A DIRECT METHOD AND ITS APPLICATION TO A LINEAR HYDROMAGNETIC STABILITY PROBLEM

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Abstract The main steps (Section 1), advantages and drawbacks (Section 3) of the direct method are discussed. A few tricks leading to easier computations and some open problems are revealed too (Section 3). The secular manifolds, the characteristic manifolds and their bifurcation sets, called the false neutral manifolds, are described (Section 2). The survey of the existing results obtained by applying the direct method to a particular Couette flow and a few new results are presented (Section 4).

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1. THE DIRECT METHOD BASED ON THE CHARACTERISTIC EQUATION

After more than hundred years of its existence, the linear theory of hydrodynamic and hydromagnetic stability is still of interest mainly due to two facts: this theory provides the necessary conditions for instability and it is much simpler than the non linear theory.

The eigenvalue problems governing the linear stability of certain fluid flows, and consisting in two points problems for systems of ordinary differential equations (ode's) with constant coefficients, were solved by the direct method and four methods based on Fourier series. Here we deal with the simplest direct method based on the characteristic equation. We point out its main steps, the advantage over other methods, and the main tricks used to simplify the computations.

The eigenvalue problems we deal with are either of the form of an ode

$$\sum_{k=0}^{n} a_{n-k} D^k u = 0, \quad x \in (-0.5, 0.5) \tag{1}$$

and n homogeneous boundary conditions

$$B_r u = 0, \quad r = \overline{1,n}, \quad at \quad x = \pm 0.5 \tag{2}$$

where $D = \frac{d}{dx}$, $u : [-0.5, 0.5] \to \mathbf{R}$, $u \in C^\infty[-0.5, 0.5]$ is the unknown function and the constant coefficients a_i depend on m physical parameters $\mathcal{R}_{\overline{1,m}}$, or of the form of a system of ode's

$$\mathbf{AU} = \mathbf{0}, \quad x \in (-0.5, 0.5) \tag{3}$$

and the boundary conditions

$$B_r U_i = 0, \quad r = \overline{1,n}, \quad i = \overline{1,s} \quad at \quad x = \pm 0.5 \tag{4}$$

where $\mathbf{A}$ is a $s \times s$ differential matrix the entries of which are polynomials, with constant coefficients depending on $\mathcal{R}_{\overline{1,m}}$, in the derivative D. The order of this system is n.

By an eigenvalue of the problem (1), (2) or (3), (4) we understand one value of the chosen parameter, say $\mathcal{R}_1$, to which nontrivial solutions (called eigenvectors or eigenfunctions) u of the problem correspond, each eigenvalue is a root of the secular equation, obtained by replacing the general solution of (1) into (2) or (3) into (4). In this way the eigenvalue depends on all other parameters. Therefore, the secular equation defines some manifolds. The most convenient (physically) *secular manifold* is called the *neutral manifold* (NM). In $\mathbf{R}^m$ it separates the domain of linear and nonlinear stability. Consequently, first our aim is to determine the secular equation.

First, consider the problem (1), (2). In order to determine the general solution of (1) we formally look for it in the form $u = e^{\lambda x}$ and replace this in (1) to obtain the characteristic equation

$$f(\lambda) = 0, \tag{5}$$

where $f(\lambda)$ is a n degree polynomial in λ, the coefficients of which depend on $\mathcal{R}_{\overline{1,m}}$. Up to a null measure set, for the points $(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) \in \mathbf{R}^m$ the roots of (5) are multiple. Denote by

$$g(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) = 0, \tag{6}$$

the equations defining some manifolds the points of which correspond to multiple roots of (5). These manifolds are referred to as *false neutral manifolds* (FNM) and they have various topological dimensions smaller than m.

Let $\lambda_{1,\cdots,p}$ be the roots of (5) and let m_j , $j = 1, \cdots, p$ be their multiplicity, such that $\sum_{j=1}^{p} m_j = n$. These multiplicities are deduced by taking into account the Viète relations and the fact that most physical parameters are real and positive. Corresponding to $\lambda_{1,\cdots,p}$, a basis for the vector space consisting of the nontrivial solutions of (1) is $e^{\lambda_1 x}$, $xe^{\lambda_1 x}$, $\cdots$, $x^{m_1-1}e^{\lambda_1 x}$, $e^{\lambda_2 x}$, $xe^{\lambda_2 x}$, $\cdots$, $x^{m_2-2}e^{\lambda_2 x}$, $\cdots$, $e^{\lambda_p x}$, $xe^{\lambda_p x}$, $\cdots$, $x^{m_p-1}e^{\lambda_p x}$. Thus, in the case of the multiple roots of (5), the general solution of (1) reads

$$u(x) = \sum_{j=1}^{p} \sum_{k=0}^{m_j-1} A_k^{(j)} x^k e^{\lambda_j x}; \tag{7}$$

in the case of the simple roots of (5), i. e. $m_1 = \cdots = m_p = 1$, $p = n$, it is

$$u(x) = \sum_{i=1}^{n} A_i e^{\lambda_i x}. \tag{8}$$

Introducing (8) and (7) into (2) we obtain the secular equation

$$F^*(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) = 0, \tag{9}$$

for the case of multiple roots of (5), and

$$F(\mathcal{R}_{1,}, \cdots, \mathcal{R}_m) = 0, \tag{10}$$

for the case of simple roots of (5).

2. CHARACTERISTIC MANIFOLDS AND THEIR BIFURCATION SETS. SECULAR AND NEUTRAL MANIFOLDS

Involving only $sinh(\lambda_i/2)$ and $cosh(\lambda_i/2)$, $i = 1, \cdots, p$, and powers of λ_i, in general, the secular equation is trascendent. It yields the dependence of $\mathcal{R}_1$ on $\mathcal{R}_2, \cdots, \mathcal{R}_m$. This represents the end of the method.

The equation (10) defines a (secular) manifold in the m-dimensional space of parameters. This manifold can have an infinity of *sheets*. From the physical point of view, the most convenient sheet is just the neutral manifold.

In (10) F is a determinant the columns of which have the same form in λ_i, i corresponding to the i-th column. Formally, (10) is satisfied on FNM because for each point of (6) at least two roots λ_i and λ_j coincide, and, therefore, the columns i and j of (10) are identical. Thus, formally, every FNM is a secular manifold. In fact, this is not true because (10) is not defined on FNM and, therefore, (10) is not entitled to serve as a secular equation for the points of FNM. Concrete examples [6] show that FNM could be physically more convenient (if it would be a secular manifold) than the true neutral manifold given by (10). Whence the name of *false* bearing by the manifolds defined by (6). In these cases the direct numerical computations are invalid. In other examples, parts of FNM proved to be limits of the secular manifolds of (10), or even of the neutral manifolds defined by (10). This is the reason why, apart from (10) we must solve all secular equations (9), corresponding to all multiplicities m_j and, so, to all manifolds (6). It is only in this way that we can deduce which points of $\mathbf{R}^m$ are secular points, indeed. Formally, the secular equations (9) are deduced from (10), namely writing the column j for λ_j while the column $j+k$ $k = 1, \cdots, m_j - 1$ are obtained by differentiating k times the $j+k$-th column of (10) with respect to λ_{j+k} and then replacing λ_{j+k} by λ_j.

The equations (9) are valid only on the manifold (6). Consequently, if some manifold defined by (6) is q dimensional, then the secular manifold of (9), when it exists, is $q-1$ dimensional. In this way, the secular manifolds are: those of dimension $m-1$ (corresponding to (10)) and those of smaller dimension (corresponding to (9) and situated on the manifolds (6)). Thus the complicated problem concerning the relative position, intersection and geometric structure of these manifolds arises [11].

Only seldom the roots of the secular equations (9) exist.

The false neutral manifolds defined by (6) are bifurcation sets for the characteristic manifold (10), the dimension of which is $m+1$ if $\lambda_{1,\cdots,n} \in \mathbf{R}$ and $m+2$ if some $\lambda_i \in \mathbf{C}$. Some among the FNM consist of bifurcarion points for some other FNM or are the bifurcation sets of these ones. We recall that the bifurcation set B of a manifold M is the projection on the parameter space of the set B_M of the bifurcation points of M, corresponding to the points where at least two sheets of M coalesce. Hence B is the set of bifurcation values corresponding to B_M.

3. ADVANTAGE AND DRAWBACKS OF THE DIRECT METHOD AND TRICKS TO ITS EASIER APPLICATION. OPEN PROBLEMS

Among the advantages of the direct method we quote :1) the very simple general form (7) of the solution of (1) and the corresponding secular equations have only a finite number of terms, therefore these solutions are *exact*. The method based on Fourier and asymptotic series involve an infinite number of terms of the solution representation and of the secular equations. Therefore, they are approximate, even if they are called exact; 2) among all methods we know, it is the only one which provides the *false secular points*. It shows how dangerous is to apply numerical methods without a theoretical support; 3) this method is the *simplest* among all methods used for problems (1), (2); 4) the direct method applies *irrespective the form of the boundary conditions*. In the case of the problem (3), (4) the coefficients of various functions are related, such that the boundary conditions can be easily written for a single component of the solution $\mathbf{U}$, which in other methods generally is impossible

[12]; 5) in the direct method, in order to get a simpler form of the secular equation, the columns of F in (10) are divided by $cosh(\lambda_i/2)$. If λ_i are purely imaginary, then this division is forbidden but the condition $cosh(\lambda_i/2) = 0$ yields secular points valid for *every boundary conditions*. This is a striking property with basic implications in applications.

When applying the Fourier series based methods the expressions of coefficients have as denominator just the expression $f(\lambda)$ from (5). Therefore they ceased to be valid for $\lambda = \lambda_i$ and, so, the direct method must be applied in order to complete this study.

Often, instead of the given problem (1), (2) or (3), (4), the problems for the even and odd part of the solution are solved. In spite of the fact that the trascendent secular equation has a finite number of terms containing the powers of λ_i and hyperbolic sine and cosine of $\lambda_i/2$, the solution of this equation is pratically impossible to obtain. In the same way, for higher-order ode's of the governing eigenvalue problems no closed-form solutions of the characteristic equation (5) are known. This is why, in carring out numerical computations, the equations (5) and (10) or (5) and (9) must be solved simultaneously [1].

In the case of ode's containing only even-order derivatives, a suitable change of variables, e. g. $\mu = \lambda^2 - a^2$, leads to a characteristic equation the degree of which is half of the initial degree. For the equation in the closed-form solution is immediate.

The direct method was applied in hydrodynamic stability theory only by us and our collaborators starting with the year 1977. Apart from very simple situations, a systematical theoretical investigation of the bifurcation of the involved manifolds is a difficult open problem. This is the case especially when more than three parameters occur. This was also remarked by Collatz in [11]. It is also in these cases that the determination of the multiplicity of the characteristic roots is another open problem. The separate numerical solution of the characteristic equation and of the secular equations require numerical methods specific to bifurcation theory. Therefore, we can avoid this by numerically solving both these equations simultaneously [1].

4. APPLICATION OF THE DIRECT METHOD TO THE CASE OF A PARTICULAR COUETTE FLOW

Consider the Couette flow of a fluid situated between two rotating coaxial cylinders situated at a very small distance. The fluid is electrically conducting and subject to an axial magnetic field. The eigenvalue problem governing the linear stability of this flow against normal mode perturbation reads [6]

$$\left\{(D^2 - a^2)^2 + Qa^2\right\}^2 v = -Ta^2(D^2 - a^2)v, \quad -0.5 < z < 0.5 \qquad (1')$$

$$Dv = (D^2 - a^2)v = \Big\{(D^2 - a^2)^2 + Qa^2\Big\}v = D\Big\{(D^2 - a^2)^2 + Qa^2\Big\}v = 0, \quad (2')$$

where T, $Q > 0$ are the dimensionless Taylor and Chandrasekhar numbers respectively, a is the positive wavenumber, z is the vertical coordinate, $D = \frac{d}{dz}$ and v is the unknown stream function. The associated characteristic equation is

$$(\lambda^2 - a^2)^4 + 2Qa^2(\lambda^2 - a^2)^2 + Ta^2(\lambda^2 - a^2) + Q^2a^4 = 0. \qquad (5')$$

With the notation $\mu = \lambda^2 - a^2$ it becomes

$$\mu^4 + 2Qa^2\mu^2 + Ta^2\mu + Q^2a^4 = 0. \qquad (5'')$$

Introduce the surfaces

$$\mathcal{C} : T = T^* \equiv 16aQ\sqrt{Q}/(3\sqrt{3}), \qquad (6'_1)$$

$$\mathcal{C}_1 : T = T^{**} \equiv (Q^2 + a^2)^2, \qquad (6'_2)$$

and denote by $\mathcal{C}^*$ their intersection, i. e.

$$\mathcal{C}^* : Q = 3a^2, \quad T = 16a^4. \qquad (6'_3)$$

The projection of $\mathcal{C}^*$ on the plane (a, Q) is

$$\mathcal{C}^*_* : Q = 3a^2. \qquad (6'_4)$$

Consider $a, Q, T > 0$. Then, for $(a, Q, T) \in \mathbf{R}^3 \setminus (C \cup C_1)$, $(5')$ has eight mutually disjoint roots $\lambda_{1,\cdots,8}$ and $(5'')$ has four mutually distinct roots $\mu_{1,\cdots 4}$, related to $\lambda_{1,\cdots,8}$ by the relations: $\lambda_{1,5} = \pm\sqrt{\mu_1 + a^2}$, $\lambda_{2,6} = \pm\sqrt{\mu_2 + a^2}$, $\lambda_{3,7} = \pm\sqrt{\mu_3 + a^2}$, $\lambda_{4,8} = \pm\sqrt{\mu_4 + a^2}$, for $\mu_i > -a^2$; $\lambda_{1,5} = \pm\sqrt{\mu_1 + a^2}$, $\lambda_{2,6} = \overline{\lambda_{1,5}}$, $\lambda_{3,7} = \pm\sqrt{\mu_1 + a^2}$, $\lambda_{4,8} = \overline{\lambda_{3,7}}$ for $\mu_i \in \mathbf{C}$ or $\mu_i < -a^2$. Since it is much easier to study $(5'')$ than $(5')$, let us relate the multiplicities of the roots of these two characteristic equations. Thus, $\mu_1 = \mu_2$ implies either $\lambda_1 = \lambda_2 > 0$, $\lambda_5 = \lambda_6$ if $\mu_1 = \mu_2 \in \mathbf{R}$, or $\lambda_1 = \lambda_2 = 0$ if $\mu_1 = \mu_2 = -a^2$, or λ_1 is purely imaginary and $\lambda_1 = -\lambda_2$, $\lambda_5 = -\lambda_6$, if $\mu_{1,2} < -a^2$. Therefore, almost everywhere in $\mathbf{R}^3$, more exactly for $(a, b, Q) \in \mathbf{R}^3 \setminus (C \cup C_1)$, $(5'')$ and $(5')$ have mutually distinct roots; for $(a, b, Q) \in C \setminus C^*$, $(5'')$ has two equal real roots $\mu_1 = \mu_2 \neq -a^2$ and $(5')$ has two pairs of equal and nontrivial roots $\lambda_1 = \lambda_2$, $\lambda_5 = \lambda_6$; for $(a, b, Q) \in C^*$, $(5'')$ has two roots equal to $-a^2$, i. e. $\mu_1 = \mu_2 = -a^2$ and $(5')$ has four trivial roots $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$; for $(a, b, Q) \in C_1 \setminus C^*$, $(5'')$ has one root equal to $-a^2$ and the others different, i. e. $\mu_1 = -a^2$ μ_2, μ_3, $\mu_4 \neq -a^2$. In this case $\lambda_1 = \lambda_5 = 0$, all other roots of $(5')$ are nonvanishing and mutually disjoint [2],[4].

In the space (μ, a, Q, T) the characteristic manifold defined by $(5'')$ has four sheets which coalesce along the surface $\mu_1 = \mu_2 = -a\sqrt{Q/3}$, $T = T^*$. In the space (λ, a, Q, T) the characteristic manifold defined by $(5')$ has eight sheets, two of them coalescing along the surface $\lambda_1 = \lambda_5 = 0$, $T = T^{**}$, $Q \neq 3a^2$;two pairs of them coalescing along the surfaces $\lambda_1 = \lambda_2 = \sqrt{a^2 - a\sqrt{Q/3}}$, $T = T^*$, $Q \neq 3a^2$ and $\lambda_5 = \lambda_6 = -\sqrt{a^2 - a\sqrt{Q/3}}$, $T = T^*$, $Q \neq 3a^2$; four of them coalesce along the curve $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$, $T = 16a^4$, $Q = 3a^2$. These results suggested us the following

Theorem 1. *The surface $\mathcal{C}$ is a bifurcation set for the characteristic manifolds $(5')$ and $(5'')$, the surface C_1 for $(5')$ but not for $(5'')$, C^* for $(5')$ C^*_* for $(5')$ taken on C. In addition, C^*_* is the bifurcation set for $C \cup C_1$.*

Proof. If the characteristic manifold defined by $(5')$ has bifurcation points, then the equation $(5')$ has multiple roots. For the case of a double root, this root satisfies the equation $(5')$ and the equation obtained by differentiating $(5')$ with respect to λ, i. e.

$$2\lambda[4(\lambda^2 - a^2)^3 + 4Qa^2(\lambda^2 - a^2) + Ta^2] = 0. \qquad (5')_1$$

If $\lambda = 0$ is the double root, $(5')$ implies $T = T^{**}$, henceC_1 is the bifurcation set for $(5')$. If the double root is one of the roots of $(5')$ and the equation

$$4(\lambda^2 - a^2)^3 + 4Qa^2(\lambda^2 - a^2) + Ta^2 = 0, \qquad (5')_2$$

then, by the Euler algorithm, it follows that the double root can be either $\lambda = \sqrt{a^2 - \frac{4aQ}{3\sqrt{T}}}$ or $\lambda = -\sqrt{a^2 - \frac{4aQ}{3\sqrt{T}}}$, both leading to $T = T^*$. Whence C is the bifurcation set for $(5')$.

If the characteristic manifold defined by $(5'')$ has bifurcation points, then the equation $(5'')$ has multiple roots. For the case of a double root, this root satisfies the equation $(5'')$ and the equation obtained by differentiating $(5'')$ with respect to μ, i. e.

$$4\mu^3 + 4Qa^2\mu + Ta^2 = 0. \qquad (5'')_1$$

Then, by the Euler algorithm, it follows that the double root is $\mu = -\frac{16}{9}\frac{Q^2a^2}{T}$, which introduced into $(5'')$ or $(5'')_1$ leads to $T = T^*$. Hence C is the single bifurcation set for $(5'')$.

As expected, C_1 is not a bifurcation set for $(5'')$ because for the points of C_1 the equation $(5'')$ has mutually disjoint roots, one of which being equal to $-a^2$ and leading to two equal solutions of $(5')$.

Assume that $(5')$ has a multiple root of multiplicity equal to 4. Then it must satisfy $(5')$, $(5')_1$ and

$$4(\lambda^2 - a^2)^3 + 4Qa^2(\lambda^2 - a^2) + Ta^2 + 2\lambda^2[12(\lambda^2 - a^2)^2 + 4Qa^2] = 0, \quad (5')_3$$

$$\lambda[12(\lambda^2 - a^2)^2 + 4Qa^2] + 16\lambda^3(\lambda^2 - a^2) = 0. \qquad (5')_4$$

Since the multiple root satisfies $(5')_2$, from $(5')_3$ it follows that it must satisfy the equation $12(\lambda^2-a^2)^2+4Qa^2=0$ and from $(5')_4$ it follows that it satisfies the equation $16\lambda^3(\lambda^2-a^2)=0$. Supposing that this root is nontrivial, it follows that it can be either $\lambda_1=a$ or $\lambda_2=-a$. In both these cases from $12(\lambda^2-a^2)^2+4Qa^2=0$ we have $Qa^2=0$, which contradicts the assumption $a,Q,T>0$. Therefore the single root of $(5')$ which can have multiplicity equal to four is $\lambda_1=0$ for which from $(5')_3$ it follows $-4a^6-4Qa^4+Ta^2=0$, while from $(5')$ we have $a^8+2Qa^6-Ta^4+Q^2a^4=0$. These two relations imply $Q=3a^2$, $T=16a^4$, hence C^* is a bifurcation set for $(5')$ (of a type different from those of $C\setminus C^*$ and $C_1\setminus C^*$).

For $(5'')$, C^* is just part of the bifurcation set.

The surface $C\cup C_1$ has two sheets but for C^*, where the two sheets coalesce. The projection of C^* on the (a,Q) plane is C^*_*, therefore C^*_* is the bifurcation set for $C\cup C_1$. This can be seen also considering the equation

$$[T-16aQ\sqrt{Q}/(3\sqrt{3})][T-(Q+a^2)^2]=0, \qquad (5')_5$$

which defines $C\cup C_1$ and which possesses a double root for $Q=3a^2$.

C^*_* is a bifurcation set for $(5')$ taken on C, i. e. for

$$(\lambda^2-a^2)^4+2Qa^2(\lambda^2-a^2)^2+16a^3Q\sqrt{Q}/(3\sqrt{3})(\lambda^2-a^2)+Q^2a^4=0, \quad (5')_1$$

because differentiating this equation with respect to λ and imposing to the solution $\lambda=0$ of the obtained equation to satisfy $(5')_1$ we obtain $Q=3a^2$ defining C^*_*. Whence, Theorem 1. The detailed geometrical structure of the characteristic manifolds and bifurcation manifolds is given in [4], [3].

The secular equations must be written separately for each of the bifurcation sets, because for each of them the form of the general solution of $(1')$ $(2')$ is different. We write them only for an even function v [2] (physical reasons show that odd v is not realistic). We have the following types of general even solution for $(1')$, $(2')$ corresponding to various multiplicities of λ_i

$$v(z)=\sum_{i=1}^{4}A_i cosh(\lambda_i z), \quad \text{for} \quad (a,Q,T)\in\mathbf{R}^3\setminus(C\cup C_1) \qquad (8')$$

$$v(z)=A_1cosh(\lambda_1 z)+B_2 z sinh(\lambda_1 z)+\sum_{i=1}^{3}A_i cosh(\lambda_i z), \quad \text{for} \quad (a,Q,T)\in C\setminus C^* \qquad (7')_1$$

$$v(z)=A_1+\sum_{i=2}^{4}A_i cosh(\lambda_i z), \quad \text{for} \quad (a,Q,T)\in C_1\setminus C^* \qquad (7')_2$$

$$v(z)=A_1+A_2z^2+\sum_{i=1}^{3}A_i cosh(\lambda_i z), \quad \text{for} \quad (a,Q,T)\in C^* \qquad (7')_3$$

Since $(7')_2$ can be, simply, obtained from $(8')$ for $\lambda_1 = 0$, we consider it no longer. In this case no secular points exist [4] and, so, the entire surface $C_1 \setminus C^*$ is a false secular manifold indeed.

The secular equation corresponding to $(8')$ reads

$$\begin{vmatrix} \lambda_1 sinh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \\ \mu_1 cosh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \\ (\mu_1^2 + Qa^2) cosh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \\ \lambda_1(\mu_1^2 + Qa^2) sinh\frac{\lambda_1}{2} & \cdot & \cdot & \cdot \end{vmatrix} = 0, \quad \text{for} \quad (a,Q,T) \in \mathbf{R}^3 \setminus (C \cup C_1). \quad (10')$$

The $i-$th lacking columns in $(10')$ is identical to the first column but with λ_1 and μ_1 replaced by λ_i and μ_i.

Formally, in the secular equation for $(a,Q,T) \in C \setminus C^*$ the first, third and columns are identical with those from $(10')$ while the second column is obtained by differentiating the second column in $(10')$ with respect to λ_2 and then replacing λ_2 by λ_1. Similarly, in the secular equation for $(a,Q,T) \in C^*$, the second column is obtained by differentiating twice the second column in $(10')$ and then replacing λ_2 by λ_1 [2], [4], the remaining columns being identical with those from $(10')$.

For an easier solution of the secular equations the notation $t_i = \lambda_i tanh(\lambda_i/2)$ is introduced. As a consequence, the new form of the secular equations will contain the product of $cosh(\lambda_i/2)$, $i = 1 \cdots 4$. If some of λ_i are purely imaginary then the corresponding $cosh(\lambda_i/2)$ vanish, the equality $\prod_{i=1}^{4} cosh(\lambda_i/2) = 0$ representing additional secular equations which do not depend on the boundary conditions. For the points of C^*, in [4] it was shown that the corresponding secular equation (in the old form) has no solution, therefore no secular points belong to C^*. Moreover, since in this case $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$ and $\lambda_{3,4,7,8}$ are complex not purely imaginary, we have neither additional secular points. For the points of $C \setminus C^*$ we have $\mu_1 = \mu_2 = -a\sqrt{Q/3}$, hence $\lambda_{1,2} = \sqrt{a^2 - a\sqrt{Q/3}} = -\lambda_{5,6}$. Consequently, taking into account that $C \setminus C^*$ is a FNM indeed, we have

Theorem 2. *No point of C_1 is secular. For $(a,Q,T) \in C \setminus C^*$, $a < \sqrt{Q/3}$, there exists the additional secular equation $cos\sqrt{a\sqrt{(Q/3)} - a^2} = 0$, which are independent of the boundary conditions $(2')$.*

In the last case the secular curves are $Q = 3[a + (2k+1)^2\pi^2/(4a)]^2$, $T = 16a[a + (2k+1)^2\pi^2/(4a)]^3$, $k \in \mathbf{N}$, and they are situated on $C \setminus C^*$.

There are a lot of theoretical open problems for $(1')$ and $(2')$. The first is the existence of solutions of $(10')$ and of the secular equation for points of $C \setminus C^*$. However, computations show that there exist infinitely many secular surfaces (sheets) defined by $(10')$ and an infinity of spatial curves situated on $C \setminus C^*$

which consist of secular points. No secular points exist on C_1, but additional secular curves (Theorem 2) exist on C. Thus, even for the very simple case of (1′), (2′), the geometry of the set of the secular points is complicated, this set consisting of surfaces and curves separated by C_1 and by C^*. A heuristical reasoning [4] shows that these curves (all of them belonging to $C \setminus C^*$) are limits for the secular surfaces of (10′) . A few numerical results [4] suggest *Conjecture. The curve* $T = T^*$, $Q = const$ *and the surface* $C \setminus C^*$, *except for the secular curves situated on* $C \setminus C^*$ *are false secular manifolds. When they exist, the secular curves situated on* $C \setminus C^*$ *are limit sets of the secular surfaces defined by* (10′) *and have some extremality properties.*

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