
d^-(\gamma) &= \min \{ d(\Sigma, m) \mid m \in res([m_0], \gamma) \}.
\end{align*}
\] (15)

**Proof** Relation (15) follows easily from Definition 4.4 and the fact that the degree \( d(\Sigma, m) \) is a monotone increasing function in the argument \( m \).

Proceeding from Corollary 3.1, using this relation we conclude that the inferior concurrency-degree \( d^-(\gamma) \) is computable.

5. CONCLUSION

In this paper we have presented a method to compute the residue (i.e., the set of minimal elements) of the reachability set of a marked P/T-net, by applying a more general algorithm from [8].
As an application of this result, we have showed how we can compute the inferior concurrency-degree of a marked P/T-net, using the residue of its reachability set.

Since Petri nets are used as suitable models for real-world parallel or distributed systems, the concurrency-degrees defined for Petri nets are an intuitive measure of the concurrency of the modelled systems, and, therefore, they have some practical importance. For instance, they are useful for the evaluation of the models in the process of designing such a system: after making a model of that system as a Petri net, the study of the concurrency-degree of the model will give information to the designers about the concurrency of that system, allowing them to notice the inefficient components of the system, and to make improvements of the model by remodelling those components.

Therefore, it is important to be able to compute the concurrency-degrees for Petri nets.

Some problems remain to be studied, for example: finding better algorithms to compute the residue of the reachability set for P/T-nets; finding better algorithms to compute the concurrency-degrees of P/T-nets; making some case studies on models of real-world systems.

References