MATHEMATICAL MODELING

AND

ORDINARY DIFFERENTIAL EQUATIONS

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Chapter 1

Introduction

1.1 What is mathematical modeling?

In science, we understand our real world by observations, collecting data, find rules inside or among them, and eventually, we want to explore the truth behind and to apply it to predict the future. This is how we build up our scientific knowledge. The above rules are usually in terms of mathematics. They are called mathematical models. One important such models is the ordinary differential equations. It describes relations between variables and their derivatives. Such models appear everywhere. For instance, population dynamics in ecology and biology, mechanics of particles in physics, chemical reaction in chemistry, economics, etc. In modern science, an important data collected by Tycho Brache leaded Kepler’s discovery of his three laws of planetary motion and the birth of Newton’s mechanics and Calculus.

Nowaday, we have many advance tools to collect data and poweful computers to ana-lyze them. It is therefore important to learn the theory of ordinary differential equation, an important language of science.

In this course, I will mainly focus on two important classes of mathematical models by ordinary differential equations:

- population dynamics in biology
- dynamics in classical mechanics

The first one studies behaviors of population of species. It can also be applied to physical mixing, chemical reactions, etc. The second one include many important examples such as harmonic oscillators, pendulum, Kepler problems, electric cricuit, etc. Basic physical laws such as growth rate, conservation laws, etc. for modeling will be introduced.

The goal is to learn (i) how to do modeling, (ii) how to solve the corresponding differential equations, (iii) how to interprete the solutions, and (iv) how to develop general theory.

1.1.1 Two Examples

I will talk two simple examples to explain mathematical models.
A falling object  A object falling down from hight $y_0$. Let $v(t)$ be its velocity at time $t$. According to Newton’s law,

$$\frac{dv}{dt} = -g.$$  \hspace{1cm} (1.1)

Here $g$ is the gravitation constant. Usually the object experiences friction. One empirical model is that the friction force per mass is inverse proportionally to its speed. Adding this frictional force, the model becomes

$$\frac{dv}{dt} = -g - \alpha v,$$  \hspace{1cm} (1.2)

where $\alpha$ is the frictional coefficient.

Cooling/Heating of an object  An object is taken out of registrater to defrose. Let $y(t)$ be its temperature at time $t$. Let the room temperature be $K$ and the initial temperature of the object is $y_0$. According to Newton’s law of cooling/heating: the rate change of $y$ is proportionally to the difference between $y(t)$ and $K$. More precisely,

$$\frac{dy(t)}{dt} = -\alpha(y(t) - K).$$  \hspace{1cm} (1.3)

Here, $\alpha$ is a conductivity coefficient. It depends on the object. This method can also be identify the dead time of a human body.

As you can see that these two models are mathematical identical. We can use one theory to cover them.

1.1.2 Methods and tools to solve the differential equations

Calculus as a tool  The main tool is Calculus. Let us solve the ODE by Calculus as the follows.

$$\frac{dy}{dt} = -\alpha(y - K)$$

$$\frac{1}{y - K} \frac{dy}{dt} = -\alpha$$

$$d \log |y - K| = -\alpha dt$$

$$\log |y - K| = -\alpha t + C$$

Here, $C$ is an integration constant.

$$|y - K| = e^C \cdot e^{-\alpha t}$$

$$y(t) - K = \pm e^C \cdot e^{-\alpha t}$$

$$y(t) - K = C_1 e^{-\alpha t}$$

Here $C_1 = \pm e^C$ is a constant. Now, we plug the initial condition: $y(0) = y_0$. We then get $C_1 = y_0 - K$ and

$$y(t) = K + (y_0 - K)e^{-\alpha t}.$$  \hspace{1cm} (1.4)
1.1. WHAT IS MATHEMATICAL MODELING?

We observe that $y(t) \to K$ as $t \to \infty$. This is true for any initial datum $y_0$. We call $K$ is a stable equilibrium. For the heating/cooling problem, the temperature $y(t)$ will eventually approach the room temperature $K$. For the falling object problem, the velocity $v(t)$ will approach a termination velocity $K = -g/\alpha$. For any time $0 < t < \infty$, in fact, $y(t)$ is a linear interpolation between $y_0$ and $K$. That is,

$$y(t) = e^{-\alpha t} y_0 + (1 - e^{-\alpha t})K$$

The time to reach half way (i.e. $(y_0 + K)/2$) requires

$$K + (y_0 - K)e^{-\alpha t} = \frac{1}{2}(y_0 + K).$$

$$e^{-\alpha t} = \frac{1}{2}.$$  

This yields $t_{hf} = \log 2/\alpha$. We thus interprete $1/\alpha$ to be the relaxation time. The solution $y(t)$ relaxes to its stable equilibrium $K$ at time scale $1/\alpha$.

**Homework.** A dead body is found at 6:30 AM with temperature $18^\circ$. At 7:30 AM, the body temperature is $16^\circ$. Suppose the surrounding temperature is $16^\circ$ and the alive people’s temperature is about $37^\circ$. Estimate the dead time.

**Using mathematical software** There are many mathematical software which can solve ODEs. We shall use Maple in this class. Let us type the following commands in Maple. To use the tool of differential equations, we need to include it by typing

```maple
> with(DEtools):
> with(plots):
> Deq:= diff(y(t),t) = r*(K-y(t));
        Deq := $\frac{dy(t)}{dt} = r(K-y(t))$
> dfieldplot(subs(r=0.5,K=5,Deq),y(t),t=-5..5,y=-2..7,arrows=slim):
```
1.2 First-order equations

1.2.1 Autonomous equation

In the previous section, we have seen two examples of first order equation of the form: \( y' = f(y) \). Such a system with \( f \) being independent of \( t \) is called an autonomous system. For these kinds of system, we can use integration technique to find its solution. Namely,

\[
\frac{y'(t)}{f(y(t))} = 1.
\]
Here, I express $y$ as a function of $t$ to emphasize the above expression is a function in $t$ on both sides. We can then integrate it in $t$. The right-hand side (RHS) becomes $t + C$, whereas the left-hand side (LHS) is

$$\int \frac{y'(t)}{f(y(t))} \, dt = \int \frac{dy}{f(y)}$$

In the last step, the change of integration variable is adopted. Suppose we can find this integral, i.e. suppose

$$\int \frac{dy}{f(y)} = \Phi(y),$$

then the solution $y$ satisfies

$$\Phi(y) = t + C.$$ 

This is an implicit expression of the solution.

**Homeworks**

1. $y' = ay^2$
2. $y' = (y - y_0)(y_1 - y)$
3. $y' = r(y - y_0)(y - y_1)(y - y_2)$
4. $y' = (y - y_0)^2(y_1 - y)$
5. $y' = (y - y_0)^3(y_1 - y)$
6. $y' = r \tanh(y)$
7. B-D: pp. 48, 12, 14, 21, 23, 28

### 1.2.2 Linear first-order equation

The linear first-order equation reads

$$y' = a(t)y + b(t). \tag{1.5}$$

We first study the homogeneous equation:

$$y' = a(t)y.$$ 

We separate $t$ and $y$ and get

$$\frac{y'}{y} = a(t).$$

The LHS is $d \log y(t)/dt$. We integrate it and get

$$\int \frac{d \log y(t)}{dt} \, dt = \int a(t) \, dt$$
This yields
\[
\log y(t) = A(t) + C_1, \text{ or } y(t) = Ce^{A(t)},
\]
where \(A'(t) = a(t)\), and \(C\) or \(C_1\) is a constant. We may choose \(A(0) = 0\). That is, \(A(t) = \int_0^t a(s) \, ds\). The constant \(C\) is \(y_0\) if we require \(y(0) = y_0\). We conclude that the solution is
\[
y(t) = y(0)e^{\int_0^t a(s) \, ds}.
\]

Next, we study the inhomogeneous equation. The method below is known as variation of constant. We guess our solution to have the form
\[
y(t) = Ce^{A(t)}.
\]
Plugging it into (1.5), we obtain
\[
C'(t)e^{A(t)} + a(t)Ce^{A(t)} = a(t)Ce^{A(t)} + b(t)
\]
This yields
\[
C'(t) = b(t)e^{-A(t)}
\]
Hence the solution is
\[
C(t) = C(0) + \int_0^t b(s)e^{-A(s)} \, ds
\]
By plugging the initial datum, we obtain \(C(0) = y(0)\). Hence, the general solution is given by
\[
y(t) = y(0)e^{A(t)} + \int_0^t b(s)e^{-A(s)+A(t)} \, ds.
\]
The idea behind the variation of constant is that the ansatz
\[
y(t) = C(t)e^{A(t)}
\]
has the property:
\[
y'(t) = C(t)A'(t)e^{A(t)} + C'(t)e^{A(t)}.
\]
In a short time, if \(C\) remains nearly unchanged, \(e^{A(t)}\) behaves like solutions of \(y' = A'(t)y\). By allowing \(C(t)\) varying, the \(C'(t)\) term can take care contribution of the source \(b(t)\) pumping into the system.

It is important to notice that the integrand \(b(s)e^{A(t)-A(s)}\) is the solution of \(y' = a(t)y\) for \(s < t\) with \(y(s) = b(s)\). This means that the source term \(b(s)\) generates a solution \(b(s)e^{A(t)-A(s)}\) at time \(s\). The total contribution of the source term from time \(0\) to \(t\) is the accumulation of these solutions, i.e. \(\int_0^t b(s)e^{A(t)-A(s)} \, ds\). This is called the Duhamel principle.
Example.

Consider

\[ y' + \frac{2}{t} y = t - 1. \]

Let

\[ A(t) = -\int \frac{2 \, dt}{t} = \ln t - 2, \]

and \( e^{-A(t)} = t^2 \). By multiplying \( e^{-A(t)} \) on both sides, we obtain

\[ \frac{d}{dt} (t^2 y) = t^2 (t - 1). \]

Integrating in \( t \), we get

\[ t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + C. \]

Hence,

\[ y(t) = \frac{t^2}{4} - \frac{t}{3} + C. \]

Homeworks

1. B-D: pp. 39, 3, 7, 10, 19, 21, 29, 33, 35

1.2.3 Integration factors and integrals

A general first-order equation can be expressed as

\[ \frac{dy}{dt} = f(t, y) \]  \hspace{1cm} (1.6)

with initial datum

\[ y(0) = y_0. \]  \hspace{1cm} (1.7)

The solution is a curve in \( t-y \) plane. This curve can be expressed explicitly like \( y = y(t) \), or in implicit form like \( \phi(t, y) = \text{const} \). In the latter case, we mean that \( y \) can be solved as a function in \( t \) locally and this function satisfies (1.6) and (1.7). We call \( \phi \) an integral of (1.6), which means that \( \phi(t, y(t)) \) remains a constant along a solution curve \( y = y(t) \). Sometimes, \( \phi \) is called an invariant.

To be more precise, consider a curve passing through \((t_0, y_0)\) defined implicitly by

\[ \phi(t, y) = \phi(t_0, y_0) = \text{const}. \]

Suppose \( \phi_y(t_0, y_0) \neq 0 \). By the implicit function theorem, we can solve \( y \) as a function of \( t \) in a neighborhood of \((t_0, y_0)\), that is \( y = y(t) \) for \( t \in (t_0 - \epsilon, t_0 + \epsilon) \) for some \( \epsilon > 0 \) with \( y(t_0) = y_0 \). Along this curve,

\[ d\phi = (\phi_t, \phi_y) \cdot (dt, dy) = 0. \]
The direction \((dt, dy)\) is the tangent direction of the curve \(y = y(t)\). The above formula simply means that \((\phi_t, \phi_y)\) is normal to the curve \(y = y(t)\). On the other hand, we can interpret the ODE \(y' = f(t, y)\) by

\[(f, -1) \cdot (dt, dy) = 0\]

That is, \((f, -1)\) is also normal to the curve. In other words, \(\phi = \text{const}\) is a solution of the ODE \(y' = f(t, y)\) is equivalent to

\[(\phi_t, \phi_y) \parallel (f, -1).\]

We can consider more general equation called the Phaffian equation:

\[M(t, y)dt + N(t, y)dy = 0.\] (1.8)

A function \(\phi(t, y)\) is called an integral of (1.8) if \(d\phi = 0\) equivalent to (1.8). That is,

\[(\phi_t, \phi_y) \parallel (M, N)\]

or equivalently, there exists a function \(\mu(t, y) \neq 0\) such that

\[(\phi_t, \phi_y) = \mu(M, N)\]

In other word, if there exists a function \(\mu(t, y) \neq 0\) such that

\[d\phi = \mu (M(t, y)dt + N(t, y)dy)\]

then \(\phi\) is an integral of (1.8). The function \(\mu\) is called an integration factor of (1.8).

**Examples**

1. Consider the equation

\[\frac{dy}{dx} = \frac{-x}{y}\]

This is equivalent to

\[xdx + ydy = 0.\]

We integrate it and get \(\phi = x^2 + y^2 = \text{Const}\) is an integral of this equation.

2. Consider the linear equation

\[y' = 2y + t.\] (1.9)

We have seen that \(\mu = e^{-2t}\) is an integration factor. In fact, the equation can be rewritten as

\[dy - 2ydt = tdt.\]

We multiply both sides by \(\mu = e^{-2t}\) to get

\[e^{-2t}(dy - 2ydt) = te^{-2t} dt\] (1.10)
1.2. **FIRST-ORDER EQUATIONS**

The left-hand side (LHS) is a total differential:

\[ e^{-2t}(dy - 2ydt) = d(e^{-2t}y) \]

The right-hand side (RHS) is also a total differential:

\[ te^{-2t} dt = d \int te^{-2t} dt \]

and

\[ \int te^{-2t} dt = -\frac{1}{2} \int tde^{-2t} = -\frac{1}{2}te^{-2t} + \frac{1}{2} \int e^{-2t} dt = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + C. \]

Hence, (1.10) can be express as

\[ d \left( e^{-2t}y + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} \right) = 0. \]

We call \( \phi := e^{-2t}y + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} \) an integral of (1.9).

3. In the linear equation (1.5)

\[ y' = a(t)y + b(t), \]

we multiply (1.5) by \( \mu(t) = e^{-A(t)} \) and use \( a(t) = A'(t) \), we obtain

\[ e^{-A(t)}y' - A'(t)e^{-A(t)}y = e^{-A(t)}b(t) \]

\[ \frac{d}{dt} (e^{-A(t)}y) = e^{-A(t)}b(t) \]

We can then integrate this formula in \( t \) to obtain the solution for (1.5). In this method, \( \mu = e^{-A(t)} \) is an integration factor and

\[ \phi = e^{-A(t)}y - \int e^{-A(t)}b(t) dt \]

is an integral.

Notice that the integration factor and the integral are not unique. For instance, in example 1 above, any function \( \mu(r^2) \) is an integration factor of the equation \( xdx + ydy = 0 \), and the antiderivative \( \psi = \int \mu \) is its integral. Here, \( r^2 = x^2 + y^2 \).

This is easily seen from the following observation. Suppose \( \phi \) is an integral and \( \mu \) is the corresponding integration factor. Consider a composition function

\[ \psi(t,y) = h(\phi(t,y)), \]

where \( h(\cdot) \) is any smooth function. Then

\[ d\psi = h'\phi = h'\mu (M(t,y)dt + N(t,y)dy) = 0. \]

Hence, \( \psi \) is another integral with a new integration factor \( h'(\phi(t,y))\mu(t,y) \).

Certainly, if both \( \phi \) and \( \psi \) are integrals of (1.8), they represent the same solutions, namely, there is one-to-one correspondence of the level sets of \( \phi \) and \( \psi \):

\[ \phi(t,y) = C_1 \text{ if and only if } \psi(t,y) = C_2. \]

Two functions \( \phi \) and \( \psi \) with this property is called function dependent. If we define a function \( h \) which maps: \( C_1 \rightarrow C_2 \), then \( \psi(t,y) = h(\phi(t,y)) \). Thus, two integrals are functional dependent and are related through a composition of function.
Homeworks

1. Find the integral curves (or integrals) of the equations

(a) \( \frac{dy}{dx} = -\frac{x^2}{y^2} \)

(b) \( \frac{dy}{dx} = \frac{x^2}{1+y^2} \)

1.2.4 Separable equations

Suppose the function \( M(t, y) \) and \( N(t, y) \) in (1.8) are separable, that is

\[
M(t, y) = f_1(t)f_2(y),
\]

\[
N(t, y) = g_1(t)g_2(y),
\]

Dividing (1.8) by \( f_2(y)g_1(t) \), then the Phaffian equation (1.8) becomes

\[
\frac{f_1(t)}{g_1(t)}\, dt + \frac{g_2(y)}{f_2(y)}\, dy = 0.
\]

We can integrate it to obtain an integral \( \phi \):

\[
\phi(t, y) := \int \frac{f_1(t)}{g_1(t)}\, dt + \int \frac{g_2(y)}{f_2(y)}\, dy.
\]

Then \( \phi(t, y) = \text{constant} \) defines a solution implicitly. In this example, \( 1/f_2(y)g_1(t) \) is an integration factor.

Examples

1. \( y' = t/y^2 \). This implies \( y^3/3 = t^2/2 + C \), or

\[
y(t) = \left( \frac{3t^2}{2} + k \right)^{1/3}.
\]

2. \( (x^2 + 1)(y^2 - 1)\, dx + xy\, dy = 0 \). The answer is

\[
y^2 = 1 + Ce^{-\frac{x^2}{2}}.
\]

3. \( y' = t^2/(1 - y^2) \). Ans.: \( -t^3 + 3y - y^3 = \text{const} \).

4. \( y' = (4x - x^3)/(4 + y^3) \). Ans. \( y^4 + 16y + x^4 - 8x^2 = \text{const} \).

5. \( y' = \frac{3x^2+4x+2}{2(y-1)} \). Ans. \( y^2 - 2y = x^3 + 2x^2 + 2x + 3 \).
Homogeneous equations  We consider the equation:

\[ P(x, y) \, dx + Q(x, y) \, dy = 0. \]

Suppose \( P \) and \( Q \) are homogeneous of degree \( n \). Following Leibnitz’s method, we define a homogeneous variable \( v = y/x \). We use \( x \) and \( v \) as our new variables. We have \( dy = d(xv) = x \, dv + v \, dx \). From homogeneity, we have \( P(x, xv) = x^n P(1, v) \) and \( Q(x, xv) = x^n Q(1, v) \). The equation becomes

\[ (P(1, v) + vQ(1, v)) \, dx + xQ(1, v) \, dv = 0. \]

We can use method of separation of variables:

\[
\frac{dv}{R(v)} + \frac{dx}{x} = 0,
\]

where

\[
R(v) = v + \frac{P(1, v)}{Q(1, v)}.
\]

The solution is

\[
\int \frac{dv}{R(v)} = -\log |x| + C.
\]

**Example.** Solve the equation

\[ y' = \frac{x + y}{x - y}. \]

Let \( v = y/x \). We can transform the equation to

\[ y' = \frac{1 + v}{1 - v}. \]

\[
\frac{dy}{dx} = \frac{d}{dx}(xv) = v + xv'.
\]

Hence, we get

\[ v + xv' = \frac{1 + v}{1 - v}. \]

\[
xv' = \frac{1 + v}{1 - v} - v = \frac{1 + v^2}{1 - v}.
\]

\[
\frac{1 - v}{1 + v^2}v' = \frac{1}{x}.
\]

We integrate both sides to get

\[
\arctan v - \frac{1}{2} \log(1 + v^2) = \log |x| + C.
\]

Or

\[
\arctan(y/x) - \frac{1}{2} \log(1 + (y/x)^2) = \log |x| + C.
\]
Homeworks  B-D: pp. 49, 30, 31

Bernoulli equation  Bernoulli equation has the form
\[ y' = a(t)y + b(t)y^n \]  (1.11)
Divide both sides by \(y^{-n}\), we obtain
\[ y^{-n}y' = a(t)y^{-n+1} + b(t). \]
Or
\[ \frac{1}{1-n} (y^{1-n})' = a(t)y^{1-n} + b(t) \]
This suggests the following change of variable:
\[ z = y^{1-n}. \]
Then
\[ z' = (1-n)a(t)z + (1-n)b(t) \]  (1.12)
which can be solved.

Homeworks  (Courant and John, Vol. II, pp. 690) Solve the following equations
1. \( xy' + y = y^2 \log x \)
2. \( xy^2(xy' + y) = a^2 \)
3. \( (1-x^2)y' - xy = axy^2. \)

* Riccati equation  (Courant and John, Vol. II, pp. 690) The Riccati equation reads
\[ y' = a(t)y^2 + b(t)y + c(t) \]  (1.13)
It can be transformed into a linear equation if we know a particular solution \( y = y_1(x) \). We introduce the new unknown
\[ u = \frac{1}{y - y_1}. \]


1.3  Modeling with First Order Equations

1.3.1  Some Examples—Homeworks
- A Conteminent problem B-D: pp. 60, 5, 6, 19.
- A Falling object problem B-D: pp. 64, 23, 27
- Compound interest problem B-D: pp. 61, 11
- Heating/Cooling problem B-D: pp. 62. 17, 18
1.3. Modelizing with First Order Equations

1.3.2 Modeling population of single species

Let us start from the simplest model.

**Simple population growth model**  Let \( y(t) \) be the population (say European population in U.S.) at time \( t \). The census data are from 1790-2000 (every 10 years). We can build a model based on the following hypothesis:

\[
\frac{dy}{dt} = \text{births} - \text{deaths} + \text{migration}. \tag{1.14}
\]

It is natural to assume that the births and the deaths are proportion to the population. Let us neglect the migration for the moment. In terms of mathematical equations, this reads

\[
\frac{dy}{dt} = ry \tag{1.15}
\]

where \( r \) is called the net growth rate, which is the natural growth rate minus the death rate. We should have \( r > 0 \) if the population is growing. We can set the initial value

\[
y(0) = y_0, \tag{1.16}
\]

the population at year 1790. With (1.15) and (1.16), we can find its solution

\[
y(t) = y_0 e^{rt}.
\]

We can find the growth rate \( r \) by fitting the data, say the census at year 1800. This yields that \( r = 0.03067 \). We find it fits very well until 1820. From then on, the discrepancy becomes larger and larger. It suggests that

- the growth rate \( r \) is treated as a constant is only valid local in time;
- environmental limitation is not taken into account.

**Logistic population model**  The above population model was proposed by Malthus (1766-1834), an economist and a mathematician. One criticism of the simple growth model is that it does not take the limit of environment into consideration. With this consideration, we should expect that there is an environmental carrying capacity \( K \) such that

- when \( y < K \), the rate \( y' > 0 \),
- when \( y > K \), the rate \( y' < 0 \).

A simple model with these considerations is the follows:

\[
y' = ry \left(1 - \frac{y}{K}\right). \tag{1.17}
\]

This is called the *logistic population model*. It was suggested by the Belgian mathematician Pierre Verhulst (1838). It is a nonlinear equation. There is another interpretation for the nonlinear term \( ry^2/K \). Namely, \( y^2 \) represents the rate of pair-interaction. The coefficient \( r/K \) is the rate of this interaction to the change of \( y \). The minus sign simply means that the pair-interaction decreases the population growth due to a competition of resource.
Exact solutions for the logistic equation  We can solve this equation by the method of separation of variable.

\[
\frac{y'(t)}{y(1 - y/K)} = r.
\]

Integrating in \( t \) yields

\[
\int \frac{y'(t)}{y(1 - y/K)} \, dt = rt + C.
\]

By change-variable formula for integration, we have

\[
\int \frac{1}{y(1 - y/K)} \, dy = rt + C.
\]

This yields

\[
\int \left( \frac{1}{y} + \frac{1}{K - y} \right) \, dy = rt + C
\]

\[
\log \left| \frac{y}{K - y} \right| = rt + C.
\]

\[
\left| \frac{y}{K - y} \right| = \frac{1}{C_1 e^{-rt}}.
\]

Here \( C_1 = e^{-C} \) is another constant. When \( 0 < y < K \), we get

\[
\frac{y}{K - y} = \frac{1}{C_1 e^{-rt}}.
\]

This yields

\[
y = \frac{K}{1 + C_1 e^{-rt}}.
\]

When \( y < 0 \) or \( y > K \), we get

\[
\frac{y}{K - y} = -\frac{1}{C_1 e^{-rt}}.
\]

This gives

\[
y = \frac{K}{1 - C_1 e^{-rt}}.
\]

When \( t = 0 \), we require \( y(0) = y_0 \). We find that in both cases, \( C_1 = |1 - K/y_0| \). Thus, the solution is

\[
y(t) = \left\{ \begin{array}{ll}
\frac{K}{1-C_1 e^{-rt}} & y_0 < 0 \text{ or } y_0 > K \\
\frac{K}{1+C_1 e^{-rt}} & 0 < y_0 < K
\end{array} \right.
\]

and \( y(t) \equiv 0 \) if \( y(0) = 0 \), \( y(t) \equiv K \) if \( y(0) = K \).
Remark. We observe that

- for initial \( y_0 \) with \( 0 < y_0 \), we have \( y(t) \to K \);
- the states \( y \equiv 0 \) and \( y(t) \equiv K \) are constant solutions.

These constant solutions are called the equilibrium states. Any solution with initial state near \( K \) will approach to \( K \) as \( t \) tends to infinity. We call \( K \) a stable equilibrium. On the other hand, if the initial state is a small perturbation of the zero state, it will leave off the zero state and never come back. We call 0 a unstable equilibrium.

Remark. When \( y_0 < 0 \), we observe that the solution \( y(t) \to \infty \) as \( t \to t^* - \), where

\[
1 - C_1 t^* = 0.
\]

We call the solution blows up at finite time. This solution has no ecological meaning.

Qualitative analysis for the logistic equation. We can analyze the properties (equilibrium, stability, asymptotic behaviors) of solutions of the logistic equation by the phase portrait analysis. First, let us notice two important facts:

- For any point \((t_0, y_0)\), there is a solution \( y(\cdot) \) passing through \((t_0, y_0)\). In other words, \( y(t_0) = y_0 \).
- No more than one solution can pass through \((t_0, y_0)\).

They are the existence and uniqueness theorem of the ODE. Let us accept this fact for the moment. Next, we can use the equilibria to classify our general solutions.

The first step is to find all equilibria of this system. Let us denote the right-hand side of (1.17) by \( f(y) \), i.e.

\[
f(y) = r y \left( 1 - \frac{y}{K} \right).
\]

An equilibrium is a constant solution \( y(t) \equiv \bar{y} \), where \( f(\bar{y}) = 0 \). In our case, the equilibria are \( y(t) \equiv 0 \) and \( y(t) \equiv K \).

The second step is to classify all other solutions. On the \( t-y \) plane, we first draw the above two constant solutions. Now, by the uniqueness, no solution can pass through these two constant solution. Therefore, the \( y \)-space is naturally partitioned into three regions

\[
I_1 = (-\infty, 0), \ I_2 = (0, K), \ I_3 = (K, \infty).
\]

If \( y(0) \in I_\ell \), then the corresponding \( y(t) \) stays in \( I_\ell \) for all \( t \).

The third step is to characterize all solutions in each regions. For any solution in \( I_2 \), we claim that \( y(t) \to K \) as \( t \to \infty \). From \( f(y) > 0 \) in \( I_2 \) and \( f(y) < 0 \) in \( I_1 \cup I_3 \), we can conclude that \( y(\cdot) \) is increasing in \( I_1 \) and decreasing in \( I_1 \) or \( I_3 \). We claim that \( y(t) \to K \) as \( t \to \infty \) for any solution in region \( I_2 \). Indeed, \( y(t) \) is increasing and has an upper bound \( K \). By the monotone convergence property of \( \mathbb{R} \), \( y(t) \) has a limit as \( t \) tends to infinity. Let us call this limit \( \bar{y} \). We claim that \( \bar{y} = K \). If not, \( \bar{y} \) must be in \((0, K)\) and hence
By the continuity of $f$, there must be an $\epsilon > 0$ and a neighborhood $I$ of $\bar{y}$ such that $f(y) > \epsilon$ for all $y \in I$. Since $\lim_{t \to -\infty} y(t) = \bar{y}$ monotonically, there must be a $t_0$ such that $y(t) \in I$ for $t \geq t_0$. On the other hand, the corresponding $y'(t) = f(y(t)) \geq \epsilon$. Hence $y(t) \geq y(t_0) + \epsilon(t - t_0)$ for all $t \geq t_0$. This contradicts $y(t)$ being bounded. Hence, we get $y(t) \to K$ as $t \to \infty$. Similarly, for solution $y(\cdot) \in I_3$, $y(t) \to K$ as $t \to \infty$.

Using the same argument, we can show that for solution in $I_1 \cup I_2$, $y(t) \to 0$ as $t \to -\infty$. This means that 0 is unstable. Indeed, for $y(0) < 0$, we have $f(y) < 0$. This implies $y(\cdot)$ is decreasing for $t > 0$. If $y(t)$ has a low bound, then $y(t)$ will have a limit and this limit $\bar{y} < 0$ and must be a zero of $f$. This is a contradiction. Hence $y(t)$ has no low bound.

To summarize, we have the following theorem.

**Theorem 3.1** All solutions of (1.17) are classified into the follows.

1. **equilibria**: $y(t) \equiv 0$ and $y(t) \equiv K$;

2. **If** $y(0) \in I_1$, then $\lim_{t \to -\infty} y(t) = 0$ and $y(t) \to -\infty$ as $t$ increases;

3. **If** $y(0) \in I_2$, then $\lim_{t \to -\infty} y(t) = 0$ and $\lim_{t \to \infty} y(t) = K$;

4. **If** $y(0) \in I_3$, then $y(t) \to \infty$ as $t$ decreases and $\lim_{t \to \infty} y(t) = K$;

The biological interpretation is the follows.

- If $y(0) < K$, then $y(t)$ will increase to a saturated population $K$ as $t \to \infty$.

- If $y(0) > K$, then $y(t)$ will decrease to the saturated population $K$ as $t \to \infty$.

- $y(t) \equiv K$ is the stable equilibrium, whereas $y(t) \equiv 0$ is an unstable equilibrium.

**Maple Practice** Below, we demonstrate some Maple commands to learn how to solve plot the solutions.

```maple
> with(plots):
> with(DEtools):
> DiffEq := diff(y(t),t)=r*y(t)*(1-y(t)/K);
> DiffEq := \frac{d}{dt} y(t) = r y(t) \left(1 - \frac{y(t)}{K}\right)
> dfieldplot(subs(r=0.1,K=5,DiffEq),y(t),t=-5..5,y=-2..7,arrows=slim,colour=y/7);
```
> fig1 := DEplot(subs(r=0.1,K=5,DiffEq),y(t),
> t=-50..50,[[y(0)=1]],y=-2..7,stepsize=.05,arrows=none,linecolour=red):
> fig2 := DEplot(subs(r=0.1,K=5,DiffEq),y(t),
> t=-50..50,[[y(0)=2]],y=-2..7,stepsize=.05,arrows=none,linecolour=blue):
> fig3 := DEplot(subs(r=0.1,K=5,DiffEq),y(t),
> t=-50..50,[[y(0)=6]],y=-2..7,stepsize=.05,arrows=none,linecolour=green):
> fig4 := DEplot(subs(r=0.1,K=5,DiffEq),y(t),
> t=-50..50,[[y(0)=-1]],y=-2..7,stepsize=.05,arrows=none,linecolour=black):
> display({fig1,fig2,fig3,fig4});
Logistic population model with harvesting  Suppose migration is considered. Let $e$ be the migration rate. We should modify the model by

$$ y' = ry \left( 1 - \frac{y}{K} \right) - ey. $$  

(1.18)

The migration rate $e$ can be positive (migrate out) or negative (migrate in).

This model is often accepted in ecology for harvesting a renewable resources such as shrimps, fishes, plants, etc. In this case, $e > 0$ is the harvesting rate which measures the harvesting effort. The quantity $ey$ is the amount from harvesting per unit time. It is called the harvesting yield per unit time.
This harvesting model is still a logistic equation
\[ y' = (r - e)y \left( 1 - \frac{ry}{(r - e)K} \right) \]  
(1.19)

with new growth rate \( r - e \). The new equilibrium is
\[ K_h = K \left( 1 - \frac{e}{r} \right), \]
which is the sustained population. When \( e < r \), we have \( 0 < K_h < K \). This means that the saturated population \( K_h \) decreases due to harvesting. When \( e > r \), then the species will be extinct due to overharvesting. Indeed, you can check that \( y(t) \equiv 0 \) is the stable equilibrium and \( y(t) \equiv K_h \) is the unstable equilibrium now. The quantity \( Y(e) = eK_h \) is called the sustained harvesting yield. An ecological goal is to maximize this sustained harvesting yield at minimal harvesting effort. We see that the maximum occurs at \( e = r/2 \). The corresponding sustained harvesting yield is
\[ Y \left( \frac{r}{2} \right) = \frac{rK}{2} = \frac{rK}{4}. \]

There is another way to model harvesting of natural resources. We may use harvesting amount \( C \) instead of the harvesting rate \( e \) as our parameter. The model now reads
\[ y' = ry \left( 1 - \frac{y}{K} \right) - C := f_C(y). \]  
(1.20)

The equilibrium (i.e. \( f_C(y) = 0 \)) occurs at \( f_C(y) = 0 \). On the \( C-y \) plane, \( f_C(y) = 0 \) is a parabola. For \( C \leq rK/4 \), there are two solutions for \( f_C(y) = 0 \):
\[ y_{\pm} = \frac{K}{2} \pm \sqrt{\frac{K^2}{4} - \frac{CK}{r}}. \]

For \( C > rK/4 \), there is no real solution. For \( C < rK/4 \), we can draw arrows on the intervals \(( -\infty, y_-), (y_-, y_+), (y_+, \infty) \) to indicate the sign of \( f_C \) in that interval. We conclude that \( y_+ \) is a stable equilibrium. We rename it \( K_h \).

To have sustained resource, we need \( K_h > 0 \). That is,
\[ \frac{K}{2} + \sqrt{\frac{K^2}{4} - \frac{CK}{r}} \geq 0. \]

This is equivalent to
\[ C \leq \frac{rK}{4}. \]

So the maximal harvesting to maintain \( K_h > 0 \) is
\[ C = \frac{rK}{4}. \]

For \( C > rK/4 \), \( y(t) \to 0 \) as \( t \) increases to some \( t^* \).
The solution for \( y' = ry(1 - \frac{y}{K}) - C \) with \( y(0) = y_0 \) is
\[
y(t) = \frac{1}{2} \left( K + \frac{\Delta}{r} \tanh \left( \frac{\Delta}{2K}(t + C_0) \right) \right)
\]
where
\[
\Delta = \sqrt{rK(rK - 4C)}, \quad C_0 = \frac{2K}{\Delta} \arctanh \left( \frac{r}{\Delta}(2y_0 - K) \right).
\]
In addition to the constraint \( C \leq K/4 \), we should also require \( y(0) > 0 \). Otherwise, there would be no harvesting at all. This would give another constraint on \( C \). You may find it by yourself.

Homeworks
1. B-D, pp. 88, 7, 16, 17
2. B-D: pp. 91, 22, 23, 25, 26, 27, 28

1.3.3 Abstract phase field models

Abstract logistic population models
We can use the following abstract model
\[
y' = f(y).
\]
(1.21)
The function \( f \) depends on \( y \) only. Such systems are called autonomous systems. We consider the initial datum
\[
y(0) = y_0
\]
(1.22)
Here \( f(y) \) has the following qualitative properties:
- \( f(y_0) = f(y_1) = 0 \),
- \( f(y) > 0 \) for \( y_0 < y < y_1 \),
- \( f(y) < 0 \) for \( y < y_0 \), or \( y > y_1 \).

First, there are two equilibrium solutions:
\[
y(t) \equiv y_0, \ y(t) \equiv y_1.
\]

For general solutions, we integrate the equation
\[
\frac{dy}{f(y)} = dt,
\]
One the left, we integrate in \( y \) from \( y_0 \) to \( y \), and on the right, we integrate in \( t \) from 0 to \( t \). We arrive at
\[
\Phi(y) - \Phi(y_0) = t
\]
where \( \Phi \) is a function such that \( \Phi'(y) = 1/f(y) \). From the properties of \( f \), we obtain that
\[
\Phi(y) : \begin{cases} 
\text{decreasing,} & \text{for } y > y_1, y < y_0 \\
\text{increasing,} & \text{for } y_0 < y < y_1.
\end{cases}
\]
Therefore, the function is invertible in each of the three regions: \((-\infty, y_0), (y_0, y_1), \) and \((y_1, \infty)\). The solution \( y(t) \) with initial datum is precisely the inversion of \( \Phi \) with \( \Phi(y_0) = 0 \).
1.3. MODELING WITH FIRST ORDER EQUATIONS

A bistable model  We consider the autonomous equation

\[ y' = f(y) \]

where \( f(y) \) has three zeros \( y_1 < y_2 < y_3 \). Assume the sign of \( f \) is \( f(y) > 0 \) for \( y < y_1, y_2 < y < y_3 \), and \( f(y) > 0 \) for \( y_1 < y < y_2, y > y_3 \). In this case, for \( y(t) \) with initial data \( y(0) \) satisfying \( y(0) < y_2, y_1 < y < y_2, y > y_3 \), we have \( y(t) \to y_1 \) as \( t \to \infty \). If \( y(0) > y_1 \), then \( y(t) \to y_3 \) as \( t \to \infty \). The states \( y_1 \) and \( y_3 \) are the two stable states. Such a model is called a bistable model. It is usually used to model phase field of some material. A simple model is \( f(y) = y(1 - y)(1/2 - y) \).

Maple tool: phase line analysis  Use Maple to draw the function \( f(y) \). The \( y \)-axis is partitioned into regions where \( f(y) > 0 \) or \( f(y) < 0 \). Those \( y^* \) such that \( f(y^*) = 0 \) are the equilibria. An equilibrium \( y^* \) is stable if \( f \) is increasing near \( y^* \) and unstable if \( f \) is decreasing there.

Asymptotic behaviors and convergent rates  Let us focus to an autonomous system which has only one equilibrium, say \( y = 0 \). That is, the rate function \( f(0) = 0 \). Let us consider two cases: \( f(y) = -\alpha y \) and \( f(y) = -\beta y^2 \) with \( y(0) > 0 \). We need minus to have \( y \equiv 0 \) a stable equilibrium.

- Case 1: \( y' = f(y) = -\alpha y \). In this case, we have seen that the solution is

\[ y(t) = y(0)e^{-\alpha t} \]

We see that the solution tends to its equilibrium 0 exponentially fast. The physical meaning of \( 1/\alpha \) is the time that the difference of solution from its equilibrium is reduced by a fixed factor \( (e^{-1}) \). We say the convergent rate to its equilibrium to be \( O(e^{-\alpha t}) \).

- Case 2: \( y' = f(y) = -\beta y^2 \). In this case,

\[ y(t) = \frac{1}{1/y(0) + \beta t} \]

We observe that \( y(t) \to 0 \) as \( t \to \infty \) with rate \( O(1/t) \).

Homework

2. Construct an ODE so that \( y(t) = 5 \) is its asymptotic solution with convergent rate \( e^{-2t} \).
3. What is the convergent rate of solutions of the ODE \( y' = -\beta y^3 \) as \( t \to \infty \)?
4. What is the convergent rate of solutions of the ODE \( y' = -\alpha y - \beta y^3 \) as \( t \to \infty \)?
5. Construct an ODE so that \( y(t) = (1 + t) \) is its asymptotic solution with convergent rate \( e^{-2t} \).

6. Construct an ODE so that \( y(t) = (1 + t) \) is its asymptotic solution with convergent rate \( t^{-1} \).

7. Search for ”bistability” in Wikipedia

### 1.3.4 An example from thermodynamics–existence of entropy

Consider a thermodynamic system: a container with fixed amount of gases inside and having one free side (a piston ) which allows volume change. The basic thermodynamic variables are the volume \( V \), the pressure \( p \), the internal energy \( e \), and the temperature \( T \). They are not independent to each other. Only two are independent. For ideal gas, they are related by the ideal gas law:

\[
pV = RT,
\]

where \( R \) is called the universal gas constant. For so called polytropic gases, the internal energy is linearly proportional to the temperature \( T \), i.e.

\[
e = c_v T
\]

where \( c_v \) is called the specific heat at constant volume. It means that the amount of energy you need to add to the system at constant volume to gain one degree increase of temperature.

We can change the volume \( V \) of the system by moving the piston. If the process is moved slowly, we image that the system has no energy exchange with external environment except the work we apply to it through the piston. Such a process is called an adiabetic process (no heat exchange with the external world). In such a process, by the conservation of energy,

\[
de = -pdV,
\]

where \(-pdV\) is the work we apply to the system. This is a Phaffin equation. Using the ideal gas law and the assumption of polytropic gas, we get

\[
\frac{c_v}{R} (pdV + V dp) = -pdV.
\]

This gives

\[
\left(1 + \frac{c_v}{R}\right) pdV + \frac{c_v}{R} V dp = 0.
\]

We divide both sides by \( c_v/R \), we get

\[
\gamma pdV + V dp = 0,
\]

where

\[
\gamma := \frac{1 + \frac{c_v}{R}}{\frac{c_v}{R}},
\]
is called the gas constant. This Phaffin equation can be integrated by using the technique of separation of variable:
\[
\frac{\gamma dV}{V} + \frac{dp}{p} = 0.
\]
Thus, we get
\[
\ln p + \gamma \ln V = C
\]
Hence,
\[
pV^\gamma
\]
is a constant. This means that each adiabatic process keeps \(pV^\gamma\) invariant (the integral of an adiabatic process). The quantity \(pV^\gamma\) labels a thermostat of the system. It is called an entropy. Notice that any function of \(pV^\gamma\) is also invariant under an adiabatic process. The one which has \(1/T\) as an integration factor for the Phaffin equation \(de + pdV = 0\) is called the physical entropy. That is
\[
TdS = de + pdV.
\]
This leads to
\[
dS = \frac{1}{T} (de + pdV)
\]
\[
= \frac{R}{pV} (\frac{c_v}{R} (pdV + Vdp) + pdV)
\]
\[
= c_v \left( \frac{dV}{V} + \frac{dp}{p} \right)
\]
\[
= c_v d \ln(pV^\gamma)
\]
\[
= dc_v \ln(pV^\gamma)
\]
Thus, the physical entropy
\[
S = c_v \ln(pV^\gamma).
\]

1.4 Existence, uniqueness

In this section, we shall state but without proof the existence and uniqueness theorems. We also show some examples and counter-examples regarding to the existence, uniqueness.

1.4.1 Local existence theorem

**Theorem 4.2 (Local existence theorem)** Suppose \(f(t, y)\) is continuous in a neighborhood of \((t_0, y_0)\). Then the initial value problem
\[
y' = f(t, y),
\]
\[
y(t_0) = y_0
\]
has a solution \(y(\cdot)\) existing on a small interval \((t_0 - \epsilon, t_0 + \epsilon)\) for some small number \(\epsilon > 0\).

This theorem states that there exists an interval (may be small) where a solution does exist. The solution may not exist for all \(t\). Let us see the following example.
Examples Consider the initial value problem
\[ y' = y^2 \]
\[ y(0) = y_0 \]

By the method of separation of variable,
\[ \frac{dy}{y^2} = dt \]
\[ \int_{y_0}^{y} \frac{dy}{y^2} = t \]
\[ -y^{-1} + y_0^{-1} = t \]
\[ y(t) = \frac{y_0}{1 - ty_0}. \]

When \( y_0 < 0 \), the solution does exist in \([0, \infty)\). But when \( y_0 > 0 \), the solution can only exist in \([0, 1/y_0]\). The solution blows up when \( t \to 1/y_0 \):
\[ \lim_{t \to 1/y_0} y(t) = \infty. \]

In the local existence theorem, it only states that the solution exists in a small region. If the solution does have a limit at the end of this small interval, we can solve the equation again to extend this solution. Eventually, we can find the maximal interval of existence.

Homeworks Find the maximal interval of existence for the problems below.

1. \( y' = 1 + y^2, \ y(0) = y_0 \)
2. \( y' = y^3, \ y(0) = y_0 \)
3. \( y' = e^y, \ y(0) = y_0 \)
4. \( y' = y \ln y, \ y(0) = y_0 > 0 \).

1.4.2 Uniqueness theorem

The initial value problem may not have a unique solution. Let us see the following problem:
\[ y' = y^{1/2}, \ y(0) = 0 \]

By the method of separation of variable,
\[ \frac{dy}{\sqrt{y}} = dt, \]
\[ \int \frac{dy}{\sqrt{y}} = t + C, \]
1.5. FIRST ORDER DIFFERENCE EQUATIONS

\[ 2\sqrt{y} = t + C \]

With the initial condition \( y(0) = 0 \), we get \( C = 0 \). Hence

\[ y(t) = \frac{t^2}{4} \]

is a solution. On the other hand, we know \( y(t) \equiv 0 \) is also a solution.

Homework.

1. Construct another example \( y' = f(y) \) which does not have uniqueness.

**Theorem 4.3** Assume that \( f \) and \( \partial f / \partial y \) are continuous in a small neighborhood of \((t_0, y_0)\). Suppose \( y_1(t) \) and \( y_2(t) \) are two solutions that solve the initial value problem

\[ y' = f(t, y), \ y(t_0) = y_0 \]

on an interval \((t_0 - \epsilon, t_0 + \epsilon)\) for some \( \epsilon > 0 \). Then

\[ y_1(t) = y_2(t), \ \text{for all } t \in (t_0 - \epsilon, t_0 + \epsilon). \]

In other word, no two solutions can pass through the same point in the \( t - y \) plane. This is very useful. For instance, in the logistic equation: \( y' = ry(1 - y/K) \), 0 and \( K \) are the only two equilibrium states. They naturally partition the domain into three regions: \( I_1 = (-\infty, 0) \), \( I_2 = (0, K) \) and \( I_3 = (K, \infty) \). By the uniqueness theorem, no solution can cross these two constant states. From this, we can obtain that the solution starting from \( y(0) > 0 \) will tend to \( K \) as \( t \to \infty \), because it will approach a constant state and this constant state can only be \( K \). We will see more applications of the uniqueness theorem in the subsequent chapters.

1.5 First Order Difference Equations

1.5.1 Euler method

Consider the first order equation

\[ y' = f(t, y). \]

If the solution is smooth (this is what we would expect), we may approximate the derivative \( y'(t) \) by a finite difference

\[ y'(t) \sim \frac{y(t + \Delta t) - y(t)}{\Delta t}. \]

Thus, we choose a time step size \( h \). Let us denote \( t^0 + nh = t^n \) and \( t^0 \) is the initial time. We shall approximate \( y(t^n) \) by \( y^n \). For \( t^n < t < t^{n+1}, y(t) \) is approximated by a linear function. Thus, we approximate \( y' = f(t, y) \) by

\[ \frac{y^{n+1} - y^n}{h} = f(t^n, y^n). \]  \( (1.23) \)
This is called the Euler method. It approximates the solution by piecewise linear function. The approximate solution $y_{n+1}$ can be computed from $y_n$. If we refine the mesh size $h$, we would expect the solution get closer to the true solution. To be more precise, let us fix any time $t$. Let us divide $[0, t]$ into $n$ subintervals evenly. Let $h = t/n$ be the step size. We use Euler method to construct $y_n$. The convergence at $t$ means that $y_n \to y(t)$ as $n \to \infty$ (with $nh = t$ fixed, hence $h \to 0$).

**Homework**

1. Use Euler method to compute the solution for the differential equation

$$y' = ay$$

where $a$ is a constant. Find the condition on $h$ such that the sequence $y_n$ so constructed converges as $n \to \infty$ and $nh = t$ is fixed.

### 1.5.2 First-order difference equation

This subsection is a computer project to study the discrete logistic map:

$$y_{n+1} = \rho y_n \left(1 - \frac{y_n}{k}\right).$$

It is derived from the Euler method for the logistic equation.

$$\frac{y_{n+1} - y_n}{h} = ry_n \left(1 - \frac{y_n}{k}\right),$$

with $\rho = 1 + rh$ and $k = K(1 + rh)/rh$. We use the following normalization: $x_n = y_n/k$ to get

$$x_{n+1} = \rho x_n (1 - x_n) := F(x_n).$$

This mapping $(F : x_n \mapsto x_{n+1})$ is called the logistic map. The project is to study the behaviors of this logistic map by computer simulations.

**Iterative map** In general, we consider a function $F : \mathbb{R} \to \mathbb{R}$. The mapping

$$x_{n+1} = F(x_n), \quad n = 0, 1, 2, \ldots,$$

is called an iterative map. We denote the composition $F \circ F$ by $F^2$.

A point $x^*$ is called a fixed point (or an equilibrium) of the iterative map $F$ if it satisfies

$$F(x^*) = x^*$$

A fixed point $x^*$ is called stable if we start the iterative map from any $x_0$ close to $x^*$, the sequence $\{F^n(x_0)\}$ converges to $x^*$. A fixed point $x^*$ is called unstable if we start the iterative map from any $x_0$ arbitrarily close to $x^*$, the sequence $\{F^n(x_0)\}$ cannot converge to $x^*$. The goal here is to study the behavior (stable, unstable) of a fix point as we vary the parameter $\rho$.

1. Find the condition on $\rho$ such that the logistic map $F$ maps $[0, 1]$ into $[0, 1]$.

2. For $\rho = 0.5, 1.5, 2.5$ find $\lim_{n \to \infty} x_n$ with $x_0 \in [0, 1]$. 

Homework  B-D: pp. 129, 14-18.

1.6 Historical Note

You can find the figures below from Wikipedia.

Data, modeling
- Tycho Brahe (1546-1601)
- Johannes Kepler (1571-1630)
- Galileo Galilei (1564-1642)

Population model
- Thomas Malthus (1766-1834)
- Pierre Verhulst (1804-1849)

Calculus and Numerical Method
- Isaac Newton
- Euler

Mathematical software
- Maple software
Chapter 2

Second Order Linear Equations

In this chapter, we study linear second-order equations of the form:

\[ ay'' + by' + cy = f(t), \ a \neq 0, \]

with constant coefficients. We shall investigate the linear oscillator model in great detail. It is a simple model for spring-mass system and the electric circuit system.

2.1 Models for linear oscillators

2.1.1 The spring-mass system

Consider a mass attached to a spring in one dimension. Let \( y \) be its location, and let \( y = 0 \) be its position at rest. The motion of the mass is governed by Newton’s force law. There are three kinds of forces the mass may be exerted.

- **Restoration force.** As the mass moves to \( y \), it is exerted a restoration force by the spring. According to Hook’s law, this restoration force is linearly proportional to \( y \) with reverse direction. That is,

  \[ F_r = -ky \]

  where \( k \) is the spring constant. The minus sign indicates that the force is opposite to the direction of the mass motion.

- **Friction force.** Suppose there is a friction. The friction force is proportional to the velocity with opposite direction. That is

  \[ F_f = -\gamma y' \]

  where \( \gamma \) is the damping (or friction) coefficient.

- **External force.** The mass may be exerted by the gravitational force, or some other external force modeled by \( f(t) \).

The Newton’s law then gives

\[ my'' = -\gamma y' - ky + f(t). \]

(2.1)
CHAPTER 2. SECOND ORDER LINEAR EQUATIONS

2.1.2 Electrical circuit system

Consider a circuit which consists of an inductor, a resistor, a capacitor and a battery. Suppose the wire is uniform. Then, according to the law of conservation of charges, the current is uniform throughout the whole circuit (i.e. it is independent of the position). Let $I(t)$ denote this current, $Q(t)$ be the charges. By the definition of current, $dQ/dt = I$. When the electric current passing through these components, there is a potential difference on the two ends of each component. Namely, the potential difference through each component is

- resistor: $\Delta V_r = RI$.
  A resistor is a dielectric material. The potential difference between the two ends of a resistance induces an electric field $E$. It drives electrons in the resistance move at current $I$. The Ohm law says that $I$ is proportional to $E$ and hence $\Delta V = Ed$, where $d$ is the length of the resistance.

- capacitor: $\Delta V_c = Q/C$.
  A capacitor is a pair of parallel plates with equal charges and opposite signature. There is an electric field $E$ induced by the charges on the two plates. It is clear that the more charges on the plates, the higher the electric field. That is, $E$ is proportional to $Q$. The potential difference on the two plates is $\Delta V = Ed$. Hence, $\Delta V$ is proportional to $Q$.

- inductor: $\Delta V_i = L \frac{dI}{dt}$.
  A inductance is a solenoid. By the Amperè law, the current on a circular wire induces a magnetic field mainly through the disk the circle surrounds. The time-varying current induces a time-varying magnetic field. By the Farady law, this time-varying magnetic field induces an electric field $E$ (electromotive force) in the opposite direction. Thus, there is a linear relation between the potential drop $\Delta V$ (which is $Ed$) and $dI/dt$.

The constants $R, C, L$ are called the resistance, conductance and inductance, respectively. From the Kirchhoff law (conservation of energy), we have

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C}Q = f(t)$$

(2.2)

where $f(t)$ is the external potential from the batery.

We notice there is an analogy between mechanical oscillators and electrical oscillators.

2.2 Methods to solve second order linear equations

We rewrite the above linear oscillator equation in an abstract form:

$$ay'' + by' + cy = f(t),$$

(2.3)

where $a, b, c$ are real constants. We should prescribe initial data:

$$y(0) = y_0, \ y'(0) = y_1$$

(2.4)
for physical consideration. We may express (2.13) in an operator form:

\[ L(D)y = f, \quad (2.5) \]

where

\[ L(D) = aD^2 + bD + c, \quad D = \frac{d}{dt}. \]

The term \( f \) is called the source term.

### 2.2.1 Homogeneous equations

Equation (2.13) without source term is called a homogeneous equation:

\[ L(D)y = 0. \quad (2.6) \]

We try a solution of the form \( y(t) = e^{\lambda t} \) (called an \textit{ansatz}) for the homogeneous equation. Plug this \textit{ansatz} into the homogeneous equation. We obtain

\[ L(D) \left( e^{\lambda t} \right) = L(\lambda) e^{\lambda t} = \left( a\lambda^2 + b\lambda + c \right) e^{\lambda t} = 0. \]

This leads to

\[ a\lambda^2 + b\lambda + c = 0. \]

The polynomial equation is called the characteristic equation for (2.13). Let \( \lambda_1, \lambda_2 \) be its two roots (possible complex roots).

There are three cases:

- **Case 1:** \( \lambda_1 \neq \lambda_2 \) and real. In this case, we have found two solutions \( y_1(t) = e^{\lambda_1 t} \) and \( y_2(t) = e^{\lambda_2 t} \).

- **Case 2:** \( \lambda_1 \neq \lambda_2 \) and complex. In this case, the two complex roots are conjugate to each other (because \( a, b, c \) are real). Let us denote \( \lambda_1 = \alpha + i\omega \) and \( \lambda_2 = \alpha - i\omega \). In this case, we can check that the real part and the imaginary part of \( e^{i\lambda t} \) are solutions. That is, \( y_1(t) = e^{\alpha t} \cos \omega t \) and \( y_2(t) = e^{\alpha t} \sin \omega t \) are two solutions.

- **Case 3:** \( \lambda_1 = \lambda_2 \). In this case, we can check \( y_1(t) = e^{\lambda_1 t} \) and \( y_2(t) = te^{\lambda_1 t} \) are two solutions. Indeed, from \( \lambda_1 \) being the double root of \( L(\lambda) = 0 \), we have \( L(\lambda_1) = 0, \) and \( L'(\lambda_1) = 0 \). By plugging \( te^{\lambda_1 t} \) into the equation (2.6), we obtain

\[ L \left( \frac{d}{dt} \right) \left( te^{\lambda_1 t} \right) = L(\lambda_1) \left( te^{\lambda_1 t} \right) + L'(\lambda_1) \left( e^{\lambda_1 t} \right) = 0. \]

### Homeworks.

1. Consider the equation \( y'' - y = 0 \).
   
   (a) Show that \( C_1 e^t + C_2 e^{-t} \) is a solution for any constants \( C_1, C_2 \).
   
   (b) Show that \( e^t \) and \( e^{-t} \) are independent. That is if \( C_1 e^t + C_2 e^{-t} = 0 \), then \( C_1 = C_2 = 0 \).
2. Show that $e^t$ and $te^t$ are independent.

3. Consider the ODE: $ay'' + by' + cy = 0$, with $a, b, c$ being real. Suppose $y(t) = y_1(t) + iy_2(t)$ be a complex solution.

(a) Show that both its real part $y_1$ and imaginary part $y_2$ are solutions too.

(b) Show any linear combination of $y_1$ and $y_2$ is also a solution.

It is important to observe that the solution set of (2.6) forms a linear space (vector space). That is, if $y_1(\cdot)$ and $y_2(\cdot)$ are two solutions of (2.6), so are their linear combinations $C_1y_1(\cdot) + C_2y_2(\cdot)$ for any two constants $C_1$ and $C_2$. In fact, you can check

$$L \left( \frac{d}{dt} \right) (C_1y_1 + C_2y_2) = C_1L \left( \frac{d}{dt} \right) y_1 + C_2L \left( \frac{d}{dt} \right) y_2 = 0.$$ 

We call this solution set the solution space of (2.6). From the existence and uniqueness for the initial value problem, we know that a general solution is uniquely determined by its initial data: $y(0)$ and $y'(0)$, which are two free parameters. Thus, the solution space of (2.6) is two dimensional. General solutions can be expressed as $C_1y_1(\cdot) + C_2y_2(\cdot)$. To solve the initial value problem (2.6), (2.4), we need to express in terms of the initial data $y_0$ and $y_1$.

There are three cases:

**Case 1.** $\lambda_1 \neq \lambda_2$ and real. A general solution for the homogeneous equation has the form $y(t) = C_1y_1(t) + C_2y_2(t)$, where

$$y_1(t) := e^{\lambda_1 t}, \quad y_2(t) := e^{\lambda_2 t}.$$ 

The constants $C_1$ and $C_2$ are determined by the initial condition (2.4):

$$C_1 + C_2 = y_0, \quad \lambda_1 C_1 + \lambda_2 C_2 = y_1.$$ 

From $\lambda_1 \neq \lambda_2$, we see that $C_1$ and $C_2$ can be solved uniquely:

$$C_1 = \frac{\lambda_2 y_0 - y_1}{\lambda_2 - \lambda_1}, \quad C_2 = \frac{y_1 - \lambda_1 y_0}{\lambda_2 - \lambda_1}.$$ 

**Case 2.** $\lambda_1 \neq \lambda_2$ and complex. In this case, they are conjugate to each other. Let us denote $\lambda_1 = \alpha + i\omega$ and $\lambda_2 = \alpha - i\omega$. We have found two solutions

$$y_1(t) = \text{Re}(e^{\lambda_1 t}) = e^{\alpha t} \cos \omega t$$
$$y_2(t) = \text{Im}(e^{\lambda_1 t}) = e^{\alpha t} \sin \omega t$$

A general solution of the form $y(t) = C_1y_1(t) + C_2y_2(t)$,

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1If you don’t know the definition of vector space, check into the Wikipedia
2.2. METHODS TO SOLVE SECOND ORDER LINEAR EQUATIONS

satisfying the initial condition (2.4) leads

\[
\begin{align*}
y_0 &= y(0) = C_1 \\
y_1 &= y'(0) = C_1\alpha + C_2\omega.
\end{align*}
\]

The constants \(C_1\) and \(C_2\) can be solved uniquely because we have \(\omega \neq 0\) in this case.

**Case 3.** \(\lambda_1 = \lambda_2\). In this case,

\[
y_1(t) := e^{\lambda_1 t} \text{ and } y_2(t) := te^{\lambda_1 t}
\]

are two independent solutions. So, general solution has the form \(C_1y_1(t) + C_2y_2(t)\). The constants \(C_1\) and \(C_2\) are determined by the initial data. This leads to

\[
\begin{align*}
C_1 &= y_0 \\
\lambda_1 C_1 + C_2 &= y_1.
\end{align*}
\]

The functions \(\{y_1(\cdot), y_2(\cdot)\}\) form a basis of the solution space. They are called the fundamental solutions of (2.6).

**Wronskian.** Suppose our initial data are set at time \(t_0\) instead of 0, i.e. we are given \(y(t_0) = y_0\) and \(y'(t_0) = y_1\). To find the corresponding solution \(y(\cdot)\), we assume \(y = C_1y_1(t) + C_2y_2(t)\). Plug into the initial conditions, we get two equations for \(C_1\) and \(C_2\):

\[
\begin{align*}
y_1(t_0)C_1 + y_2(t_0)C_2 &= y_0 \\
y_1'(t_0)C_1 + y_2'(t_0)C_2 &= y_1
\end{align*}
\]

To have solution, we need to require the determinant

\[
W(y_1, y_2)(t_0) := \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0
\]

We call this determinant the Wronskian of \(y_1\) and \(y_2\). In the homeworks below, you will check the Wronskian of the fundamental solutions \(y_1\) and \(y_2\) is never zero. This means that you can impose initial data at any time \(t_0\).

**Homework.**

1. Let \(\lambda_1 \neq \lambda_2\). Show that the \(W(e^{\lambda_1 t}, e^{\lambda_2 t}) \neq 0\) for all \(t\).

2. Let \(\lambda = \alpha + i\omega\). Find the Wronskians \(W(e^{\lambda t}, e^{-\lambda t})\) and \(W(e^{\alpha t} \cos \omega t, e^{\alpha t} \sin \omega t)\).

3. Let \(\lambda \in \mathbb{C}\). Show that the \(W(e^{\lambda t}, te^{\lambda t}) \neq 0\) for all \(t\).


5. Solve the initial value problem \(y'' - y' - 2y = 0\), \(y(0) = \alpha\), \(y'(0) = 2\). Then find \(\alpha\) so that the solution approaches zero as \(t \to \infty\).
6. Consider the ODE
\[ y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0. \]
(a) Determine the values of \( \alpha \) for which all solutions tend to zero as \( t \to \infty \).
(b) Determine the values of \( \alpha \) for which all solutions become unbounded as \( t \to \infty \).

7. B-D: pp. 164, 26

2.3 Linear oscillators

2.3.1 Harmonic oscillators

To understand the physical meaning of the solutions of the linear oscillation systems, let us first consider the case when there is no damping term (i.e. friction or resistance). That is
\[ L \left( \frac{d}{dt} \right) y = a \frac{d^2y}{dt^2} + cy = 0. \] (2.7)

We call such system a harmonic oscillator or free oscillator. The corresponding characteristic equation \( a\lambda^2 + c = 0 \) has two characteristic roots
\[ \lambda_1 = -i \sqrt{\frac{c}{a}}, \quad \lambda_2 = i \sqrt{\frac{c}{a}}, \]
which are pure imaginary due to both \( a, c > 0 \) in a harmonic oscillator. Let us denote
\[ \omega_0 = \sqrt{\frac{c}{a}} \] (2.8)

Then the general solution for (2.7) is
\[ C_1 e^{-i\omega_0 t} + C_2 e^{i\omega_0 t}. \]

Its real part forms the real solution of (2.7). It has the form
\[ y(t) = B_1 \cos \omega_0 t + B_2 \sin \omega_0 t, \]
where \( B_i \) are real. We may further simplify it as
\[ y(t) = A \cos(\omega_0 t + \theta_0) \] (2.9)

where
\[ A = \sqrt{B_1^2 + B_2^2}, \quad \cos(\theta_0) = B_1/A, \quad \sin(\theta_0) = -B_2/A, \]
\( A \) is called the amplitude and \( \theta_0 \) is the initial phase. They are related to the initial data \( y_0 \) and \( y_1 \) by \[ y_0 = A \cos(\theta_0), \quad y_1 = \omega_0 A \cos(\theta_0). \]
This motion is called harmonic oscillation or free oscillation. It is important to note that through a transformation:

\[ y = \cos \theta \]

the ODE (2.7) is converted to a linear motion with constant speed:

\[
\frac{d^2\theta}{dt^2} = 0, \quad \frac{d\theta}{dt} = \omega_0
\]

Its solution solution is given by \( \theta(t) = \theta_0 + \omega_0 t \). So it can be viewed as a circular motion with constant angular speed.

### 2.3.2 Damping

In this section, we consider (2.13) with damping term:

\[
ay'' + by' + cy = 0.
\]

The coefficient \( b > 0 \). We recall that the homogeneous equation has two independent solutions \( e^{\lambda_1 t} \) and \( e^{\lambda_2 t} \), where

\[
\lambda_1 = -\frac{b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = -\frac{b - \sqrt{b^2 - 4ac}}{2a},
\]

are the two roots of the characteristic equation \( a\lambda^2 + b\lambda + c = 0 \). We have the following cases: \( \Delta = b^2 - 4ac < 0, = 0 \) or \( > 0 \).

**Case 1. damped free oscillation**  When \( b^2 - 4ac < 0 \), we rewrite

\[
\lambda_1 = -\alpha + i\omega, \quad \lambda_2 = -\alpha - i\omega,
\]

where \( \alpha = b/2a > 0, \omega = \sqrt{4ac - b^2}/2a > 0 \). Then two independent solutions are

\[
y_1(t) = e^{-\alpha t} \cos(\omega t), \quad y_2(t) = e^{-\alpha t} \sin(\omega t).
\]

So, the general solutions for the homogeneous equation oscillate (the damper is not so strong and the oscillation is still maintained), but their amplitudes damp to zero exponentially fast at rate \( b/2a \). The relaxation time is \( \tau := 2a/b \). Thus, the smaller \( b \) is (weaker damper), the longer the relaxation time is. But, as long as \( b > 0 \), the solution decays to zero eventually.

In the spring-mass system, \( a = m, b = \gamma, c = k \). The free oscillation frequency is \( \omega_0^2 = k/m \). The effective oscillation \( y_i = e^{-(\gamma/2m)t} e^{i\omega t} \) has frequency \( \omega = \sqrt{4mk - \gamma^2}/2m < \sqrt{k/m} = \omega_0. \) Thus, the damping slows down the oscillation frequency. The frequency \( \omega \) is called the *quasifrequency*.

**Case 2. Critical damping**  When \( b^2 - 4ac = 0 \), the eigenvalue \( \lambda_1 = -b/2a \) is a double root. In additional to the solution \( y_1(t) = e^{\lambda_1 t} \), we can check

\[
y_2(t) = te^{\lambda_1 t}
\]

is another solution. You may check that this solution still decays to zero as \( t \to \infty \). Certainly it is slower than \( y_1(t) \). A concrete example is \( y'' + 2y' + y = 0. \)
Case 3. Overdamping  When \( b^2 - 4ac \geq 0 \), \( \lambda_i \) are real and negative. The two independent solutions

\[ y_i(t) = e^{\lambda_i t} \rightarrow 0, \text{ as } t \rightarrow \infty, \quad i = 1, 2. \]

We call this is overdamping. It means that the damper is too strong so that the solution has no oscillation at all and decays to 0 exponentially fast. The decay rate is \( O(e^{-\alpha t}) \), where \( \alpha = b/2a \). The quantity \( 1/\alpha \) is called the relaxation time. As a concrete example, consider \( y'' + 3y' + y = 0 \). One eigenvalue is \( \lambda_1 = -3/2 + \sqrt{5}/2 \). The other is \( \lambda_2 = -3/2 - \sqrt{5}/2 \). We see the solution \( y_1(t) = e^{\lambda t} \) decays slower than \( y_2(t) := e^{\lambda_2 t} \).

Homeworks.

1. Consider the ODE \( my'' + \gamma y' + ky = 0 \). Show that the energy defined by

\[ E(t) := \frac{m}{2} y'(t)^2 + \frac{1}{2} ky(t)^2 \]

satisfies \( E'(t) \leq 0 \).

2. Consider the ODE \( y'' + \alpha y' + \omega_0^2 y = 0 \) with \( \alpha, \omega > 0 \). In the critical case \( \alpha = 2\omega_0 \), there is a solution \( y^*(t) = te^{-\omega_0 t} \). When \( \alpha < 2\omega_0 \), construct a solution \( y_\alpha \) such that \( y_\alpha \rightarrow y^* \) as \( \alpha \rightarrow 2\omega_0 \).

3. B-D, pp. 204: 21

4. B-D, pp. 205: 26

2.3.3 Forcing and Resonance

In this section, we study forced vibrations. We will study two cases: free vibration with periodic forcing and damped vibration with periodic forcing.

Free vibration with periodic forcing  Let us consider the free vibration with a periodic forcing

\[ y'' + \omega_0^2 y = F_0 \cos(\Omega t). \]

We have two subcases.

Case 1. \( \Omega \neq \omega_0 \).

It is reasonable to guess that there is a special solution which is synchronized with the periodic external forcing. Thus, we try a special solution of the form \( C \cos(\Omega t) \). By plugging into the equation, we can find the coefficient \( C = F_0/(\alpha(\Omega^2 - \omega_0^2)) \). Thus, the function

\[ y_p(t) = \frac{F_0}{a(\Omega^2 - \omega_0^2)} \cos(\Omega t) \]
is a special solution. Let us still abbreviate $F_0/(\Omega^2 - \omega_0^2)$ by $C$. The general solution can be expressed as

$$y(t) = C \cos(\Omega t) + A \cos(\omega_0 t) + B \sin(\omega_0 t)$$

$$= C \cos((\omega_l - \omega_h)t) + A \cos((\omega_l + \omega_h)t) + B \sin((\omega_l + \omega_h)t)$$

$$= C (\cos(\omega_l t) \cos(\omega_h t) + \sin(\omega_l t) \sin(\omega_h t))$$

$$+ A (\cos(\omega_l t) \cos(\omega_h t) - \sin(\omega_l t) \sin(\omega_h t))$$

$$+ B (\sin(\omega_l t) \cos(\omega_h t) + \cos(\omega_l t) \sin(\omega_h t))$$

$$= [(C + A) \cos(\omega_l t) + B \sin(\omega_l t)] \cos(\omega_h t)$$

$$+ [B \cos(\omega_l t) + (C - A) \sin(\omega_l t)] \sin(\omega_h t)$$

$$= \tilde{A} \cos(\omega_l t - \Omega) \cos(\omega_h t) + \tilde{B} \cos(\omega_l t - \Omega) \sin(\omega_h t),$$

where

$$\omega_h = \frac{\omega_0 + \Omega}{2}, \quad \omega_l = \frac{\omega_0 - \Omega}{2}$$

indicate low and high frequencies, respectively; and

$$(C + A, B) = \tilde{A}(\cos(\Omega), \sin(\Omega)), \quad (C - A, B) = \tilde{B}(\cos(\Omega), \sin(\Omega)).$$

Let us take the case when $\Omega \sim \omega_0$. In this case,

$$C = \frac{F_0}{a(\Omega^2 - \omega_0^2)}$$

is very large, and hence $\tilde{A}$ is very large. We concentrate on the solution $y(t) = \tilde{A} \cos(\omega_l t - \Omega) \cos(\omega_h t)$. In this solution, we may view $\tilde{A} \cos(\omega_l t - \Omega)$ as the amplitude of the high frequency wave $\cos(\omega_h t)$. This amplitude itself is a low frequency wave, which is the *envelope* of the solution $y(t)$. We call it the *modulation wave*. This phenomenon occurs in acoustics when two tuning forks of nearly equal frequency are sound simultaneously.

**Case 2.** $\Omega = \omega_0$.

In this case, we try a special solution of this form:

$$y_p = Ct \cos(\omega_0 t) + Dt \sin(\omega_0 t).$$

By plugging into the equation, we find a special solution

$$y_p = Rt \sin(\omega_0 t), \quad R := \frac{F_0}{2a\omega_0}$$

The general solution is

$$y(t) = R t \sin(\omega_0 t) + A \cos(\omega_0 t + \theta_0)$$

(2.12)

The amplitude of this solution increases linearly in time. Such a phenomenon is called *resonance*. 
Damped vibrations with periodic forcing  We consider a damped vibration system with periodic forcing:

\[ y'' + by' + cy = F_0 \cos(\Omega t). \]

The two eigenvalues of the corresponding homogeneous system are

\[ \lambda_1 = -b + \sqrt{b^2 - 4c}, \quad \lambda_2 = -b - \sqrt{b^2 - 4c}. \]

As before, we have three cases: (1) overdamping, (2) critical damping, (3) underdamping:

- **Overdamping case:** \( b^2 - 4c > 0 \). In this case, \( \lambda_1 \) and \( \lambda_2 \) are real and negative. \( y_i(t) = e^{\lambda_i t}, \ i = 1, 2 \) are two independent solutions for the homogeneous equation.

- **Critical damping:** \( b^2 = 4c \). In this case, \( \lambda_1 = \lambda_2 = -b/2 \). The two independent solutions for the homogeneous equation are

\[ y_1(t) = e^{-bt/2}, \ y_2(t) = te^{-bt/2}. \]

- **Underdamping case:** \( b^2 - 4c < 0 \). In this case,

\[ \lambda_1 = -b + i\omega, \ \lambda_2 = -b - i\omega, \]

where

\[ \omega = \sqrt{4c - b^2}. \]

The two independent solutions of the homogeneous equations are \( y_1(t) = e^{-bt} \cos(\omega t) \) and \( y_2(t) = e^{-bt} \sin(\omega t) \).

To find a special solution for the inhomogeneous equation, we try

\[ y_p = C \cos(\Omega t) + D \sin(\Omega t). \]

By plugging into the equation, we find

\[-\Omega^2(C \cos(\Omega t) + D \sin(\Omega t)) + b\Omega(-C \sin(\Omega t) + D \cos(\Omega t)) + c(C \cos(\Omega t) + D \sin(\Omega t)) = F_0 \cos(\Omega t).\]

This yields

\[-\Omega^2 C + b\Omega D + cC = F_0\]
\[-\Omega^2 D - b\Omega C + cD = 0\]

This solves \( C \) and \( D \):

\[ C = (c - \Omega^2)F_0/\Delta, \quad D = b\Omega F_0/\Delta, \]

where

\[ \Delta = (c - \Omega^2)^2 + b^2\Omega^2. \]

Notice that \( \Delta \neq 0 \) whenever there is a damping. Let

\[ A := \sqrt{C^2 + D^2} = \frac{F_0}{\Delta}, \quad \Omega_0 = \arctan \left( \frac{-b\Omega}{c - \Omega^2} \right). \]
Then
\[ y_p = C \cos(\Omega t) + D \sin(\Omega t) \]
\[ = A \cos(\Omega_0 \cos(\Omega t) - A \sin(\Omega_0 \sin(\Omega t)) \]
\[ = A \cos(\Omega t + \Omega_0) \]

Thus, a special solution is again a cosine function with amplitude \( A \) and initial phase \( \Omega_0 \). The general solution is
\[ y(t) = A \cos(\Omega t + \Omega_0) + C_1 y_1(t) + C_2 y_2(t). \]

Notice that \( y(t) \to A \cos(\Omega t + \Omega_0) \) as \( t \to \infty \) because both \( y_1(t) \) and \( y_2(t) \) tend to 0 as \( t \to \infty \). We call the solution \( A \cos(\Omega t + \Omega_0) \) the steady-state solution or the forced response. This solution synchronized with the external periodic forcing.

**Remarks.**

- We notice that the amplitude \( A \) has maximum when \( \Omega = \omega_0 := \sqrt{c} \), that is, the external forcing has the same period as the internal period \( \omega_0 \).

- We also notice that \( A \to \infty \) only when \( b = 0 \) (no damping) and \( c = \Omega^2 \). This is the resonance case. Otherwise, there is no resonance. In other word, general solutions approach the forced responded solution, even in the case of resonance with damping.

**Homework.**

Find a special solution for the following equations

1. Compute the general solution of the given equation.
   
   (a) \( y'' + 4y = 3 \cos 2t \).
   
   (b) \( y'' + 9y = \sin t + \sin 2t + \sin 3t \).
   
   (c) \( y'' + 4y = \cos^2 t \).

2. Solve the initial value problem \( y'' + 4y = 3 \cos 2t + \cos t \), \( y(0) = 2 \), \( y'(0) = 1 \).

3. Consider the ODE \( y'' + \omega_0^2 y = \cos \omega t \) with \( \omega \sim \omega_0 \), say \( \omega = \omega_0 + \Delta \omega \). For each \( \Delta \omega \), find a particular solution of this equation so that its limit approaches the resonant solution as \( \Delta \omega \to 0 \).

4. B-D, pp. 215: 15,16, 18
2.4 Inhomogeneous equations

Now, we study the inhomogeneous equation with general forcing term \( f \):

\[
ay'' + by' + cy = f(t).
\]  

(2.13)

We may abbreviate it by an operator notation:

\[
L \left( \frac{d}{dt} \right) [y] = f,
\]

where \( L(s) = as^2 + bs + c \). From the theory for homogeneous equations, we know that we can find two independent solutions Let \( y_1(\cdot) \) and \( y_2(\cdot) \) be a set of fundamental solutions of the homogeneous equation

\[
L \left( \frac{d}{dt} \right) [y] = 0.
\]

Suppose \( y_p(\cdot) \) is a special solution of (2.13), then so is \( y_p + C_1 y_1 + C_2 y_2 \) for any constants \( C_1 \) and \( C_2 \). This is because the linearity of the equation. Namely,

\[
L[y_p + C_1 y_1 + C_2 y_2] = L[y_p] + C_1 L[y_1] + L[y_2] = f + 0 + 0.
\]

From the existence and uniqueness of ODEs, we know the solution sets depends on two parameters. We can conclude that the solution set \( S \) to (2.13) is \( S = y_p + S_0 \), where \( S_0 \) is the solution space corresponding to the homogeneous equation. In other words, the solution set of (2.13) is an affine space. The choice of the special solution is not unique. If \( y_q \) is another special solution, then a solution \( y = y_p + z \) with \( z \in S_0 \), then \( y = y_q + (y_p - y_q + z) \) and \( y_p - y_q + z \in S_0 \). Thus, it is sufficient to find just one special solution. Then we can construct all solutions with the helps of the fundamental solutions \( y_1 \) and \( y_2 \).

We introduction two methods to find a special solution. In latter chapter, we will further introduce the method of Laplace transform to find special solutions.

2.4.1 Method of undetermined coefficients

In the case when the source term is of the form:

\[
t^k e^{\lambda t},
\]

we can use the following method of undetermined coefficient to find a special solution. We use examples to explain.

1. \( f(t) = t^k \). We try \( y_p(t) \) to be a polynomial of degree \( k \). That is

\[
y_p(t) = a_k t^k + a_{k-1} t^{k-1} + \cdots + a_0.
\]

Plug this into equation, we obtain a polynomial equations. Equate both sides and we can determine the coefficients. There are \( k + 1 \) linear equations for \( k + 1 \) knowns \( a_k, \ldots, a_0 \).
Example 1. Let \( f(t) = t \). We try \( y_p = a_1 t + a_0 \). By \( L(D)y_p = t \), we get
\[
 a \cdot 0 + b \cdot (a_1) + c \cdot (a_1 t + a_0) = t.
\]
This yields
\[
 ca_1 = 1 \\
 ba_1 + ca_0 = 0.
\]
Hence, \( y_p = t/c - b/c^2 \) is a special solution.

2. \( f(t) = t^k e^{\alpha t} \), where \( \alpha \) is real. We have two subcases.

- \( \alpha \neq \lambda_i \), where \( \lambda_i, i = 1, 2 \) are roots of the characteristic equation \( a\lambda^2 + b\lambda + c = 0 \). We try
\[
 y_p(t) = (a_k t^k + a_{k-1} t^{k-1} + \cdots + a_0) e^{\alpha t}.
\]
Plug into equation, we can determine the coefficients \( a_i \).

Example 2. Find a special solution for \( y'' - y = te^{2t} \). We choose \( y_p(t) = (at + b)e^{2t} \). Plug this into the equation, we get
\[
 4(at + b)e^{2t} + 4ae^{2t} - (at + b)e^{2t} = te^{2t}
\]
This yields
\[
 3a = 1 \\
 4b + 4a - b = 0.
\]
Hence, \( a = 1/3 \) and \( b = -4/9 \).

- \( \alpha = \lambda_1 \). This is a resonant case. We try
\[
 y_p = t(a_k t^k + \cdots + a_0) e^{\lambda_1 t}
\]

Example 3. Let us consider \( y'' - y = e^t \) as an example. We try \( y_p = ate^t \). We have
\[
 y'_p = ae^t + (at)e^t \\
 y''_p = 2ae^t + (at)e^t
\]
The equation \( y'' - y = e^t \) yields
\[
 (at)e^t + 2ae^t - (at)e^t = e^t.
\]
This gives
\[
 a - a = 0 \\
 2a = 1
\]
Hence, \( y_p = \frac{1}{2} te^t \) is a special solution.
3. \( f(t) = t^k e^{\alpha t} \cos(\Omega t) \), or \( t^k e^{\alpha t} \sin(\Omega t) \). In this case, we introduce a complex forcing term

\[
f(t) = t^k e^{\lambda t}, \quad \lambda := \alpha + i\Omega.
\]

The real part of a solution to this complex forcing term is a special solution to the forcing term \( t^k e^{\alpha t} \cos(\omega t) \). For this complex forcing term, it can be reduced to the previous case.

**Homework.**

1. Find a special solution for \( y'' - y = te^t \).
2. Find a special solution for \( y'' - 2y' + y = e^t \).
3. Find a special solution for \( y'' - 2y' + y = te^t \).
4. Find a special solution for \( y'' + 4y = te^{it} \).
5. Find a special solution for \( y'' + y = te^{it} \).

**Homework.**

1. B-D, pp. 184: 12, 27
2. B-D, pp. 185: 29, 31, 32, 33
3. B-D, pp. 186: 34

### 2.4.2 Method of Variation of Constants

We use variation of constants to solve the inhomogeneous equation (2.13). For the simplicity, we may assume the coefficient \( a \) of (2.13) is 1. Suppose \( y_1(\cdot) \) and \( y_2(\cdot) \) are two independent solutions of the homogeneous equation (2.6). We assume the solution of (2.13) has the form

\[
\begin{pmatrix}
y(t) \\
y'(t)
\end{pmatrix} = C_1(t) \begin{pmatrix}
y_1(t) \\
y_1'(t)
\end{pmatrix} + C_2(t) \begin{pmatrix}
y_2(t) \\
y_2'(t)
\end{pmatrix}
\]

The first equation is \( y(t) = C_1(t)y_1(t) + C_2(t)y_2(t) \). We differentiate it and combine the resulting equation with the second equation \( y'(t) = C_1(t)y_1'(t) + C_2(t)y_2'(t) \). We get

\[
C_1'y_1 + C_2'y_2 = 0.
\]

Plugging (2.14) this into (2.13), we obtain

\[
L \left( \frac{d}{dt} \right) y = C_1L \left( \frac{d}{dt} \right) y_1 + C_2L \left( \frac{d}{dt} \right) y_2 + (C_1'y_1' + C_2'y_2') = f
\]

This leads to

\[
(C_1'y_1' + C_2'y_2') = f
\]
2.4. INHOMOGENEOUS EQUATIONS

Equations (2.15) and (2.16) give a first-order differential equation for $C_1$ and $C_2$:

$$\begin{pmatrix} C'_1(t) \\ C'_2(t) \end{pmatrix} = \Phi(t)^{-1} \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad (2.17)$$

where

$$\Phi(t) := \begin{pmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{pmatrix}. \quad (2.18)$$

By integrating (2.17), we obtain

$$\begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix} = \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} + \int_0^t \Phi(s)^{-1} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds$$

$$= \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} + \int_0^t \frac{1}{W(y_1, y_2)(s)} \begin{pmatrix} y'_2(s) & -y_2(s) \\ -y'_1(s) & y_1(s) \end{pmatrix} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds$$

$$= \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} + \int_0^t \frac{1}{W(y_1, y_2)(s)} \begin{pmatrix} -y_2(s)f(s) \\ y_1(s)f(s) \end{pmatrix} ds$$

Thus, a special solution is given by the following expression

$$y_p(t) = -y_1(t) \int_0^t \frac{y_2(s)f(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_0^t \frac{y_1(s)f(s)}{W(y_1, y_2)(s)} ds. \quad (2.19)$$

**Example.** Solve the equation

$$y'' - y = f(t)$$

with initial data

$$y(0) = 0, \quad y'(0) = 0.$$

**Answer.** The homogeneous equation $y'' - y = 0$ has fundamental solutions $y_1(t) = e^{-t}$ and $y_2(t) = e^t$. The corresponding Wronskian

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 2.$$}

Thus, the special solution

$$y_p(t) = -e^{-t} \int_0^t \frac{e^s f(s)}{2} ds + e^t \int_0^t \frac{e^{-s} f(s)}{2} ds$$

$$= \int_0^t \sinh(t - s) f(s) ds$$

You may check this special solution satisfies the initial conditions $y(0) = y'(0) = 0$.

**Example.** Find a particular solution of

$$y'' + y = \csc t$$
for $t$ near $\pi/2$. Ans. The fundamental solutions corresponding to the homogeneous equation is

$$y_1(t) = \cos t, \quad y_2(t) = \sin t.$$ 

The Wronskian $W(y_1, y_2)(t) = 1$. A special solution is given by

$$y_p(t) = -y_1(t) \int_{\pi/2}^{t} \frac{y_2(s)f(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_{\pi/2}^{t} \frac{y_1(s)f(s)}{W(y_1, y_2)(s)} ds$$

$$= -\cos t \int_{\pi/2}^{t} \sin(s) \csc(s) \, ds + \sin t \int_{\pi/2}^{t} \cos(s) \csc(s) \, ds$$

$$= -(t - \pi/2) \cos t + \sin t \cdot \ln \sin t.$$ 

**Homeworks.**

1. B-D, pp. 190: 5, 7, 10, 22
2. B-D, pp. 191: 23, 24, 25, 26, 27
Chapter 3

Linear Systems with Constant Coefficients

3.1 Initial value problem for $n \times n$ linear systems

3.1.1 Examples

An general $n \times n$ linear system of differential equation is of the form

$$y'(t) = Ay(t) + f(t),$$

(3.1)

where

$$y = \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad f = \begin{pmatrix} f^1 \\ f^2 \\ \vdots \\ f^n \end{pmatrix},$$

Its initial value problem is to study (3.1) with initial condition:

$$y(0) = y_0.$$  

(3.2)

We list some important examples below.

**Reduction to first-order systems**  A general high-order ODE can be reduced to systems of first-order equation by introducing high derivatives as new unknowns. For example, the linear second-order ODE

$$ay'' + by' + cy = f$$

(3.3)

can be rewritten as

$$\begin{cases} y' = v \\ av' = -bv - cy + f \end{cases}$$

If $(y, v)$ is a solution of this first-order system, then from $v = y'$, we have $v' = y''$. From the second equation, we conclude that $y$ satisfies $ay'' + by' + cy = f$. Conversely, if $y$ satisfies (3.3), then $y$ is twice differentiable. Let us name $y' = v$. Then $v' = y''$. From (3.3), $av' + bv + cy = f$. Hence, these two equations are equivalent.
In general, an nth-order equation
\[ y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) \]
is equivalent to the following system
\[
\begin{align*}
    y'_1 &= y_2 \\
    y'_2 &= y_3 \\
    &\vdots \\
    y'_n &= f(t, y_1, y_2, \ldots, y^n)
\end{align*}
\]

**Rotation in \( \mathbb{R}^3 \)** Let \( \Omega \) be a vector in \( \mathbb{R}^3 \). Consider the equation in \( \mathbb{R}^3 \):
\[ y'(t) = \Omega \times y(t) \quad (3.4) \]
This equation can be written as the form \( y' = Ay \) with
\[ A = \begin{pmatrix} 0 & -\omega_3 & -\omega_2 \\ \omega_3 & 0 & -\omega_1 \\ \omega_2 & \omega_1 & 0 \end{pmatrix}. \]
Physically, if \( y \) denote a velocity, then the term \( \Omega \times y \) is a force pointing to the direction perpendicular to \( y \) and \( \Omega \). We will see later that this is a rotation.

**Coupled spring-mass systems** Consider a coupled spring-mass system. The system contains two masses and springs. The mass \( m_1 \) is connected on its two ends to wall and a mass \( m_2 \) respectively by springs with spring constants \( k_1 \) and \( k_2 \). The mass \( m_2 \) is connected to mass \( m_1 \) on one end and to the wall on the other end by a spring with spring constant \( k_3 \). Let \( x_i \) be the position of the mass \( m_i \). Then the equations for \( x_i \) are
\[
\begin{align*}
    m_1 x_1'' &= -k_1 x_1 - k_2 (x_1 - x_2) \\
    m_2 x_2'' &= -k_3 x_2 - k_2 (x_2 - x_1)
\end{align*}
\]
We can rewrite this equation into a \( 4 \times 4 \) system of linear equation by introducing \( y = (x_1, x'_1, x_2, x'_2)^t \).

**Homework.**

**3.1.2 Linearity and solution space**
We shall first study the homogeneous equation
\[ y' = Ay. \quad (3.5) \]
Since the equation is linear in \( y \), we can see the following linear property of the solutions. Namely, if \( y_1 \) and \( y_2 \) are solutions of (3.5), so does their linear combination: \( \alpha_1 y_1 + \alpha_2 y_2 \), where \( \alpha_1, \alpha_2 \) are any two scalar numbers. Therefore, if \( S_0 \) denotes the set of all solutions of (3.5), then \( S \) is a vector space.

In the case of inhomogeneous equation (3.1), suppose we have already known a particular solution \( y_p \), then so is \( y_p + y \) for any \( y \in S_0 \). On the other hand, suppose \( z \) is a solution of the inhomogeneous equation:

\[
z' = Az + f
\]

then \( z - y_p \) satisfies the homogeneous equation (3.5). Hence, \( z - y_p = y \) for some \( y \in S \). We conclude that the set of all solutions of the inhomogeneous equation (3.1) is the affine space

\[
S = y_p + S_0.
\]

To determine the dimension of the solution, we notice that all solutions are uniquely determined by their initial data (the uniqueness theorem),

\[
y(0) = y_0 \in \mathbb{R}^n.
\]

Hence, \( S \) is \( n \) dimensional. We conclude this argument by the following theorem.

**Theorem 3.1** The solution space \( S_0 \) for equation (3.5) is a two-dimensional vector space. The solution space for equation (3.1) is the affine space \( y_p + S_0 \), where \( y_p \) is a particular solution of (3.1).

**Basis** In a vector space \( V \), a set of vectors \( v_1, \ldots, v_n \) is called a basis of \( V \) if

- \( V \) is spanned by \( v_1, \ldots, v_n \), that is,

\[
V = \{ \sum_{i=1}^{n} C_i v_i | C_i \in \mathbb{R} \}
\]

- \( v_1, \ldots, v_n \) are independent, that is, if \( \sum_{i=1}^{n} C_i v_i = 0 \), then \( C_i = 0 \) for all \( i = 1, \ldots, n \).

This two conditions implies that any vector \( v \) can be represented as \( v = \sum_{i=1}^{n} C_i v_i \) uniquely.

Our goal in this section is to construct a basis \( \{ y_1, \ldots, y_n \} \) in \( S_0 \). A general solution in \( S_0 \) can be represented as

\[
y(t) = \sum_{i=1}^{n} C_i y_i(t).
\]

For an initial value problem with \( y(t_0) = y_0 \), the coefficients \( C_i \) are determined by the linear equation

\[
\sum_{i=1}^{n} y_i(t_0) C_i = y_0.
\]
or

\[ Y(t_0)C = y_0 \]

where

\[ Y(t) = [y_1(t), y_2(t), \ldots, y_n(t)], C = [C_1, \ldots, C_n]^t. \]

If \( y_1, \ldots, y_n \) are independent, then \( C_i \) can be solved uniquely. Such a set of solutions \( \{y_1, \ldots, y_n\} \) is called a fundamental solution of (3.5).

So, two main issues are:

- How to find a set of solutions \( y_1, \ldots, y_n \) in \( S_0 \)?
- How to know they are independent?

We shall answer these questions in the next section for \( 2 \times 2 \) system.

3.2 \( 2 \times 2 \) systems

3.2.1 Independence and Wronskian

In the solution space \( S_0 \), two solutions \( y_1 \) and \( y_2 \) are called independent if \( C_1y_1(t) + C_2y_2(t) = 0 \) implies \( C_1 = C_2 = 0 \). This definition is for all \( t \), but based on the uniqueness theorem, we only need to check this condition at just one point. We have the following theorem.

**Theorem 3.2** Suppose \( y_1 \) and \( y_2 \) are solutions of (3.5). If \( y_1(t_0) \) and \( y_2(t_0) \) are independent in \( \mathbb{R}^2 \), then \( y_1(t) \) and \( y_2(t) \) are independent in \( \mathbb{R}^2 \) for all \( t \) in the maximal interval of existence for both \( y_1 \) and \( y_2 \) which contains \( t_0 \).

**Proof.** Let \( t_1 \) be a point lying in the maximal interval of existence containing \( t_0 \). Suppose \( y_1(t_1) \) and \( y_2(t_1) \) are linearly dependent, then there exist constants \( C_1 \) and \( C_2 \) such that

\[ C_1y_1(t_1) + C_2y_2(t_1) = 0. \]

Let \( y = C_1y_1 + C_2y_2 \). Notice that both \( y \) and the zero constant solution have the same value at \( t_1 \). By the uniqueness theorem, \( y \equiv 0 \) on the maximal interval of existence containing \( t_1 \), hence, containing \( t_0 \). This contradicts to \( y_1(t_0) \) and \( y_2(t_0) \) being independent. \( \square \)

Given any two solutions \( y_1 \) and \( y_2 \), we can define the Wronskian

\[ W(y_1, y_2)(t) = \text{det}(y_1(t), y_2(t)) = \begin{vmatrix} y_{1,1} & y_{2,1} \\ y_{1,2} & y_{2,2} \end{vmatrix} \]  \hspace{1cm} (3.6)

to test the independence of them.

**Theorem 3.3** Let \( y_1 \) and \( y_2 \) be two solutions of (3.5). Let us abbreviate the Wronskian \( W(y_1, y_2)(t) \) by \( W(t) \). We have
(i) 
\[ \frac{dW}{dt} = (tr A)W \]

(ii) \( W(t_0) \neq 0 \) if and only if \( W(t) \neq 0 \) for all \( t \).

**Proof.** Let \( Y = (y_1, y_2) \). Then we have
\[ Y' = AY. \]

The Wronskian \( W(t) \) is \( det Y(t) \). We differentiate \( W \) in \( t \), we get
\[ W' = y_{1,1}y_{2,2} - y'_{1,2}y_{2,1} - y'_{2,1}y_{1,2} + y'_{2,2}y_{1,1} \]
\[ = \sum_k (a_{1,k}y_{k,1}y_{2,2} - a_{1,k}y_{k,2}y_{2,1} - a_{2,k}y_{k,1}y_{1,2} + a_{2,k}y_{k,2}y_{1,1}) \]
\[ = (a_{1,1} + a_{2,2})(y_{1,1}y_{2,2} - y_{1,2}y_{2,1}) \]
\[ = tr(A)W. \]

Since \( W(t) = W(t_0) \exp(tr(A)(t - t_0)) \), we see that \( W(t_0) \neq 0 \) if and only if \( W(t) \neq 0 \).

### 3.2.2 Finding exact solutions

For the homogeneous equation
\[ y'(t) = Ay(t) \]
we try a solution of the form \( y(t) = e^{\lambda t}v \), where \( v \in \mathbb{R}^2 \) is a constant. Plugging into (3.5), we get
\[ \lambda v e^{\lambda t} = Av e^{\lambda t}. \]

We find that \( y(t) = e^{\lambda t}v \) is a solution of (3.5) if and only if
\[ Av = \lambda v. \quad (3.7) \]

That is, \( \lambda \) is the eigenvalue and \( v \) is the corresponding eigenvector. The eigenvalue \( \lambda \) satisfies the following characteristic equation
\[ \det (\lambda I - A) = 0. \]

In two dimension, this is
\[ \lambda^2 - T\lambda + D = 0, \]
where
\[ T = a + d, \quad D = ad - bc \]
are the trace and determinant of \( A \), respectively. The eigenvalues are
\[ \lambda_1 = \frac{T + \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2}. \]

There are three possibilities for the eigenvalues:
• $T^2 - 4D > 0$. Then $\lambda_1 \neq \lambda_2$ and are real.

• $T^2 - 4D < 0$. Then $\lambda_1, \lambda_2$ and are complex conjugate.

• $T^2 - 4D = 0$. Then $\lambda_1$ is a double root.

**Case 1.**

$\lambda_i$ are real and there are two independent real eigenvectors $v_1$ and $v_2$. The corresponding two independent solutions are

$$y_1 = e^{\lambda_1 t}v_1, \quad y_2 = e^{\lambda_2 t}v_2.$$ 

A general solution has the form

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

If the initial data is $y_0$, then

$$C_1 v_1 + C_2 v_2 = y_0.$$ 

Let $T$ be the matrix $(v_1, v_2)$. Then

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = T^{-1} y_0.$$ 

The $0$ state is an equilibrium. Its behavior is determined by the sign of the eigenvalues:

• $\lambda_1, \lambda_2 < 0$: all solutions tend to 0 as $t \to \infty$. We call 0 state a sink. It is a stable equilibrium.

• $\lambda_1, \lambda_2 > 0$: all solutions tend to infinity as $t \to \infty$. In fact, all solutions tend to the 0 state as $t \to -\infty$. We call 0 state a source. It is an unstable equilibrium.

• $\lambda_1 \lambda_2 < 0$. Let us take $\lambda_1 < 0$ and $\lambda_2 > 0$ as an example for explanation. A general solution has the form

$$y(t) = C_1 e^{\lambda_1 t}v_1 + C_2 e^{\lambda_2 t}v_2.$$ 

We have $e^{\lambda_1 t} \to 0$ and $e^{\lambda_2 t} \to \infty$ as $t \to \infty$. Hence if $y(0) \in M_s := \{\gamma v_1, \gamma \in \mathbb{R}\}$, then the corresponding $C_2 = 0$, and $y(t) \to 0$ as $t \to \infty$. We call the line $M_s$ a stable manifold. On the other hand, if $y(0) \in M_u := \{\gamma v_2, \gamma \in \mathbb{R}\}$, then the corresponding $C_1 = 0$ and $y(t) \to 0$ as $t \to -\infty$. We call the line $M_u$ an unstable manifold. For any other $y_0$, the corresponding $y(t)$ has the following asymptotics:

$$y(t) \to v_1\text{-axis, as } t \to -\infty,$$

$$y(t) \to v_2\text{-axis, as } t \to +\infty.$$ 

That is, all solutions approach the stable manifold as $t \to \infty$ and the unstable manifold as $t \to -\infty$. The 0 state is the intersection of the stable and unstable manifolds. It is called a saddle point.

• $\lambda_1 = 0$ and $\lambda_2 \neq 0$. In this case, a general solution has the form: $y(t) = C_1 v_1 + C_2 e^{\lambda_2 t}v_2$. The equilibrium $\{\bar{y} | A\bar{y} = 0\}$ is a line: $\{C_1 v_1 | C_1 \in \mathbb{R}\}$. When $\lambda_2 < 0$, then all solutions approach $C_1 v_1$. This means that the line $C_1 v_1$ is a stable line.
Example 1. Consider the matrix

\[ A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}. \]

The corresponding characteristic equation is

\[ \det (\lambda I - A) = (\lambda - 1)^2 - 4 = 0. \]

Hence, the two eigenvalues are

\[ \lambda_1 = 3, \quad \lambda_2 = -1. \]

The eigenvector \( v_1 \) corresponding to \( \lambda_1 = 3 \) satisfies

\[ (A - \lambda_1 I) v_1 = 0. \]

This gives

\[ v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \]

Similarly, the eigenvector corresponding to \( \lambda_2 = -1 \) is

\[ v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \]

Example 2.

\[ y = Ay, \quad A = \begin{pmatrix} 8 & -11 \\ 6 & -9 \end{pmatrix}. \]

The eigenvalues of \( A \) are roots of the characteristic equation \( \det (\lambda I - A) = 0 \). This yields two eigenvalues \( \lambda_1 = -3 \) and \( \lambda_2 = 2 \). The corresponding eigenvectors satisfy

\[ (A - \lambda_i) v_i = 0. \]

For \( v_1 \), we have

\[ \begin{pmatrix} 8 + 3 & -11 \\ 6 & -9 + 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

This yields

\[ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

Similarly, we obtain

\[ v_2 = \begin{pmatrix} 11 \\ 6 \end{pmatrix}. \]

The general solution is

\[ y(t) = C_1 e^{-3t} v_1 + C_2 e^{2t} v_2. \]

The line in the direction of \( v_1 \) is a stable manifold, whereas the line in \( v_2 \) direction is an unstable manifold. The origin is a saddle point.
Example 3.

\[ y = Ay, \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. \]

The eigenvalues of \( A \) are \( \lambda_1 = 0 \) and \( \lambda_2 = -5 \). The corresponding eigenvectors are \( v_1 = (2, -1)^t \) and \( v_2 = (1, 2)^t \). The general solutions are \( y(t) = C_1(2, -1)^t + C_2e^{-5t}(1, 2)^t \). All solutions approach the line \( C_1(2, -1)^t \).

Case 2.

\( \lambda_i \) are complex conjugate.

\[ \lambda_1 = \alpha + i\omega, \quad \lambda_2 = \alpha - i\omega. \]

Since \( A \) is real-valued, the corresponding eigenvectors are also complex conjugate:

\[ w_1 = u + iv, \quad w_2 = u - iv. \]

We have two independent complex-valued solutions: \( z_1 = e^{\lambda_1 t}w_1 \) and \( z_2 = e^{\lambda_2 t}w_2 \).

Since our equation (3.5) has real coefficients, its real-valued solution can be obtained by taking the real part (or pure imaginary part) of the complex solution. In fact, suppose \( z(t) = x(t) + iy(t) \) is a complex solution of the real-value ODE (3.5). Then

\[ \frac{d}{dt}(x(t) + iy(t)) = A(x(t) + iy(t)). \]

By taking the real part and the imaginary part, using the fact that \( A \) is real, we obtain

\[ \frac{dx}{dt} = Ax(t), \quad \frac{dy}{dt} = Ay(t) \]

Hence, both the real part and the imaginary part of \( z(t) \) satisfy the equation.

Now, let us take the real part and the imaginary part of one of the above solution:

\[ z_1(t) = \left(e^{\alpha t}(\cos \omega t + i \sin \omega t)\right)(u + iv) \]

Its real part and imaginary part are respectively

\[ y_1(t) = e^{\alpha t}(\cos \omega t u - \sin \omega t v), \quad y_2(t) = e^{\alpha t}(\sin \omega t u + \cos \omega t v) \]

The other solution \( z_2 \) is the complex conjugate of \( z_1 \). We extract the same real solutions from \( z_2 \).

You may wonder now whether \( u \) and \( v \) are independent. Indeed, if \( v = cu \) for some \( c \in \mathbb{R} \), then

\[ A(u + iv) = \lambda_1(u + iv) \]

gives

\[ A(1 + ic)u = \lambda_1(1 + ic)u \]
3.2. 2 × 2 SYSTEMS

\[
\mathbf{A}\mathbf{u} = \lambda_1 \mathbf{u} = (\alpha + i\omega)\mathbf{u}
\]

This yields
\[
\mathbf{A}\mathbf{u} = \alpha \mathbf{u}, \quad \text{and} \quad \omega \mathbf{u} = 0,
\]
because \(\mathbf{A}\) is real. This implies \(\omega = 0\) if \(\mathbf{u} \neq 0\). This contradicts to that the eigenvalue \(\lambda_1\)
has nontrivial imaginary part.

From the independence of \(\mathbf{u}\) and \(\mathbf{v}\), we conclude that \(\mathbf{y}_1\) and \(\mathbf{y}_2\) are also independent,
and constitute a real basis in the solution space \(S\). A general solution is given by
\[
\mathbf{y}(t) = C_1\mathbf{y}_1(t) + C_2\mathbf{y}_2(t),
\]
where \(C_i\) are real and are determined by the initial data:
\[
C_1\mathbf{y}_1(0) + C_2\mathbf{y}_2(0) = \mathbf{y}_0,
\]
\[
C_1\mathbf{u} + C_2\mathbf{v} = \mathbf{y}_0.
\]
Let \(T\) be the matrix \((\mathbf{u}, \mathbf{v})\), then
\[
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = T^{-1}\mathbf{y}_0.
\]

We may use another parameters to represent the solution.
\[
\mathbf{y}(t) = C_1 e^{\omega t} (\cos(\omega t)\mathbf{u} - \sin(\omega t)\mathbf{v}) + C_2 e^{\omega t} (\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{v})
\]
\[
= e^{\alpha t} ((C_1 \cos(\omega t) + C_2 \sin(\omega t))\mathbf{u} + (C_2 \cos(\omega t) - C_1 \sin(\omega t))\mathbf{v})
\]
\[
= A e^{\alpha t} (\cos(\omega t - \omega_0)\mathbf{u} + \sin(\omega t - \omega_0)\mathbf{v}),
\]
where \((C_1, C_2) = A(\cos(\omega_0), \sin(\omega_0))\).

Example 1.  Consider the matrix
\[
\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -4 & -1 \end{pmatrix},
\]
The characteristic equation is \(\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^2 - \lambda - 2 = 0\). The roots are \(\lambda_1 = (1 + i\sqrt{7})/2\)
and \(\lambda_2 = (1 - i\sqrt{7})/2\). The corresponding eigenvectors are
\[
\mathbf{v}_1 = \begin{pmatrix} -2 \\ 3 + i\sqrt{7} \end{pmatrix} := \mathbf{u} + i\mathbf{w}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 3 - i\sqrt{7} \end{pmatrix} := \mathbf{u} - i\mathbf{w}.
\]
We get two complex-valued solutions \(\mathbf{z}_1 = e^{\lambda_1 t}\mathbf{v}_1\) and \(e^{\lambda_2 t}\mathbf{v}_2\). The real solutions are their
real parts and imaginary parts. They are
\[
\mathbf{y}_1 = e^{t/2} (\cos(\omega t)\mathbf{u} - \sin(\omega t)\mathbf{w}),
\]
\[
\mathbf{y}_2 = e^{t/2} (\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{w})
\]
where \(\omega = \sqrt{7}/2\). To study the solution structure, we divide it into the following cases.

- \(\alpha = 0\): The eigenvalues are pure imaginary. All solutions are ellipses.
- \(\alpha < 0\): The solution are spirals and tend to 0 as \(t \to \infty\). The 0 state is a spiral sink.
- \(\alpha > 0\): The solution are spirals and tend to 0 as \(t \to -\infty\). The 0 state is a spiral source.
Case 3.

\( \lambda_1 = \lambda_2 \) are real and only one eigenvector. Let us see examples first to get some intuition.

1. **Example 1.** Consider the matrix

\[
A = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix},
\]

where \( r \) is a constant. The eigenvalue of \( A \) is \( r \) and \( v_1 = (1, 0)^T \) is the corresponding eigenvector. The \( y_2 \) component satisfies single equation

\[
y_2' = ry_2.
\]

We obtain \( y_2(t) = C_2e^{rt} \) and by plugging this into the first equation

\[
y_1' = ry_1 + C_2e^{rt},
\]

we find \( y_1(t) = C_2te^{rt} \) is a special solution. The general solution of \( y_1 \) is \( y_1(t) = C_2te^{rt} + C_1e^{rt} \). We can express these general solutions in vector form:

\[
y(t) = C_1e^{rt}(1, 0)^T + C_2\begin{pmatrix} e^{rt}(0, 1)^T + te^{rt}(1, 0)^T \end{pmatrix}.
\]

2. **Example 2** Consider the matrix

\[
A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.
\]

The characteristic equation

\[
0 = \det(\lambda I - A) = (\lambda - 1)(\lambda - 3) + 1 = (\lambda - 2)^2.
\]

has a double root \( \lambda = 2 \). The corresponding eigenvector satisfies

\[
(A - 2I)v = 0
\]

\[
\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

This yields a solution, called \( v_1 \):

\[
v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

This is the only eigenvector. The solution \( e^{2t}v_1 \) is a solution of the ODE. To find the other independent solution, we expect that there is a resonant solution \( te^{2t} \) in the
3.2. $2 \times 2$ SYSTEMS

Direction of $v_1$. Unfortunately, $te^{2t}v_1$ is not a solution unless $v_1 = 0$. Therefore, we try to another

$$y(t) = te^{2t}v_1 + e^{\mu t}v_2,$$

for some unknown vector $v_2$. We plug it into the equation $y' = Ay$ to find $v_2$:

$$y' = (e^{2t} + 2te^{2t})v_1 + \mu e^{\mu t}v_2,$$

we obtain

$$2v_1te^{2t} + v_1e^{2t} + \mu e^{\mu t}v_2 = A(v_1te^{2t} + v_2e^{\mu t}).$$

Using $Av_1 = 2v_1$, we get

$$v_1e^{2t} + \mu e^{\mu t}v_2 = Av_2e^{\mu t}.$$

This should be valid for all $t$. Hence, we get $\mu = 2$ and

$$(A - 2I)v_2 = v_1.$$

That is

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$ 

This gives $v_1 + v_2 = -1$. So,

$$v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

is a solution.

Now, we find two solutions

$$y_1 = e^{2t}v_1,$$
$$y_2 = te^{2t}v_1 + e^{2t}v_2.$$

**Remark.** The double root case can be thought as a limiting case where the second eigenvalues $\lambda_2 \to \lambda_1$ and $v_2 \to v_1$. In this case,

$$\frac{1}{\lambda_2 - \lambda_1} \left( e^{\lambda_2 t}v_2 - e^{\lambda_1 t}v_1 \right)$$

is also a solution. This is equivalent to differentiate

$$e^{\lambda t}v$$

in $\lambda$ at $\lambda_1$, where $v(\lambda)$ is the eigenvector corresponding to $\lambda$. This derivative is

$$te^{\lambda_1 t}v_1 + e^{\lambda_1 t}\frac{\partial v}{\partial \lambda}.$$

The new vector $\frac{\partial v}{\partial \lambda}$ is denoted by $v_2$. By plugging $te^{\lambda_1 t}v_1 + e^{\lambda_1 t}v_2$ into the equation, we obtain $v_2$ should satisfies

$$(A - \lambda_1 I)v_2 = v_1.$$
Let us return to study general equation: \( y' = Ay \) with eigenvalues of \( A \) being a double root (i.e. \( \lambda_1 = \lambda_2 \)). Let us call the corresponding eigenvector \( v_1 \). Our goal is to find another vector \( v_2 \) such that
\[
(A - \lambda_1 I)v_2 = v_1.
\]

The characteristic equation is
\[
p(\lambda) := \det(\lambda I - A) = (\lambda - \lambda_1)^2 = 0.
\]
The Caley-Hamilton theorem states that \( A \) satisfies the matrix equation:
\[
p(A) = 0.
\]

This can be seen from the following argument. Let \( Q(\lambda) \) be the matrix
\[
Q(\lambda) = \begin{pmatrix} d & -b \\ -c & a - \lambda \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} - \lambda I.
\]

Then
\[
(A - \lambda I)Q(\lambda) = p(\lambda)I.
\]

We immediately get \( p(A) = 0 \).

Now, suppose \( \lambda_1 \) is a double root of \( A \). We can find an eigenvector \( v_1 \) such that
\[
Av_1 = \lambda_1 v_1.
\]

We cannot find two independent eigenvectors corresponding to \( \lambda_1 \) unless \( A \) is an identity matrix. Yet, we can find another vector \( v_2 \) such that
\[
(A - \lambda_1 I)v_2 = v_1.
\]

The solvability of \( v_2 \) comes from the follows. Let \( N_k \) be the null space of \( (A - \lambda_1 I)^k \), \( k = 1, 2 \). We have the following mapping
\[
N_2 \xrightarrow{A - \lambda_1 I} N_1 \xrightarrow{A - \lambda_1 I} \{0\}
\]

We have seen that the only eigenvector is \( v_1 \). Thus, \( N_1 = \langle v_1 \rangle \), the span of \( v_1 \). From the Caley-Hamilton theorem, \( N_2 = \mathbb{R}^2 \). The kernel of \( A - \lambda_1 I \) is \( N_1 \). From a theorem of linear map: the sum of the dimensions of range and kernel spaces equals the dimension of the domain space. The domain space is \( \mathbb{R}^2 \) and the dimension of the kernel space is 1. Hence, the range space has dimension 1. We conclude that the range of \( A - \lambda_1 I \) is \( N_1 \). Hence, there exists a \( v_2 \in N_2 \) such that
\[
(A - \lambda_1 I)v_2 = v_1.
\]

The matrix \( A \), as represented in the basis \( v_1 \) and \( v_2 \), has the form
\[
A[v_1, v_2] = [v_1, v_2] \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}
\]
This is called the Jordan canonical form of \( A \). We can find two solutions from this form:

\[
\begin{align*}
    y_1(t) & = e^{\lambda_1 t}v_1, \\
    y_2(t) & = te^{\lambda_1 t}v_1 + e^{\lambda_1 t}v_2
\end{align*}
\]

You can check the Wronskian \( W[y_1, y_2](t) \neq 0 \). Thus, \( y_1 \) and \( y_2 \) form a fundamental solution. The general solution has the form

\[
y(t) = C_1y_1(t) + C_2y_2(t).
\]

The stability of the equilibrium, the 0 state lies on the sign of \( \lambda_1 \).

- \( \lambda_1 < 0 \): the 0 state is a stable equilibrium.
- \( \lambda_1 > 0 \): the 0 state is an unstable equilibrium.
- \( \lambda_1 = 0 \): the general solution reads
  \[
y(t) = C_2tv_2 + C_1v_1
\]
  The 0 state is “unstable.”

**Homeworks.**

1. B-D, pp. 398: 1, 3, 7, 9
2. B-D, pp. 400: 31, 32
3. B-D, pp. 410: 1, 5, 9, 13, 25
4. B-D, pp. 413: 31

### 3.2.3 Stability

We can plot a stability diagram on the plane of the two parameters \( T \) and \( D \), the trace and the determinant of \( A \). The eigenvalues of \( A \) are

\[
\lambda_1 = \frac{T + \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T - \sqrt{T^2 - 4D}}{2}.
\]

Let \( \Delta := T^2 - 4D \). On the \( T-D \) plane, the parabola \( \Delta = 0 \), the line \( D = 0 \) and the line \( T = 0 \) partition the plane into the following regions. The status of the origin is as the follows.

- \( \Delta > 0, D < 0 \), the origin is a saddle point.
- \( \Delta > 0, D > 0, T > 0 \), the origin is an unstable node (source).
- \( \Delta > 0, D > 0, T < 0 \), the origin is an stable node (sink).
• $\Delta < 0$, $T < 0$, the origin is a stable spiral point.
• $\Delta < 0$, $T > 0$, the origin is an unstable spiral point.
• $\Delta < 0$, $T = 0$, the origin is an stable circular point.
• $\Delta = 0$, $T < 0$, the origin is a stable node.
• $\Delta = 0$, $T > 0$, the origin is an unstable node.
• $\Delta = 0$, $T = 0$, the origin is an unstable node.

Homework.
1. B-D, pp. 493: 13, 19

3.3 Linear systems in three dimensions

Consider the $3 \times 3$ linear system

$$ y' = Ay, $$

where

$$ y = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. $$

We look for three independent solutions of the form $e^{\lambda t}v$. By plugging this into the equation, we find $\lambda$ and $v$ have to be the eigenvalue and eigenvector of $A$:

$$ Av = \lambda v. $$

The eigenvalue satisfies the characteristic equation

$$ \det (\lambda I - A) = 0. $$

This is a third order equation because we have a $3 \times 3$ system. One of its roots must be real. The other two roots can be both real or complex conjugate. We label the first one by $\lambda_3$ and the other two by $\lambda_1$ and $\lambda_2$. The corresponding eigenvectors are denoted by $v_i$, $i = 1, 2, 3$. It is possible that $\lambda_1 = \lambda_2$. In this case, $v_1$ and $v_2$ are the vectors to make $A$ in Jordan block. That is

$$ Av_1 = \lambda_1 v_1 $$
$$ Av_2 = \lambda_1 v_2 + v_1 $$

The general solution is

$$ y(t) = C_1 y_1(t) + C_2 y_2(t) + C_3 y_3(t). $$
3.3. LINEAR SYSTEMS IN THREE DIMENSIONS

The solution \( y_1 \) and \( y_2 \) are found exactly the same way as that in two dimensions. The solution \( y_3(t) = e^{\lambda_3 t}v_3 \). If \( \lambda_3 < 0 \), then the general solution tends to the plane spanned by \( v_1 \) and \( v_2 \). Let us denote this plane by \( \langle v_1, v_2 \rangle \). On the other hand, if \( \lambda_3 > 0 \), the solution leaves the plane \( \langle v_1, v_2 \rangle \). The origin \( 0 \) is asymptotically stable only when the real part of \( \lambda_i \)

\[ \text{Re } \lambda_i < 0, i = 1, 2, 3. \]

Example.

Consider

\[
A = \begin{pmatrix}
0 & 0.1 & 0 \\
0 & 0 & 0.2 \\
0.4 & 0 & 0
\end{pmatrix},
\]

The characteristic equation is

\[ \lambda^3 - 0.008 = 0. \]

The roots are

\( \lambda_3 = 0.2, \lambda_1 = 0.2e^{i2\pi/3}, \lambda_2 = 0.2e^{-i2\pi/3}. \)

The eigenvectors are

\[
v_3 = \begin{pmatrix}
1/2 \\
1 \\
1
\end{pmatrix},
\]

\[
v_1 = \begin{pmatrix}
-1 + i\sqrt{3} \\
-2 - i2\sqrt{3} \\
4
\end{pmatrix},
\]

\[
v_2 = \begin{pmatrix}
-1 - i\sqrt{3} \\
-2 + i2\sqrt{3} \\
4
\end{pmatrix}.
\]

We denote \( v_1 = u_1 + iu_2 \) and \( v_2 = u_1 - iu_2 \). We also denote \( \lambda_1 = \alpha + i\omega \), where \( \alpha = -0.1 \) and \( \omega = \sqrt{0.03} \). Then the fundamental solutions are

\[
y_1(t) = e^{\alpha t}(\cos(\omega t)u_1 - \sin(\omega t)u_2)
\]

\[
y_2(t) = e^{\alpha t}(\sin(\omega t)u_1 + \cos(\omega t)u_2)
\]

\[
y_3(t) = e^{\lambda_3 t}v_3
\]

3.3.1 Rotation in three dimensions

An important example for \( 3 \times 3 \) linear system is the rotation in three dimensions. The governing equation is

\[
y'(t) = \Omega \times y
\]

\[
= \begin{pmatrix}
0 & -\omega_3 & -\omega_2 \\
\omega_3 & 0 & -\omega_1 \\
\omega_2 & \omega_1 & 0
\end{pmatrix} y
\]

We have many examples in the physical world represented with the same equation.

- Top motion in classical mechanics: \( y \) is the angular momentum and \( \Omega \times y \) is the torque.
• Dipole motion in a magnetic field: $y$ is the angular momentum which is proportional to the magnetic dipole.

• A particle motion under Coriolis force: $y$ is the velocity and $-2\Omega \times y$ is the Coriolis force.

• Charge particle motion in magnetic field: $y$ is the velocity. The term $\Omega \times y$ is a force pointing to the direction perpendicular to $y$ and $\Omega$. This is the Lorentz force in the motion of a charge particle in magnetic field $\Omega$.

• Spin motion in magnetic field: $y$ is the spin and $\Omega$ is the magnetic field.

We may normalize $\Omega = \omega \hat{z}$. In this case, the equation becomes

$$
\begin{align*}
y'^1 &= -\omega y^2 \\
y'^2 &= \omega y^1 \\
y'^3 &= 0 
\end{align*}
$$

The solution reads:

$$
y(t) = R(t)y(0), \quad \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\
\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1 \end{pmatrix}
$$

It is a rotation about the $z$ axis with angular velocity $\omega$.

**Motion of a charge particle in constant electric magnetic field** The force exerted by a charged particle is known as the Lorentz force

$$
F = q(E + v \times B)
$$

The motion of the charged particle in this E-M field is governed by

$$
m\ddot{r} = F.
$$

Suppose the EM field is constant with $E$ only in $z$ direction and $B$ in $x$ direction. Then the motion is on $y - z$ plane if it is so initially. We write the equation in each components:

$$
\begin{align*}
m\ddot{y} &= qB\dot{z}, \\
m\ddot{z} &= qE - qB\dot{y}.
\end{align*}
$$

Let

$$
\omega := \frac{qB}{m},
$$

the equations are rewritten as

$$
\begin{align*}
\dot{y} &= \omega \dot{z}, \\
\dot{z} &= \omega \left(\frac{E}{B} - \dot{y}\right).
\end{align*}
$$

The particle started from zero velocity has the trajectory

$$
y(t) = \frac{E}{\omega B}(\omega t - \sin \omega t), \quad z(t) = \frac{E}{\omega B}(1 - \cos \omega t).
$$

This is a **cycloid**.
3.4. FUNDAMENTAL MATRICES AND \( \exp(tA) \)

**Homework**

Consider the equation

\[ p \left( \frac{d}{dt} \right) y(t) = 0, \]

where \( y \) is scalar. Let us consider

\[ p(s) = (s - 1)^3. \]

Show that

\[ y_1(t) = e^t, \quad y_2(t) = te^t, \quad y_3(t) = t^2 e^t. \]

are three independent solutions.

**Homeworks.**

1. B-D, pp. 429, 17,18

### 3.4 Fundamental Matrices and \( \exp(tA) \)

#### 3.4.1 Fundamental matrices

We have seen that the general solution to the initial value problem:

\[ y'(t) = Ay(t), \quad y(0) = y_0, \]

can be express as \( y(t) = C_1 y_1(t) + \cdots + C_n y_n, \) where \( y_1, \ldots, y_n \) are \( n \) independent solutions. The matrix \( Y(t) = [y_1(t), \ldots, y_n(t)] \) is called a fundamental matrix. The solution \( y \) is expressed as \( y(t) = Y(t)C, \) where \( C = (C_1, \ldots, C_n)^t. \) By plugging \( y(t) = Y(t)C \) into the equation \( y' = Ay, \) we obtain

\[ Y'C = AYC \]

This is valid for all \( C. \) Hence, we conclude that the fundamental matrix satisfies

\[ Y'(t) = AY(t). \tag{3.8} \]

From \( y(0) = Y(0)C, \) we obtain \( C = Y(0)^{-1}y(0). \) Thus,

\[ y(t) = Y(t)Y(0)^{-1}y(0). \]

The matrix \( \Phi(t) := Y(t)Y(0)^{-1} \) is still a fundamental matrix and satisfies \( \Phi(0) = I. \)

**Homework**

1. Consider an \( n \times n \) matrix ODE

\[ Y'(t) = AY(t) \]

Let \( W(t) = \det Y(t). \) Show that

\[ W'(t) = tr(A)W(t) \]

where \( tr(A) := \sum_{i,j} a_{i,j}. \)

**Hint:** \( (detA)' = \sum_{i,j} a'_{i,j}A_{i,j}, \) where \( A_{i,j} \) is the cofactor of \( A. \)
3.4.2 \( \exp(A) \)

Let us consider the space of all complex-valued matrices \( \mathcal{M}_n = \{ A | A \text{ is a complex valued } n \times n \text{ matrix} \} \). We can define a norm on \( \mathcal{M}_n \) by

\[
\| A \| := \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2}
\]

The norm \( \| \cdot \| \) has the properties:

- \( \| A \| \geq 0 \) and \( \| A \| = 0 \) if and only if \( A = 0 \).
- \( \| A + B \| \leq \| A \| + \| B \| \).
- \( \| AB \| \leq \| A \| \| B \| \).

The proof of the last assertion is the follows.

\[
\| AB \| ^2 = \sum_{i,j} \left| \sum_k a_{ik} b_{kj} \right|^2 \\
\leq \sum_{i,j} \left( \sum_k |a_{ik}|^2 \right) \left( \sum_k |b_{kj}|^2 \right) \\
= \sum_i \left( \sum_k |a_{ik}|^2 \right) \sum_j \left( \sum_k |b_{kj}|^2 \right) \\
= \| A \|^2 \| B \|^2
\]

With this norm, we can talk about theory of convergence. The space \( \mathcal{M}_n \) is equivalent to \( \mathbb{C}^{n^2} \). Thus, it is complete. This means that every Cauchy sequence converges to a point in \( \mathcal{M}_n \).

Now we define the exponential function in \( \mathcal{M}_n \) as the follows.

\[
\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \tag{3.9}
\]

**Theorem 3.4** The exponential function has the following properties:

- \( \exp(A) \) is well-defined.
- The function \( \exp(tA) \) is differentiable and \( \frac{d}{dt} \exp(tA) = A \exp(tA) \).
- \( \exp(0) = I \).

**Proof.** This series converges because \( \mathcal{M}_n \) is complete and this series is a Cauchy series:

\[
\| \sum_{n}^{m} \frac{1}{k!} A^k \| \leq \sum_{n}^{m} \frac{1}{k!} \| A \|^k < \varepsilon,
\]

if \( n < m \) are large enough.
3.4. FUNDAMENTAL MATRICES AND EXP(TA)  

Notice that the series 

\[
\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n.
\]

convergence uniformly for \( t \) in any bounded set in \( \mathbb{R} \). Further, the function \( \exp(tA) \) is differentiable in \( t \). This is because the series obtained by the term-by-term differentiation 

\[
\sum_{n=1}^{\infty} \frac{1}{(n-1)!} t^{n-1} A^n
\]

converges uniformly for \( t \) in any bounded set in \( \mathbb{R} \). And the derivative of \( \exp(tA) \) is the term-by-term differentiation of the original series:

\[
\frac{d}{dt} \exp(tA) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} t^{n-1} A^n
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} t^{n-1} A^{n-1}
\]

\[
= A \exp(tA).
\]

We have seen that the fundamental solution \( Y(t) \) of the equation \( y' = Ay \) satisfies \( Y' = AY \). From the above theorem, we see that \( \exp(tA) \) is a fundamental solution satisfying \( \exp(0) = I \).

Below, we compute \( \exp(tA) \) for some special \( A \).

1. \( A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \). In this case,

\[
A^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}
\]

and

\[
\exp(A) = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}.
\]

If \( \lambda_1 \) and \( \lambda_2 \) are complex conjugate and \( \lambda_1 = \alpha + i\omega \), then

\[
\exp(tA) = e^{\alpha t} (\cos \omega t + i \sin \omega t) I.
\]

2. \( A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \). In this case,

\[
A^2 = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}
\]

\[
A^3 = \begin{pmatrix} 0 & \omega^3 \\ -\omega^3 & 0 \end{pmatrix}
\]

\[
A^4 = \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{pmatrix}
\]
Hence,
\[
\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix}
\]

3. \( A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \). The matrix \( A = \lambda I + N \), where

\[
N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

is called a nilponent matrix. \( N \) has the property

\[ N^2 = 0. \]

Thus,

\[ A^n = (\lambda I + N)^n = \lambda^n I + n\lambda^{n-1}N \]

With this,

\[
\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} t^n (\lambda^n I + n\lambda^{n-1}N)
\]

\[
= \exp(\lambda t)I + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \lambda^{n-1} t^n N
\]

\[
= \exp(\lambda t)I + t \exp(t\lambda) N
\]

\[
= \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}
\]

For general \( 2 \times 2 \) matrices \( A \), we have seen that there exists a matrix \( T = [v_1, v_2] \) such that

\[ AT = TA \]

where \( A \) is either diagonal matrix (case 1) or a Jordan matrix (Case 3). Notice that

\[ A^n = (TA^{-1}T^{-1})^n = TA^nT^{-1} \]

Hence, the corresponding exponential function becomes

\[
\exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} t^n TA^n T^{-1}
\]

\[
= T(\sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n) T^{-1}
\]

\[
= T \exp(tA) T^{-1}
\]
A fundamental matrix $Y$ is given by

$$ Y = [v_1, v_2] \exp(tA) $$

In the case of Jordan form, we get

$$ [y_1(t), y_2(t)] = [v_1, v_2] \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} $$

$$ = [e^{\lambda t}v_1, te^{\lambda t}v_1 + e^{\lambda t}v_2] $$

This is identical to the fundamental solution we obtained before.

**Homeworks.**

- B-D, pp. 420: 3, 18
- B-D, pp. 428, 6, 17, 18, 21
- Show that if $AB = BA$, then $\exp(A + B) = \exp(A) \exp(B)$. In particular, use this result to show $\exp((t - s)A) = \exp(tA) \exp(sA)^{-1}$.

### 3.5 Nonhomogeneous Linear Systems

We consider the inhomogeneous linear systems:

$$ y'(t) = Ay(t) + f(t), \quad y(0) = y_0. \quad (3.10) $$

We use variation of parameters to solve this equation. Let $\Phi(t) = \exp(tA)$ be the fundamental solution for the homogeneous equation. To find a particular solution for the inhomogeneous equation, we consider

$$ y(t) = \Phi(t)u(t). $$

We plug this into equation. We get

$$ \Phi' u + \Phi u' = A\Phi u + f $$

Using $\Phi' = A\Phi$, we get

$$ \Phi u' = f $$

Hence, a particular of $u$ is

$$ u(t) = \int_0^t \Phi(s)^{-1}f(s) \, ds $$

Thus a particular solution $y_p(t)$ is

$$ y_p(t) = \Phi(t) \int_0^t \Phi(s)^{-1}f(s) \, ds = \int_0^t \Phi(t)\Phi(s)^{-1}f(s) \, ds $$
This special solution has 0 initial data. The solution for initial condition $y(0) = y_0$ has the following expression:

$$y(t) = \Phi(t)y_0 + \int_0^t \Phi(t)\Phi(s)^{-1}f(s)\,ds$$  \hspace{1cm} (3.11)

Notice that the matrix exponential function also satisfies the exponential laws. We can rewrite the above expression as

$$y(t) = \Phi(t)y_0 + \int_0^t \Phi(t-s)f(s)\,ds.$$  \hspace{1cm} (3.12)

**Homeworks.**

- B-D, pp. 439, 13.
Chapter 4

Methods of Laplace Transforms

The method of Laplace transform converts a linear ordinary differential equation with constant coefficients to an algebraic equation. The core of this differential equation then lies in the roots of the corresponding algebraic equation. In application, the method of Laplace transform is particularly useful to handle general source terms.

4.1 Laplace transform

For function \( f \) defined on \([0, \infty)\), we define its Laplace transformation by

\[
\mathcal{L}f(s) = F(s) := \int_0^\infty f(t)e^{-st} \, dt
\]

\( \mathcal{L} \) is a linear transformation which maps \( f \) to \( F \). For those functions \( f \) such that

\[
|f(t)| \leq Ce^{\alpha t}
\]

for some positive constants \( C \) and \( \alpha \), the above improper integral converges uniformly and absolutely for complex \( s \) lies in a compact set in \( \{ s \in \mathbb{C} | \text{Re} s > \alpha \} \):

\[
\int_0^\infty |f(t)e^{-st}| \, dt \leq C \int_0^\infty e^{\alpha t} e^{-st} \, dt = \frac{C}{s - \alpha}
\]

Here, we have used that

\[
e^{-(s-\alpha)t}|_\infty = 0
\]

due to \( \text{Re} s > \alpha \). We call functions with this growth condition (4.1) admissible. Since the integration allows \( f \) being discontinuous, the admissible functions include all piecewise discontinuous functions.

4.1.1 Examples

1. When \( f(t) \equiv 1 \), \( \mathcal{L}(1) = 1/s \).
2. $\mathcal{L}(e^{\lambda t}) = 1/(s - \lambda)$. This is because
\[
\mathcal{L}(e^{\lambda t}) = \int_0^\infty e^{\lambda t} e^{-st} dt = \int_0^\infty e^{-(s-\lambda)t} dt = \frac{1}{s - \lambda}.
\]
Indeed, this is valid for any complex number $\lambda$ and $s \in \mathbb{C}$ with $Re s > \lambda$.

3. The function
\[
h(t) = \begin{cases} 
1 & \text{for } t \geq 0 \\
0 & \text{for } t < 0
\end{cases}
\]
is called the Heaviside function. It has a discontinuity at $t = 0$ with jump $h(0^+) - h(0^-) = 1$. The Laplace transform of
\[
\mathcal{L}(h(t-a)) = \int_0^\infty h(t-a)e^{-st} dt = \int_a^\infty e^{-st} dt = e^{-as}\mathcal{L}(1) = \frac{e^{-as}}{s},
\]
for any $a \geq 0$.

4. When $f(t) = t^n$,
\[
\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt = -\frac{1}{s} \int_0^\infty t^n de^{-st} = -\frac{1}{s}(t^n e^{-st})_0^\infty - \int_0^\infty nt^{n-1}e^{-st} dt
\]
\[
= \frac{n}{s} \mathcal{L}(t^{n-1}) = \frac{n!}{s^{n+1}}.
\]
Alternatively, we also have
\[
\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt = \int_0^\infty (-\frac{d}{ds})^n e^{-st} dt
\]
\[
= (-\frac{d}{ds})^n \int_0^\infty e^{-st} dt = (-\frac{d}{ds})^n \frac{1}{s} = \frac{n!}{s^{n+1}}.
\]

5. $\mathcal{L}(t^n e^{\lambda t}) = \frac{n!}{(s-\lambda)^{n+1}}$. Indeed,
\[
\mathcal{L}(t^n e^{\lambda t}) = \int_0^\infty t^n e^{\lambda t} e^{-st} ds = \int_0^\infty t^n e^{-(s-\lambda)t} ds = \frac{n!}{(s-\lambda)^{n+1}}.
\]
4.1. LAPLACE TRANSFORM

6. \( \mathcal{L}(e^{\pm j\omega t}) = \frac{1}{s \mp j\omega} \), \( \mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2} \), \( \mathcal{L}(\sin \omega t) = \frac{j\omega}{s^2 + \omega^2} \). Indeed,

\[
\mathcal{L}(\cos \omega t) = \frac{1}{2} \mathcal{L}(e^{j\omega t} + e^{-j\omega t}) = \frac{1}{2} \left( \frac{1}{s - j\omega} + \frac{1}{s + j\omega} \right) = \frac{s}{s^2 + \omega^2}.
\]

7. We shall apply the method of Laplace transform to solve the initial value problem:

\[ y' + y = t, \; y(0) = y_0. \]

We apply Laplace transform both sides.

\[ \mathcal{L}(y') = \int_0^\infty e^{-st}y'(t) \, dt = -y(0) + s \int_0^\infty e^{-st}y(t) \, dt \]

Let us denote \( \mathcal{L}y = Y \). We have

\[ sY - y_0 + Y = \frac{1}{s^2} \]

Hence

\[
Y(s) = \frac{1}{s+1} \left( y_0 + \frac{1}{s^2} \right) = \frac{y_0}{s+1} - \frac{1}{s} - \frac{1}{s+1}
\]

Hence

\[ y(t) = y_0 e^{-t} + t - 1 + e^{-t} \]

4.1.2 Properties of Laplace transform

Let us denote the Laplace transform of \( f \) by \( F \). That is, \( F = \mathcal{L}f \).

1. \( \mathcal{L} \) is linear.

2. \( \mathcal{L} \) is one-to-one, that is \( \mathcal{L}(f) = 0 \) implies \( f = 0 \). Hence, \( \mathcal{L}^{-1} \) exists.

3. Translation:

\[
\mathcal{L}(f(t-a)) = e^{-as}F(s),
\]

\[
\mathcal{L}^{-1}F(s + a) = e^{-at}f(t),
\]

where \( f(t-a) := 0 \) for \( 0 < t < a \).

Further, given a function \( f(t) \), we require \( f(t) = 0 \) for \( t < 0 \), we have

\[ \mathcal{L}(f(t-a)) = e^{-as}(\mathcal{L}f)(s). \]

Thus, the term \( e^{-as} \) in the \( s \)-space represents a translation in the time domain.
4. Dilation:
\[ \mathcal{L}(f(bt)) = \frac{1}{b} F\left(\frac{s}{b}\right), \quad \mathcal{L}^{-1}F(bs) = \frac{1}{b} f\left(\frac{t}{b}\right). \]

5. Differentiation:
\[ \mathcal{L}(f'(t)) = sF(s) - f(0), \quad \mathcal{L}^{-1}F'(s) = -tf(t). \quad (4.2) \]

6. Integration:
\[ \mathcal{L}\left(\int_{0}^{t} f(\tau) \, d\tau\right) = \frac{F(s)}{s}, \quad \mathcal{L}^{-1}\left(\int_{s}^{\infty} F(s_1) \, ds_1\right) = \frac{f(t)}{t}. \]

7. Convolution:
\[ \mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g), \]
where
\[ (f * g)(t) = \int_{0}^{t} f(\tau)g(t - \tau) \, d\tau \]

Proof.
\[ \mathcal{L}(f * g) = \int_{0}^{\infty} e^{-st} \int_{0}^{t} f(\tau)g(t - \tau) \, d\tau \, dt \]
\[ = \int_{0}^{\infty} \int_{0}^{t} e^{-\sigma \tau} f(\tau) e^{-s(t-\tau)} g(t - \tau) \, d\tau \, dt \]
\[ = \int_{0}^{\infty} \int_{\tau}^{\infty} dt \left( e^{-\sigma \tau} f(\tau) e^{-s(t-\tau)} g(t - \tau) \right) \]
\[ = \int_{0}^{\infty} e^{-\sigma \tau} f(\tau) \, d\tau \int_{\tau}^{\infty} e^{-s(t-\tau)} g(t - \tau) \, dt = \mathcal{L}(f)\mathcal{L}(g) \]

Homeworks.
2. Find the Laplace transforms of
   (a) \( \cosh(at) \) (ans. \( s/(s^2 - a^2) \)).
   (b) \( \sinh(at) \), (ans. \( a/s^2 - a^2 \)).
   (c) \( (-t)^n f(t) \) (ans. \( F^{(n)}(s) \)).
3. B-D, pp. 331: 27,28
4. Find the Laplace transforms of
   (a) \( B_0(2t) - B_0(2t - 1) \), where \( B_0(t) = 1 \) for \( 0 \leq t < 1 \) and \( B_0(t) = 0 \) otherwise.
   (b) \( f(t) = \sum_{k=0}^{\infty} B(2t - k) \).
(c) Let \( f_0(t) = t(1 - t) \) for \( 0 \leq t < 1 \) and \( f(t) = 0 \) elsewhere. Let \( f(t) \) be the periodic extension of \( F_0 \) with period 1. Find \( \mathcal{L}f_0, \mathcal{L}f, \mathcal{L}f' \) and \( \mathcal{L}f'' \).

5. Prove
\[
\mathcal{L} \left( \int_0^t f(\tau) d\tau \right) = \frac{F(s)}{s}, \quad \mathcal{L}^{-1} \left( \int_s^\infty F(s_1) ds_1 \right) = \frac{f(t)}{t},
\]

6. Let \( f(t) \) be a period function with period \( p \). Let
\[
f_0 = \begin{cases} 
  f(t) & \text{for } 0 < t < p \\
  0 & \text{elsewhere}
\end{cases}
\]

Let \( F(s) \) denote for \( \mathcal{L}f \). Show that
\[
\mathcal{L}f_0 = \mathcal{L}f - e^{-ps} \mathcal{L}f = (1 - e^{-ps})F(s).
\]

### 4.2 Laplace transform for differential equations

#### 4.2.1 General linear equations with constant coefficients

A linear differential equations of order \( n \) with constant coefficients has the form:
\[
(a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0)y = f(t),
\]
where \( D = \frac{d}{dt} \). We may abbreviate this equation by
\[
P(D)y = f.
\]

For order \( n \) equations, We need to assume \( a_n \neq 0 \) and need impose \( n \) conditions. The initial value problem imposes the following conditions:
\[
y(0) = y_0, \ y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1}.
\]

When the source term \( f(t) \equiv 0 \), the equation
\[
P(D)y = 0
\]
is called the homogeneous equation. The equation (4.3) is called the inhomogeneous equation.

We shall accept that this initial value problem has a unique solution which exists for all time. Such existence and uniqueness theory is the same as that for the \( 2 \times 2 \) systems of linear equations. Therefore, we will not repeat here. Instead, we are interested in the cases where the source terms have discontinuities or impulses. Such problems appear in circuit problems where a power supply is only provided in certain period of time, or a hammer punches the mass of a mass-spring system suddenly, or a sudden immigration of population in the population dynamics. For linear systems with constant coefficients, the Laplace transform is a useful tool to get exact solution. The method transfers the linear differential equations with constant coefficients to an algebraic equation, where the source with discontinuities is easily expressed. The solution is found through solving the algebraic equation and by the inverse Laplace transform.
4.2.2 Laplace transform applied to differential equations

Given linear differential equation with constant coefficients (4.3):

\[ P(D)y = f \]

we can perform Laplace transform both side:

\[ \mathcal{L}(P(D)y) = \mathcal{L}f \]

We claim that

\[ \mathcal{L}(P(D)y) = P(s) \cdot Y(s) - I(s) = F(s) \]  \hspace{1cm} (4.6)

where \( Y(s) = (\mathcal{L}y)(s), \) \( F(s) = \mathcal{L}f(s) \) and

\[ I(s) = \sum_{i=1}^{n} \sum_{k=i}^{n} a_k y^{(k-i)}(0)s^{i-1}. \]

In other words, the function \( Y(s) \) of the Laplace transform of \( y \) satisfies an algebraic equation.

To show this, we perform

\[ \mathcal{L}(D^k y) = \int_0^\infty D^k y e^{-st} \, dt = \int_0^\infty e^{-st} dy^{(k-1)} = -y^{(k-1)}(0) + s\mathcal{L}(D^{k-1}y). \]

Thus,

\[ \mathcal{L}(D^k y) = (-y^{(k-1)}(0) - sy^{(k-2)}(0) - \cdots - s^{k-1}y(0)) + s^k \mathcal{L}y. \]

Now, \( P(D) = \sum_{k=0}^{n} a_k D^k, \) we have

\[ \mathcal{L}(P(D)y) = \sum_{k=0}^{n} a_k \mathcal{L}(D^k y) = -\sum_{k=1}^{n} a_k \sum_{i=1}^{k} y^{(k-i)}(0)s^{i-1} + P(s)\mathcal{L}y \]

The equation

\[ P(s) \cdot Y(s) - I(s) = F(s) \]

can be solved with

\[ Y(s) = \frac{F(s) + I(s)}{P(s)}. \]

Let us call

\[ G(t) = \mathcal{L}^{-1}\left(\frac{1}{P(s)}\right) \]

called the Green’s function. Then in the case of \( I(s) \equiv 0, \) we have

\[ y(t) = \mathcal{L}^{-1}\left(\frac{1}{P(s)} \cdot F(s)\right) = (G \ast f)(t) \]

Thus, the solution is the convolution of the Green’s function and the source term.
4.2. LAPLACE TRANSFORM FOR DIFFERENTIAL EQUATIONS

4.2.3 Generalized functions and Delta function

The delta function $\delta(t)$ is used to represent an impulse which is defined to be

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{otherwise.} \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) \, dt = 1.$$  

The $\delta$-function can be viewed as the limit of the finite impulses

$$\delta(t) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} B_0 \left( \frac{t}{\epsilon} \right)$$

where $B_0(t) = 1$ for $0 \leq t < 1$ and $B_0(t) = 0$ otherwise. This limit is taken in the integral sense. Namely, for any smooth function $\phi$ with finite support (i.e. the nonzero domain of $\phi$ is bounded), the meaning of the integral:

$$\int \delta(t) \phi(t) \, dt := \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \left( \frac{1}{\epsilon} B_0 \left( \frac{t}{\epsilon} \right) \right) \phi(t) \, dt.$$  

Since the latter is $\phi(0)$, we therefore define $\delta$ to be the generalized function such that

$$\int \delta(t) \phi(t) \, dt = \phi(0)$$

for any smooth function $\phi$ with finite support. The function $\phi$ here is called a test function. Likewise, a generalized function is defined how it is used. Namely, it is defined how it acts on smooth test functions. For instance, the Heaviside function is a generalized function in the sense that

$$\int h(t) \phi(t) \, dt := \int_0^\infty \phi(t) \, dt.$$  

The function $f(t) := a_1 \delta(t - t_1) + a_2 \delta(t - t_2)$ is a generalized function. It is defined by

$$\int f(t) \phi(t) \, dt := a_1 \phi(t_1) + a_2 \phi(t_2).$$

All ordinary functions are generalized functions. In particular, all piecewise smooth functions are generalized functions. For such a function $f$, it is un-important how $f$ is defined at the jump points. All it matters is the integral

$$\int f(t) \phi(t) \, dt$$

with test function $\phi$. For piecewise smooth function $f$, the jump point makes no contribution to the integration.
One can differentiate a generalized function. The generalized derivative of a generalized function is again a generalized function in the following sense:

\[ \int D_t f(t) \phi(t) \, dt := - \int f(t) \phi'(t) \, dt \]

The RHS is well-defined because \( f \) is a generalized function. You can check that \( D_t \delta(t) = \delta'(t) \).

If \( f \) is a piecewise smooth function having jump at \( t = a \) with jump height \([f]_a := \lim_{t \to a^+} f(t) - \lim_{t \to a^-} f(t)\). Let \( f'(t) \) be the ordinary derivative of \( f \) in the classical sense. \( f'(t) \) is defined everywhere except at \( t = a \). This \( f'(t) \) is a piecewise smooth function and hence it is a generalized function. From the definition of the generalized derivative, one has

\[ (D_t f)(t) = f'(t) + [f]_a \delta(t - a). \]

To see this,

\[ \int (D_t f) \phi \, dt := - \int_{-\infty}^{\infty} f(t) \phi'(t) \, dt = -\left( \int_{-\infty}^{a} + \int_{a}^{\infty} \right) f(t) \phi'(t) \, dt \]

These integrals are

\[
\begin{align*}
- \int_{-\infty}^{a} f(t) \phi'(t) \, dt &= - f(a-) \phi(a) + \int_{-\infty}^{a} f'(t) \phi(t) \, dt \\
- \int_{a}^{\infty} f(t) \phi'(t) \, dt &= f(a+) \phi(a) + \int_{a}^{\infty} f'(t) \phi(t) \, dt
\end{align*}
\]

Hence,

\[
\int (D_t f) \phi \, dt = (f(a+) - f(a-)) \phi(a) + \int_{-\infty}^{\infty} f'(t) \phi(t) \, dt = \int \left[ [f]_a \delta(t - a) + f'(t) \right] \phi(t) \, dt
\]

You can check that \( D_t \delta \) is a generalized function. It is defined by

\[ \int (D_t \delta) \phi(t) \, dt := -\phi(0) \]

Let us abbreviate \( D_t \delta \) by \( \delta'(t) \) in later usage.

Similarly, one can take indefinite integral of a generalized function.

\[
\int \left( \int_{-\infty}^{t} f(\tau) \, d\tau \right) \phi(t) \, dt := \int f(\tau) \left( \int_{\tau}^{\infty} \phi(t) \, dt \right) \, d\tau
\]

for any test function \( \phi \) such that \( \int \phi = 0 \). The Heaviside function \( h(t) \) can be viewed as the integral of the delta function, namely,

\[ h(t) = \int_{0}^{t} \delta(\tau) \, d\tau \]

It is easy to check that
4.2. LAPLACE TRANSFORM FOR DIFFERENTIAL EQUATIONS

1. \( \mathcal{L}\delta = \int \delta(t)e^{-st} \, dt = 1. \)

2. \( \mathcal{L}\delta' = s, \)

3. \( \mathcal{L}h = 1/s. \)

Let us go back to the differential equation:

\[ P(D)y = f. \]

with initial data \( y(0), y^{(n-1)}(0) \) prescribed. We recall that the Laplace transform of this equation gives

\[ \mathcal{L}(P(D)y) = P(s) \cdot Y(s) - I(s) = F(s) \quad (4.7) \]

where \( Y(s) = (\mathcal{L}y)(s), F(s) = \mathcal{L}f(s) \) and

\[ I(s) = \sum_{i=1}^{n} \sum_{k=i}^{n} a_k y^{(k-i)}(0)s^{i-1}. \]

The Green’s function is defined to be

\[ G = \mathcal{L}^{-1}\left( \frac{1}{P(s)} \right). \quad (4.8) \]

There are two possible cases which can produce a solution to be a Green’s function.

- \( I(s) \equiv 0 \text{ and } F(s) \equiv 1: \) That is,

\[ P(D)G(t) = \delta(t), \quad G(0) = G'(0) = \cdots = G^{(n-1)}(0) = 0. \]

Taking the Laplace transform on both sides, using

\[ \mathcal{L}\delta = 1, \]

we have \( P(s)\mathcal{L}G = 1, \) or \( \mathcal{L}G = 1/P(s), \) or

\[ G = \mathcal{L}^{-1}\left( \frac{1}{P(s)} \right). \]

*The Green’s function corresponds to solution with impulse source and zero initial data.*

- \( I(s) = 1 \text{ and } F(s) \equiv 0: \) That is

\[ P(D)G(t) = 0 \text{ for } t > 0, \quad G(0) = G'(0) = \cdots = 0, \quad G^{(n-1)}(0) = \frac{1}{a_n}. \]
Remark. Notice that the Green’s functions obtained by the above two methods are identical. Indeed, let us see the following simplest example. The function $e^{at}$ are the solution (Green’s function) of both problems:

(i) $y' - ay = \delta$, $y(0) = 0$,

(ii) $y' - ay = 0$, $y(0) = 1$.

Indeed, in the first problem, the equation should be realized for $t \in \mathbb{R}$. The corresponding initial data is $y(0^-) = 0$. While in the second problem, the equation should be understood to be hold for $t > 0$ and the initial data understood to be $y(0^+) = 1$. This is classical sense. With this solution $e^{at}$, if we define

$$y(t) = \begin{cases} e^{at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

then $D_x y + ay = \delta$. This means that this extended function is a solution of (i) and the derivative in (i) should be interpreted as weak derivative.

Examples

1. Suppose $P(D) = (D + 1)(D + 2)$. Then

$$\frac{1}{P(s)} = \frac{1}{s + 1} - \frac{1}{s + 2}$$

Hence,

$$G(t) = e^{-t} - e^{-2t}.$$

2. If $P(D) = (D + 1)^2$, then

$$G(t) = \mathcal{L}^{-1} \left( \frac{1}{(s + 1)^2} \right) = \mathcal{L}^{-1} \left( -\frac{d}{ds} \frac{1}{s + 1} \right) = t\mathcal{L}^{-1} \left( \frac{1}{s + 1} \right) = te^{-t}.$$

3. Suppose $P(D) = (D^2 + \omega^2)$. Then

$$G(t) = \mathcal{L}^{-1} \left( \frac{1}{s^2 + \omega^2} \right) = \frac{\sin \omega t}{\omega}$$

In these two examples, we notice that $G(0) = 0$ but $G'(0+) = 1$. This is consistent to $G'(0-) = 0$. Indeed, $G'$ has a jump at $t = 0$ and the generalized derivative of $G'$ produces the delta function.

With the Green’s function, using convolution, one can express the solution of the equation $P(D)y = f$ with zero initial condition by

$$y(t) = (G * f)(t) = \int_0^t G(t - \tau)f(\tau)\,d\tau.$$

A physical interpretation of this is that the source term $f(t)$ can be viewed as

$$f(t) = \int_0^t f(\tau)\delta(t - \tau)\,d\tau$$
the superposition of delta source $\delta(t - \tau)$ with weight $f(\tau)$. This delta source produces a solution $G(t - \tau)f(\tau)$. By the linearity of the equation, we have the solution is also the superposition of these solutions:

$$y(t) = \int_0^t G(t - \tau)f(\tau) \, d\tau.$$  

Next, let us see the case when $f \equiv 0$ and the initial data are not zero. We have seen that the contribution of the source is

$$Y(s) = \frac{I(s)}{P(s)}, \text{ where } I(s) = \sum_{i=1}^n \sum_{k=i}^n a_k y^{(k-i)}(0) s^{i-1}.$$  

When $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$ and $y^{(n-1)}(0) = 1$, the corresponding $I(s) = a_n y^{(n-1)}(0) = a_n$. Hence, $y(t) = \mathcal{L}^{-1}(I(s)/P(s)) = a_n G(t)$.

**Case 1.** Suppose $P(s)$ has $n$ distinct roots $\lambda_1, \ldots, \lambda_n$. Then

$$\frac{1}{P(s)} = \sum_{k=1}^n \frac{A_k}{s - \lambda_k}, \text{ where } A_k = \frac{1}{P'(-\lambda_k)}.$$

The corresponding Green’s function is

$$G(t) = \sum_{k=1}^n A_k e^{\lambda_k t}.$$  

In general, suppose $P(s)$ has $n$ distinct roots $\lambda_1, \ldots, \lambda_n$. Then

$$\frac{1}{P(s)} = \sum_{k=1}^n \frac{A_k}{s - \lambda_k}, \text{ where } A_k = \frac{1}{P'(-\lambda_k)}.$$

The corresponding Green’s function is

$$G(t) = \sum_{k=1}^n A_k e^{\lambda_k t}.$$  

**Case 2.** When $P(s)$ has multiple roots, say $P(s) = \prod_{i=1}^\ell (s - \lambda_i)^{k_i}$. Then

$$\frac{1}{P(s)} = \sum_{i=1}^\ell \sum_{j=1}^{k_i} \sum_{m=0}^{j-1} \frac{A_{i,j,m} s^m}{(s - \lambda_i)^j},$$

It can be shown that (see (4.2))

$$\mathcal{L}^{-1} \left( \frac{s^m}{(s - \lambda_i)^j} \right) = \frac{d^m}{ds^m} \mathcal{L}^{-1} \left( \frac{1}{(s - \lambda_i)^j} \right).$$
On the other hand,

\[
\mathcal{L}^{-1} \left( \frac{1}{(s - \lambda_i)^j} \right) = \mathcal{L}^{-1} \left( \frac{1}{j!} \left( -\frac{d}{ds} \right)^j \left( \frac{1}{s - \lambda_i} \right) \right) = \frac{1}{j!} t^j \mathcal{L}^{-1} \left( \frac{1}{s - \lambda_i} \right) = \frac{1}{j!} t^j e^{\lambda_i t}.
\]

Thus,

\[
G(t) = \sum_{i=1}^{\ell} \sum_{k=1}^{k_i} \sum_{j=1}^{j-1} \sum_{m=0}^{A_{i,j,m}} \frac{1}{j!} \frac{d^m}{dt^m} (t^j e^{\lambda_i t})
\]

Let us see the Laplace inverse of \( s^i/P(s) \). You can check it is \( D^i G(t) \). With this, we can write the general solution as the follows.

**Theorem 4.5** The solution to the initial value problem

\[ P(D)y = f \]

with prescribed \( y(0), ..., y^{(n-1)} \) has the following explicit expression:

\[
y(t) = \mathcal{L}^{-1} \left( \frac{I(s)}{P(s)} + \frac{F(s)}{P(s)} \right) = \sum_{i=1}^{n} \sum_{k=i}^{n} a_k y^{(k-i)}(0) G^{(i-1)}(t) + (G * f)(t)
\]

**Homeworks.**

1. B-D, pp. 344: 1, 10, 14, 15, 16
2. Prove \( \mathcal{L}(\delta^{(i)}) = s^i \).
3. Find the Green’s function for the differential operator \( P(D) = (D^2 + \omega^2)^m \).
4. Find the Green’s function for the differential operator \( P(D) = (D^2 - k^2)^m \).
5. Suppose \( G = \mathcal{L}^{-1}(1/P(s)) \) the Green’s function. Show that

\[
\mathcal{L}^{-1} \left( \frac{s^i}{P(s)} \right) = D^i G(t).
\]
6. B-D, pp. 352: 13, 18, 19, 21, 22, 23
Chapter 5

Nonlinear oscillators

We shall discuss two types of nonlinear oscillators: conservative and non-conservative.

5.1 Conservative nonlinear oscillators and the energy method

A conservative one-dimensional oscillator is modeled by the following equation:

\[ \ddot{y} = F(y) \]  \hspace{1cm} (5.1)

where \( F(y) \) is the restoration force. We may integrate \( F \) once and define the potential \( V(y) \) by

\[ V'(y) = -F(y). \]

With the help of potential, the total energy defined by

\[ E(t) = \frac{1}{2} \dot{y}(t)^2 + V(y(t)) \]  \hspace{1cm} (5.2)

is conserved. To see that, we multiplies (5.1) by \( \dot{y} \):

\[ \dot{y} \ddot{y} = -V'(y) \dot{y}, \]

By chain rule, this is

\[ \frac{d}{dt} \left( \frac{\dot{y}(t)^2}{2} + V(y(t)) \right) = 0. \]

Thus, \( E(t) = E \) for some constant \( E \). This \( E \) is the conserved energy.

5.1.1 Examples

Simple pendulum  Let us consider a pendulum with arm length \( l \). One of its end is fixed. The other end has a mass \( m \) swinging with angle \( \theta \) from the verticle line. The acceleration of the mass is \( l \ddot{\theta} \). Here, the dot denotes for \( d/dt \). The gravitational force in the direction of motion is \(-mg \sin \theta\). Therefore, its equation of motion reads

\[ ml\ddot{\theta} = -mg \sin(\theta), \]

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or

\[
\ddot{\theta} = -\kappa^2 \sin(\theta), \quad \kappa = \sqrt{\frac{g}{l}}. \tag{5.3}
\]

**Motion on a given curve in a plane** A curve \((x(s), y(s))\) in a plane can be parametrized by its arc length \(s\). If the curve is prescribed as we have in the case of simple pendulum, then the motion is described by just a function \(s(t)\). By Newton’s law, the motion is governed by

\[
m\dddot{s} = f(s),
\]

where \(f(s)\) is the force in the tangential direction of the curve. For instance, suppose the curve is given by \(y = y(s)\), and suppose the force is the uniform gravitational force \(-mg(0, 1)\), then the force in the tangential direction is

\[
f(s) = \left(\frac{dx}{ds}, \frac{dy}{ds}\right) \cdot [-mg(0, 1)] = -mg \frac{dy}{ds}.
\]

Thus, the equation of motion is

\[
\dddot{s} = -g \frac{dy}{ds}. \tag{5.4}
\]

For simple pendulum, \(s = l\theta, (x(\theta), y(\theta)) = (l \sin \theta, -l \cos \theta)\), and

\[
\frac{dy}{ds} = \frac{dy}{d\theta} \frac{d\theta}{ds} = -g \sin \theta.
\]

Hence, the equation of motion is

\[
ml\dddot{\theta} = -mg \sin \theta,
\]

or in terms of \(s\),

\[
m\dddot{s} = -mg \sin \left(\frac{s}{l}\right).
\]

**Duffing oscillator** A linear oscillator satisfies

\[
my'' = -V'(y)
\]

with restoration potential \(V(y) = \frac{k}{2}y^2\). A nonlinear oscillator has restoration potential

\[
V(y) = \frac{y^4}{4} - \frac{\delta y^2}{2}.
\]

is called a duffing oscillator. It is a double well potential.
5.1. CONSERVATIVE NONLINEAR OSCILLATORS AND THE ENERGY METHOD

**Energy method** To find the trajectory with a prescribed energy $E$, we express $\dot{y}$ in terms of $y$ and $E$:

$$\dot{y} = \pm \sqrt{2(E - V(y))}.$$  

Using separation of variables, we reach

$$\pm \frac{dy}{\sqrt{2(E - V(y))}} = dt.$$ 

The solution is given implicitly by

$$\pm \int \frac{dy}{\sqrt{2(E - V(y))}} = t - t_0.$$  

The two constants $E$ and $t_0$ are determined by the initial data.

5.1.2 Phase plane and autonomous systems

We may express the second order equation as a system of first-order ODE:

$$\frac{d}{dt} \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \dot{y}(t) \\ F(y(t)) \end{pmatrix} \quad (5.5)$$

In abstract form:

$$\dot{y} = F(y) \quad (5.6)$$

Such system is called autonomous system, i.e. $F(y)$ is independent of $t$. In this case, if $y(t)$ is a solution, so does $y(t + t_0)$ for any constant $t_0$. Therefore, we are mainly interested in the trajectory $\{y(t) | t \in \mathbb{R}\}$ in the phase space, not the graph $\{(t, y(t)) | t \in \mathbb{R}\}$.

A constant $\bar{y}$ is called an equilibrium of (5.6) if $F(\bar{y}) = 0$. In this case, $y(t) \equiv \bar{y}$ is a constant solution.

For a conservative system: $\ddot{y} = -V'(y)$, we can obtain the trajectories on the phase plane by energy method. The trajectories are given by the following implicit expression:

$$C_E = \{(y, \dot{y}) | \frac{1}{2} \dot{y}^2 + V(y) = E\}$$

**Phase diagram of harmonic oscillators** The equation for a harmonic oscillator is

$$\ddot{y} + \omega^2 y = 0.$$  

The potential $V(y) = \frac{1}{2} \omega^2 y^2$. The energy is

$$E = \frac{1}{2} \dot{y}^2 + \frac{1}{2} \omega^2 y^2$$

which are ellipses on the phase plane. The only equilibrium is $(y, \dot{y}) = (0, 0)$.

For a fixed $E > 0$, the ellipse $\frac{1}{2} \dot{y}^2 + \frac{1}{2} \omega^2 y^2 = E$ can be solved for $\dot{y}$:

$$\dot{y} = \pm \sqrt{2E - \omega^2 y^2}.$$
This first-order equation can be solved by separation of variable:

\[ \frac{dy}{\sqrt{2E - \omega^2 y^2}} = \pm dt. \]

Normalizing this equation:

\[ t = \int \frac{dy}{\sqrt{2E - \omega^2 y^2}} = \frac{1}{\omega} \int \frac{dy}{\sqrt{1 - \frac{\omega^2 y^2}{2E}}} \]

This yields

\[ \sin^{-1} \left( \frac{\omega y}{\sqrt{2E}} \right) = \omega t + C. \]

Hence

\[ y = \frac{\sqrt{2E}}{\omega} \sin(\omega(t + t_0)), \]

where \( t_0 \) is a free parameter.

**Homeworks.**

- B-D, pp. 502: 11, 14, 17, 22.

## 5.2 Simple pendulum

### 5.2.1 global structure of phase plane

We are interested in all possible solutions as a function of its parameters \( E \) and \( t_0 \). The constant \( t_0 \) is unimportant. For the system is autonomous, that is its right-hand side \( F(y) \) is independent of \( t \). This implies that if \( y(t) \) is a solution, so is \( y(t - t_0) \) for any \( t_0 \). The trajectories \( (y(t), \dot{y}(t)) \) and \( (y(t - t_0), \dot{y}(t - t_0)) \) are the same curve in the phase plane (i.e. \( y-\dot{y} \) plane). So, to study the trajectory on the phase plane, the relevant parameter is \( E \). We shall take the simple pendulum as a concrete example for explanation. In this case, \( V(y) = -\cos(y)g/l. \)

As we have seen that

\[ \frac{\dot{y}^2}{2} + V(y) = E, \] (5.7)

the total conserved energy. We can plot the equal-energy curve on the phase plane.

\[ C_E := \{ (y, \dot{y}) \mid \frac{\dot{y}^2}{2} - \frac{g}{l} \cos y = E \} \] (5.8)

This is the trajectory with energy \( E \). These trajectories can be classified into the follow categories.
5.2. **SIMPLE PENDULUM**

1. *No trajectory:* For $E < -g/l$, the set $\{(y, \dot{y})|\frac{\dot{y}^2}{2} - \frac{g}{l} \cos y = E\}$ is empty. Thus, there is no trajectory with such $E$.

2. *Equilibria:* For $E = -g/l$, the trajectories are isolated points $(2n\pi, 0)$, $n \in \mathbb{Z}$. These correspond to equibria, namely they are constant state solutions $y(t) = 2n\pi$, for all $t$.

3. *Bounded solutions.* For $-g/l < E < g/l$, the trajectories are bounded closed orbits. Due to periodicity of the cosine function, we see from (5.8) that $(y, \dot{y})$ is on $C_E$ if and only if $(y + 2n\pi, \dot{y})$ is on $C_E$. We may concentrate on the branch of the trajectory lying between $(-\pi, \pi)$, since others are simply duplications of the one in $(-\pi, \pi)$ through the mapping $(y, \dot{y}) \mapsto (y + 2n\pi, \dot{y})$.

   For $y \in (-\pi, \pi)$, we see that the condition
   $$\frac{\dot{y}^2}{2} - \frac{g}{l} \cos y = E$$
   implies
   $$E + \frac{g}{l} \cos y \geq 0,$$
   or
   $$\cos y \geq -\frac{El}{g}.$$ 

   This forces $y$ can only stay in $[-y_1, y_1]$, where
   $$y_1 = \cos^{-1}(-\frac{El}{g}).$$

   The condition $-g/l < E < g/l$ is equivalent to $0 < y_1 < \pi$. The branch of the trajectory $C_E$ in the region $(-\pi, \pi)$ is a closed orbit:
   $$\dot{y} = \begin{cases} \sqrt{2(E + \frac{g}{l} \cos y)} & \text{for } \dot{y} > 0, \\ -\sqrt{2(E + \frac{g}{l} \cos y)} & \text{for } \dot{y} < 0 \end{cases}$$

   The solution is bounded in $[-y_1, y_1]$. The two end states of this orbit $(\pm y_1, 0)$, where the velocity $\dot{y}$ is zero and the corresponding angle $y$ has the largest absolute value. The value $y_1$ is called the amplitude of the pendulum.

   We integrate the upper branch of this closed orbit by using the method of separation of variable:
   $$\int_{0}^{y} \frac{dy}{\sqrt{2(E + \frac{g}{l} \cos y)}} = \int dt = \pm(t - t_0)$$

   We may normalize $t_0 = 0$ because the system is autonomous (that is, the right-hand side of the differential equation is independent of $t$). Let us denote
   $$t_1 := \int_{0}^{y_1} \frac{dy}{\sqrt{2(E + \frac{g}{l} \cos y)}}.$$
Let us call
\[ \psi(y) := \int_0^y \frac{dy}{\sqrt{2(E + \frac{g}{l} \cos y)}}. \]

Then \( \psi(y) \) is defined for \( y \in [-y_1, y_1] \) with range \([-t_1, t_1]\). The function \( \psi \) is monotonic increasing (because \( \psi'(y) > 0 \) for \( y \in (-y_1, y_1) \)). Hence, its inversion \( y(t) = \phi(t) \) is well-defined for \( t \in [-t_1, t_1] \). This is the solution \( y(t) \) in the upper branch of \( C_E \) in \((-\pi, \pi)\). We notice that at the end point of this trajectory, \( \dot{y}(t_1) = 0 \).

Therefore, for \( t > t_1 \), we can go to the lower branch smoothly:
\[-\int_{y_1}^{y} \frac{dy}{\sqrt{2(E + \frac{g}{l} \cos y)}} = t - t_1.\]

This yields
\[-\left( \int_{y_1}^{0} + \int_{0}^{y} \right) \frac{dy}{\sqrt{2(E + \frac{g}{l} \cos y)}} = t - t_1,\]

The first integral is \( t_1 \), whereas the second integral is \( -\psi(y) \). Thus,
\[ \psi(y) = 2t_1 - t. \]

As \( y \) varies from \( y_1 \) to \(-y_1\), \( 2t_1 - t \) varies from \( t_1 \) to \(-t_1\), or equivalently, \( t \) varies from \( t_1 \) to \( 3t_1 \). Hence, the solution for \( t \in [t_1, 3t_1] \) is
\[ y(t) := \phi(2t_1 - t). \]

We notice that
\[ y(t) = \phi(2t_1 - t) = y(2t_1 - t) \text{ for } t \in [2t_1, 3t_1] \]

At \( t = 3t_1 \), \( y(3t_1) = -y_1 \) and \( \dot{y}(3t_1) = 0 \). We can continue the time by integrating the upper branch of \( C_E \) again. This would give the same orbit. Therefore, we can extend \( y \) periodically with period \( T = 2t_1 \) by:
\[ y(t) = y(t - 2nT) \text{ for } 2nT \leq t \leq 2(n + 1)T. \]

4. **Another equilibria**: For \( E = g/l \), the set \( C_E \) contains isolated equilibria:
\[ \{(2n + 1)\pi, 0)|n \in \mathbb{Z}\} \subset C_E = \]

In addition, we can also solve \( \dot{y} \) on \( C_E \):
\[ \dot{y} = \pm \sqrt{2(1 + \cos(y)) \frac{g}{l}}. \]

This can be consider as a limiting of the above case with \( E \to g/l \) from below.
5. **Unbounded solution**: For $E > g/l$, there are two branches of $C_E$, the upper one ($\dot{y} > 0$) and the lower one ($\dot{y} < 0$). The upper branch: $\dot{y} = \sqrt{2(E + \frac{g}{l} \cos(y))} > 0$ is defined for all $y \in \mathbb{R}$. By using the method of separation of variable, we get

$$
\int_0^y \frac{dy}{\sqrt{2 \left(E + \frac{g}{l} \cos(y)\right)}} = t
$$

Let us call the left-hand side of the above equation by $\psi(y)$. Notice that $\psi(y)$ is a monotonic increasing function defined for $y \in (-\infty, \infty)$, because $\psi'(y) > \frac{1}{2(E - g/l)} > 0$. The range of $\psi$ is $(-\infty, \infty)$. Its inversion $\phi(t)$ is the solution $y = \phi(t)$. Let

$$
T := \int_0^{2\pi} \frac{dy}{\sqrt{2 \left(E + \frac{g}{l} \cos(y)\right)}}
$$

From the periodicity of the cosine function, we have for $2n\pi \leq y \leq 2(n + 1)\pi$,

$$
t = \psi(y) = \left(\int_0^{2\pi} + \cdots + \int_{2n\pi}^{2n\pi} + \int_0^y\right) \frac{dy}{\sqrt{2 \left(E + \frac{g}{l} \cos(y)\right)}}
$$

This yields

$$
t = nT + \psi(y - 2n\pi).
$$

Or

$$
y(t) = 2n\pi + \phi(t - nT), \text{ for } t \in [nT, (n + 1)T].
$$

### 5.2.2 Period

Let us compute the period for case 2 in the previous subsection. Recall that

$$
T = \int_{-y_1}^{y_1} \frac{dy}{\sqrt{2 \left(E + \frac{g}{l} \cos(y)\right)}} = \sqrt{\frac{g}{2l}} \int_{-y_1}^{y_1} \frac{dy}{\sqrt{\frac{E}{g} + \cos(y)}}
$$

$$
= \sqrt{\frac{g}{2l}} \int_{-y_1}^{y_1} \frac{dy}{\sqrt{\cos(y) - \cos(y_1)}} = \sqrt{\frac{g}{l}} \int_{-y_1}^{y_1} \frac{dy}{\sqrt{\sin^2 \frac{y_1}{2} - \sin^2 \frac{y}{2}}}
$$

where $0 < y_1 = \arccos(-El/g) < \pi$ is the amplitude of the pendulum. By the substitution

$$
u = \frac{\sin(y/2)}{\sin(y_1/2)},
$$

the above integral becomes

$$
T = 2\sqrt{\frac{l}{g}} \int_{-1}^{1} \frac{du}{\sqrt{(1 - u^2)(1 - k^2u^2)}}
$$

(5.9)
where \( k = \sin(y_1/2) \). This integral is called an elliptic integral. This integral cannot be expressed as an elementary. But we can estimate the period by using

\[
1 \geq 1 - k^2u^2 \geq 1 - k^2
\]

for \(-1 \leq u \leq 1\) and using \( \int_{-1}^{1} 1/\sqrt{1-u^2} \, du = \pi \), the above elliptic integral becomes

\[
2\pi \sqrt{\frac{l}{g}} \leq T \leq 2\pi \sqrt{\frac{l}{g}} \left( \frac{1}{1 - k^2} \right)
\]  

(5.10)

**Homework.**

Using Taylor expansion for \((1 - k^2u^2)^{-1/2}\), expand the elliptic integral

\[
f(k) = \int_{-1}^{1} \frac{du}{\sqrt{(1 - u^2)(1 - k^2u^2)}}
\]

in Taylor series in \( k \) for \( k \) near 0. You may use Maple to do the integration.

### 5.3 Cycloidal Pendulum – Tautochrone Problem

#### 5.3.1 The Tautochrone problem

The period of a simple pendulum depends on its amplitude \( y_1 \). A question is that can we design a pendulum such that its period is independent of its amplitude. An ancient Greek problem called tautochrone problem answers this question. The tautochrone problem is to find a curve down which a bead placed anywhere will fall to the bottom in the same amount of time. Thus, such a curve can provide a pendulum with period independent of its amplitude. The answer is the cycloid. The cycloidal pendulum oscillates on a cycloid. The equation of a cycloid is

\[
\begin{align*}
x &= l(\theta + \pi + \sin \theta) \\
y &= -l(1 + \cos \theta)
\end{align*}
\]

Its arc length is

\[
s = \int \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta \\
= l \int \sqrt{(1 + \cos \theta)^2 + \sin^2 \theta} \, d\theta \\
= 2l \int \cos \left( \frac{\theta}{2} \right) \, d\theta \\
= 4l \sin \left( \frac{\theta}{2} \right).
\]

\footnote{Indeed, \( k = \sin(y_1/2) \)}
The force
\[
\frac{dy}{ds} = \frac{dy}{d\theta} \frac{d\theta}{ds} = \frac{l \sin \theta}{2l \cos \left( \frac{\theta}{2} \right)} = \sin \left( \frac{\theta}{2} \right) = \frac{s}{4l}.
\]

The equation of motion on cycloidal pendulum is
\[
\ddot{s} = -\frac{g}{4l} s,
\]
a linear equation! Its period is \( T = 2\pi \sqrt{\frac{l}{g}} \), which is independent of the amplitude of the oscillation.

**Which planar curves produce linear oscillators?**

The equation of motion on a planar curve is
\[
\ddot{s} = -g \frac{dy}{ds}.
\]

The question is: what kind of curve produce linear oscillator. In other word, which curve gives \( \frac{dy}{ds} = ks \). This is an ODE for \( y(s) \). Its solution is
\[
y(s) = \frac{k}{2} s^2.
\]

Since \( s \) is the arc length of the curve, we have
\[
x'(s)^2 + y'(s)^2 = 1.
\]

Hence, \( x'(s) = \pm \sqrt{1 - k^2 s^2} \). We use the substitution: \( s = \sin(\theta/2)/k \). Then
\[
y = \frac{k}{2} s^2 = \frac{1}{2k} \sin^2 \left( \frac{\theta}{2} \right) = \frac{1}{4k} (1 - \cos \theta).
\]
\[
x = \int \sqrt{1 - k^2 s^2} ds = \frac{1}{2k} \int \cos^2 \left( \frac{\theta}{2} \right) d\theta = \frac{1}{4k} \int (1 + \cos \theta) d\theta = \frac{1}{4k} (\theta + \sin \theta).
\]

Thus, the planar curve that produces linear restoration tangential force is a cycloid.


5.3.2 The Brachistochrone

The Brachistochrone problem is to find a curve on which a ball sliding down under gravitation to a point with depth \( h \) takes least time. The word “brachistochrone” means the “the shortest time delay” in Greek. It was one of the oldest problem in Calculus of Variation. Its solution is a section of a cycloid. This was founded by Leibnitz, L’Hospital, Newton and two Bernoullis. Suppose the curve starts from \( A \). Let \( s \) be the arc length of the curve. The equation of motion is
\[
m\ddot{s} = -gy'(s).
\]
This gives the conservation of energy
\[ \frac{1}{2}ms^2 + gy(s) = E. \]

At point A, we take \( s = 0, \dot{s} = 0 \) and \( y(0) = 0 \). With this normalization, \( E = 0 \). Thus, we have the speed
\[ v = \dot{s} = \sqrt{-gy}. \]

Notice that \( y \leq 0 \) under our consideration. The travelling time from A to B is given by
\[ T_A^B = \int_0^s \frac{1}{s} \, ds = \int_0^s \sqrt{-gy} \, ds \]

Bernoulli approximated \( v \) by piecewise constant function. By Snell’s law
\[ \frac{\sin \phi}{v} = \frac{\sin \phi_r}{v_r} \]

In the limit, we have
\[ \frac{\sin \phi}{v} = \text{const.} \]

where \( \phi \) is the angle between the curve and the \( y \)-axis. The tangent of the curve is \((dx/ds, dy/ds)\). Hence, the angle \( \phi \) satisfies \( \cos(\phi) = (dx/ds, dy/ds) \cdot (0, -1) = -dy/ds \). Thus, we arrive at
\[ \frac{\sin \phi}{v} = \frac{\sqrt{1 - (dy/ds)^2}}{\sqrt{-gy}} = \sqrt{k} \, (\text{const.}) \]

The constant \( k \) is to be determined later. This yields
\[ 1 - \left( \frac{dy}{ds} \right)^2 = -gky. \]

We may rewrite this equation as
\[ \frac{dy}{ds} = \sqrt{1 + gky}, \quad \text{or} \quad \frac{d(1 + gky)}{\sqrt{1 + gky}} = gks. \]

This yields
\[ 2\sqrt{1 + gky} = gks + C, \]

When \( s = 0 \), we have \( y(0) = 0 \). Hence the constant \( C = 2 \).
\[ gky = \left( \frac{2 + gks}{2} \right)^2 - 1. \]

For \( x(s) \), we have
\[ \frac{dx}{ds} = \sqrt{1 - (dy/ds)^2} = \sqrt{1 - (1 + gky)} = \sqrt{-gky} = \sqrt{1 - \left( 1 + \frac{gks}{2} \right)^2}. \]
We use the substitution:

\[ 1 + \frac{gks}{2} = \sin \frac{\theta}{2}. \]

Then we arrive that

\[ \frac{dx}{d\theta} = \frac{dx}{ds} \frac{ds}{d\theta} = \sqrt{1 - \sin^2(\theta/2)} \frac{1}{gk} \cos(\theta/2). \]

Hence,

\[ x = \frac{1}{gk} \int \cos^2(\theta/2) d\theta = \frac{1}{gk} \left( \frac{\theta + \sin \theta}{2} + C \right). \]

We can re-express \( y \) in terms of \( \theta \) as

\[
y = \frac{1}{gk} \left( -1 + \left( 1 + \frac{gks}{2} \right)^2 \right) = \frac{1}{gk} \left( -1 + \sin^2(\theta/2) \right) = -\frac{1}{gk} \frac{1 + \cos \theta}{2}.
\]

We find that \( \theta = -\pi \) corresponds to \( y = 0 \). For \( \theta = -\pi \), it should correspond \( x = 0 \). This yields \( C = \pi/2 \). Hence,

\[ x = \frac{1}{gk} \left( -\pi + \theta + \sin \theta \right). \]

This again gives a cycloid. The bottom of the cycloid occurs at \( \theta = 0 \), where \( y = -1/gk \). Therefore, the constant \( k \) is determined by the depth of the final point \( B \), which is \( h \). Thus, \( k = 1/gh \).

### 5.3.3 Construction of a cycloidal pendulum

To construct a cycloidal pendulum, we take \( l = 1 \) for explanation. We consider the evolute of the cycloid

\[ x = \pi + \theta + \sin \theta, \quad y = -1 - \cos \theta. \] \hspace{1cm} (5.11)

In geometry, the evolute \( E \) of a curve \( C \) is the set of all centers of curvature of that curve. On the other hand, if \( E \) is the evolute of \( C \), then \( C \) is the involute of \( E \). An involute of a curve \( E \) can be constructed by the following process. We first wrape \( E \) by a thread with finite length. One end of the thread is fixed on \( E \). We then unwrap the thread. The trajectory of the other end as you unwrap the thread forms the involute of \( E \). We shall show below that the evolute \( E \) of a cycloid \( C \) is again a cycloid. With this, we can construct a cycloidal pendulum as follows. We let the mass \( P \) is attached by a thread of length \( 4 \) to one of the cusps of the evolute \( E \). Under the tension, the thread is partly coincide with the evolute and lies along a tangent to \( E \). The mass \( P \) then moves on the cycloid \( C \).

Next, we show that the motion of the mass \( P \) lies on the cycloid \( C \). The proof consists of three parts.

1. The evolute of a cycloid is again a cycloid. Suppose \( C \) is expressed by \((x(\theta), y(\theta))\).

We recall that the curvature of \( C \) at a particular point \( P = (x(\theta), y(\theta)) \) is defined by 
\[
\frac{d\alpha}{ds}, \quad \alpha = \arctan \left( \frac{\dot{y}(\theta)}{\dot{x}(\theta)} \right)
\]
the inclined angle of the tangent of \( C \) and \( ds = \sqrt{\dot{x}^2 + \dot{y}^2} \, d\theta \) is the infinitesimal arc length. Thus, the curvature, as expressed by parameter \( \theta \), is given by
\[
\kappa = \frac{d\alpha}{ds} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\left(1 + \left( \frac{\dot{y}}{\dot{x}} \right)^2 \right)^{3/2}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.
\]

The center of curvature of \( C \) at \( P = (x, y) \) is the center of the osculating circle that is tangent to \( C \) at \( P \). Suppose \( PP' = (\xi, \eta) \) is its coordinate. Then \( PP' \) is normal to \( C \) (the normal \((n_x, n_y) = (-\dot{y}, \dot{x})/\sqrt{\dot{x}^2 + \dot{y}^2}\) and the radius of the osculating circle is \(1/\kappa\). Thus, the coordinate of the center of curvature is
\[
\xi = x + \frac{1}{\kappa} n_x = x - \frac{\dot{y}^2 + \dot{x}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}},
\]
\[
\eta = y + \frac{1}{\kappa} n_y = y + \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}.
\]

When \((x(\theta), y(\theta))\) is given by the cycloid equation (5.11),
\[
x = \pi + \theta + \sin \theta, \quad y = -1 - \cos \theta, \quad -\pi \leq \theta \leq \pi,
\]
we find that its evolute
\[
\xi = \pi + \theta - \sin \theta, \quad \eta = 1 + \cos \theta, \quad (5.12)
\]
is also a cycloid.

2. The evolute of \( C \) is the envelope of its normals. We want to find the tangent of the evolute \( E \) and show it is identical to the normal of \( C \). To see this, we use arc length \( s \) as a parameter on \( C \). With this, the normal \((n_x, n_y) = (-y', x')\) and the curvature \( \kappa = x' y'' - y' x'' \), where \( ' \) is \( d/ds \). The evolute is
\[
\xi = x - \rho y', \quad \eta = y + \rho x', \quad (5.13)
\]
where \( \rho = 1/\kappa \). Thus, the evolute \( E \) is also parametrized by \( s \). Since \( x'^2 + y'^2 = 1 \), we differentiate it in \( s \) to get \( x' x'' + y' y'' = 0 \). This together with \( \kappa = x' y'' - y' x'' \) yield
\[
x'' = -y' / \rho, \quad y'' = x' / \rho.
\]

Differentiating (5.13) in \( s \), we can get the tangent of the evolute \( E \):
\[
\xi' = x' - \rho y'' - \rho' y' = -\rho' y', \quad \eta' = y' + \rho x'' + \rho' x' = \rho' x', \quad (5.14)
\]
Therefore,
\[
\xi' x' + \eta' y' = 0.
\]

This means that the tangent \((\xi', \eta')\) of the evolute at the center of curvature is parallel to the normal direction \((-y', x')\) of the curve \( C \). Since both of them pass through \((\xi, \eta)\), they are coincide. In other words, the normal to the curve \( C \) is tangent to the evolute \( E \) at the center of curvature.
3. The end point of the thread $P$ lies on the cycloid $C$. We show that the radius of curvature plus the length of portion on $E$ where the thread is attatched to is 4. To see this, we denote the arclength on the evolute $E$ by $\sigma$. The evolute $E$, as parametrized by the arc length $s$ of $C$ is given by (5.13). Its arclength $\sigma$ satisfies

$$\left( \frac{d\sigma}{ds} \right)^2 = \xi'^2 + \eta'^2 = (-\rho'y')^2 + (\rho'x')^2 = \rho'^2$$

Here, we have used (5.14). Hence, $\sigma'^2 = \rho'^2$. We take $s = 0$ at $\theta = \pi ((x, y) = (\pi, -2))$. We choose $s > 0$ when $\theta > \pi$. We take $\sigma(0) = 0$ which corresponds to $(\xi, \eta) = (\pi, 2)$. We call this point $A$ (the cusp of the cycloid $E$). We also choose $\sigma(s) > 0$ for $s > 0$. Notice that $\rho'(s) < 0$. From these normalization, we have

$$\sigma'(s) = -\rho'(s).$$

Now, as the mass moves along $C$ to a point $P$ on $C$, the center of curvature of $C$ at $P$ is $Q$ which is on the evolute $E$. We claim that

length of the arc $AQ$ on $E$ + the length of the straight line $PQ = 4$.

To see that, the first part above is

$$\int_0^s \sigma' ds = -\int_0^s \rho' ds = \rho(0) - \rho(s).$$

The second part is simply the radius of curvature $\rho(s)$. Hence the above sum is $\rho(0) = 4$.

**Homework.**

1. Given a family of curves $\Gamma_\lambda : \{(x(t, \lambda), y(t, \lambda)) | t \in \mathbb{R} \}$, a curve $E$ is said to be the *envelop* of $\Gamma_\lambda$ if

(a) For each $\lambda$, $\Gamma_\lambda$ is tangent to $E$. Let us denote the tangent point by $P_\lambda$.

(b) The envelop $E$ is made of $P_\lambda$ with $\lambda \in \mathbb{R}$.

Now consider the family of curves to be the normal of a cycliod $C$, namely

$$\Gamma_\theta = (x(\theta) + tnx(\theta), y(\theta) + tny(\theta)),$$

where $(x(\theta), y(\theta))$ is given by (5.11) and $(nx, ny)$ is its normal. Using this definition of envelop, show that the envelop of $\Gamma_\theta$ is the cycloid given by (5.12).

### 5.4 The orbits of planets and stars

#### 5.4.1 Centrally directed force and conservation of angular momentum

The motion of planets or stars can be viewed as a particle moving under a centrally directed field of force:

$$\mathbf{F} = F(r)\hat{e}_r,$$
where \( r \) is the distance from the star to the center, \( \mathbf{r} \) is the position vector from the center to the star and 
\[
\hat{e}_r = \frac{\mathbf{r}}{r},
\]
is the unit director. The equation of motion of the star is 
\[
\ddot{\mathbf{r}} = F(r)\hat{e}_r.
\]
Define the angular momentum \( \mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}} \). We find 
\[
\frac{d\mathbf{L}}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = F(r)\mathbf{r} \times \hat{e}_r = 0.
\]
Hence, \( \mathbf{L} \) is a constant. A function in the state space \((\mathbf{r}, \dot{\mathbf{r}})\) is called an integral if it is unchanged along any orbits. The integrals can be used to reduce number of unknowns of the system. The conservation of angular momentum provides us three integrals. Let us write \( \mathbf{L} = L\mathbf{n} \) where \( L = |\mathbf{L}| \) and \( \mathbf{n} \) is a unit vector. The position vector \( \mathbf{r} \) and the velocity \( \dot{\mathbf{r}} \) always lie on the plane which is perpendicular to \( \mathbf{n} \). This plane is called the orbital plane. We use polar coordinates \((r, \theta)\) on this plane. Thus, by using the integrals \( \mathbf{n} \), which has two parameters, we can reduce the number of unknowns from 6 to 4, that is, from \((\mathbf{r}, \dot{\mathbf{r}})\) to \((r, \theta, \dot{r}, \dot{\theta})\). To find the equation of motion on this plane, we express 
\[
\mathbf{r} = r\hat{e}_r = r(\cos \theta, \sin \theta).
\]
Define 
\[
\hat{e}_\theta := (-\sin \theta, \cos \theta)
\]
be the unit vector perpendicular to \( \hat{e}_r \). Then a particle motion on a plane with trajectory \( \mathbf{r}(t) \) has the following velocity 
\[
\dot{\mathbf{r}} = \dot{r}\hat{e}_r + \dot{\theta}\hat{e}_\theta = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta.
\]
where \( \dot{r} \) is the radial speed and \( r\dot{\theta} \) is the circular speed. Here, we have used 
\[
\dot{\hat{e}}_r = \frac{d}{dt}(\cos \theta, \sin \theta) = \dot{\theta}\hat{e}_\theta.
\]
The acceleration is 
\[
\ddot{\mathbf{r}} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta.
\]
Here, we have used \( \dot{\hat{e}}_\theta = -\hat{e}_r \). In this formula, \( \ddot{r} \) is the radial acceleration, and \( -r\dot{\theta}^2 \) is the centripetal acceleration. The term 
\[
r(2\dot{r}\dot{\theta} + r\ddot{\theta}) = \frac{d}{dt}(r^2\dot{\theta})
\]
is the change of angular momentum. Indeed, the angular momentum is 
\[
\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}} = r\hat{e}_r \times (\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) = r^2\dot{\theta}\mathbf{n}.
\]
The equation of motion $\ddot{r} = F(r)\hat{e}_r$ gives

$$\ddot{r} - r\dot{\theta}^2 = F(r), \quad (5.15)$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0. \quad (5.16)$$

These are the two second-order equations for the unknowns $(r, \theta, \dot{r}, \dot{\theta})$. The $\theta$ equation (5.16) can be integrated and gives the conservation of angular momentum

$$r^2\dot{\theta} = \text{constant} = L. \quad (5.17)$$

If we prescribe an $L$, the trajectory lies on the set

$$\{(r, \theta, \dot{r}, \dot{\theta}) \mid \dot{\theta} = L/r^2\}.$$

We may project this set to the $(r, \theta, \dot{r})$-space and our unknowns now are reduced to $(r, \theta, \dot{r})$. The equations of motion in this space are (5.15) and (5.17).

The integral $L$ can be used to eliminate $\dot{\theta}$ from the first equation. We get

$$\ddot{r} = F(r) + \frac{L^2}{r^3}, \quad (5.18)$$

where the second term on the right-hand side is the centrifugal force. Notice that this equation is independent of $\theta$. Thus, given initial data $(r_0, \theta_0, \dot{r}_0)$ at time $t = 0$, we can find $r(t)$ and $\dot{r}(t)$ from (5.18) by using $(r_0, \dot{r}_0)$ only. We can then use $r^2\dot{\theta} = L$ to find $\theta(t)$:

$$\theta(t) = \theta_0 + \int_0^t \frac{L}{r(t)^2} \, dt.$$

The equation (5.18) can be solved by the energy method. We multiply (5.18) by $\dot{r}$ on both sides to obtain

$$\frac{d}{dt} \left( \frac{1}{2}r^2 + \Phi(r) + \frac{1}{2} \frac{L^2}{r^2} \right) = 0,$$

where $\Phi$ with $\Phi'(r) = -F(r)$ is the potential. We obtain the law of conservation of energy:

$$\frac{1}{2}r^2 + \Phi(r) + \frac{1}{2} \frac{L^2}{r^2} = \text{constant} = E. \quad (5.19)$$

This energy is another integral. A prescribed energy $E$ defines a surface in the $(r, \theta, \dot{r})$-space. Since the energy $\frac{1}{2}r^2 + \Phi(r) + \frac{1}{2} \frac{L^2}{r^2}$ is independent of $\theta$ (a consequence of centrally forcing), this energy surface is a cylinder $C_E \times \mathbb{R}_\theta$, where $C_E$ is the curve defined by (5.19) on the phase plane $r-\dot{r}$.

The equation of motion with a prescribed energy $E$ is

$$\frac{dr}{dt} = \pm \sqrt{2(E - \Phi(r)) - \frac{L^2}{r^2}}. \quad (5.20)$$
It is symmetric about the $r$-axis. Let us suppose that $r_1$ and $r_2$ ($r_1 < r_2$) are two roots of the right-hand side of the above equation:

$$2(E - \Phi(r)) - \frac{L^2}{r^2} = 0$$

and no other root in between. Then the curve defined by (5.20) is a closed curve connecting $(r_1, 0)$ and $(r_2, 0)$. The radial period is defined to be the time the particle travels from $(r_1, 0)$ to $(r_2, 0)$ and back. That is,

$$T_r = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E - \Phi(r)) - \frac{L^2}{r^2}}}$$

Next, we shall represent this orbit on the orbital plane $(r, \theta)$. From the conservation of angular momentum

$$\frac{d\theta}{dt} = \frac{L}{r^2} \neq 0,$$

we can invert the function $\theta(t)$ and use $\theta$ as our independent variable instead of the time variable $t$. The chain rule gives

$$\frac{d}{dt} = \frac{L}{r^2} \frac{d}{d\theta}.$$

The equation of motion now reads

$$\frac{L}{r^2} \frac{d}{d\theta} \left( \frac{L}{r^2} \frac{d}{d\theta} \right) - \frac{L^2}{r^3} = F(r). \quad (5.21)$$

The energy equation (5.20) becomes

$$\frac{dr}{d\theta} = \pm \frac{r^2}{L} \sqrt{2(E - \Phi(r)) - \frac{L^2}{r^2}}. \quad (5.22)$$

We can integrate this equation by separation of variable to obtain the trajectory $r = r(\theta)$ in the orbital plane. Sometimes, it is convenient to introduce $u = 1/r$ to simplify the equation (5.21):

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(\frac{1}{u})}{L^2 u^2}. \quad (5.23)$$

Multiplying $du/d\theta$ on both sides, we get the conservation of energy in $u$ variable:

$$\frac{1}{2} \left( \frac{du}{d\theta} \right)^2 + \frac{u^2}{2} + \frac{\Phi}{L^2} = \frac{E}{L^2}. \quad (5.24)$$

Next, we check the variation of $\theta$ as $r$ changes for a radial period. The roots of the right-hand side of (5.22) are equilibria. From (5.20) and (5.22), we see that $dr/d\theta = 0$ if and only if $dr/dt = 0$. Hence these roots are exactly $r_1$ and $r_2$ in (5.20). The orbit $r = r(\theta)$ defined by (5.20) must lie between its two extremals where $dr/d\theta = 0$. That is, the orbit $r = r(\theta)$ must lie between the inner circle $r \equiv r_1$ and the outer circle $r \equiv r_2$. The inner radius $r_1$ is called the pericenter distance, whereas $r_2$ the apocenter distance.
As the particle travels from pericenter to apocenter and back (i.e. one radial period $T_r$), the azimuthal angle $\theta$ increases by an amount
\[ \Delta \theta = 2 \int_{r_1}^{r_2} \frac{d\theta}{dr} dr = 2 \int_{r_1}^{r_2} \frac{L}{r^2} \frac{dt}{dr} dr = 2L \int_{r_1}^{r_2} \frac{dt}{r^4} \sqrt{2(E - \Phi(r)) - L^2/r^2}. \]

The azimuthal period is defined as the time that $\theta$ varies $2\pi$:
\[ T_\theta := \frac{2\pi}{\Delta \theta T_r}. \]

In general, $2\pi/\Delta \theta$ is not a rational number. Hence, the orbit may not be closed.

Below, we see some concrete examples. We shall find the trajectory of the motion $r = r(\theta)$.

**Quadratic potential**

The potential generated by a homogeneous sphere has the form $\Phi(r) = \frac{1}{2} \Omega^2 r^2$, where $\Omega$ is a constant. The force in Cartesian coordinate is $F = -\Omega^2 (x, y)$. Hence the equation of motion is
\[ \ddot{x} = -\Omega^2 x, \quad \ddot{y} = -\Omega^2 y. \]

We notice that the $x$ and $y$ components are decoupled. Its solution is
\[ x(t) = a \cos(\Omega t + \theta_x), \quad y(t) = b \cos(\Omega t + \theta_y), \quad \text{Eq.} (5.25) \]
where $a, b$ and $\theta_x, \theta_y$ are constants. The orbits are ellipses.

The energy equation is
\[ \frac{1}{2} \dot{r}^2 + \frac{\Omega^2}{2} r^2 + \frac{1}{2} \frac{L^2}{r^2} = E. \]

Its contour curves are bounded and symmetric about $r$ and $\dot{r}$ axis. The solution is
\[ \dot{r} = \pm \sqrt{2E - \Omega^2 r^2 - \frac{L^2}{r^2}}. \]

The trajectory intersect $\dot{r} = 0$ at $r_1$ and $r_2$, where $r_i$ satisfies $2E - \Omega^2 r^2 - \frac{L^2}{r^2}$. This yields
\[ r_i^2 = \frac{E \pm \sqrt{E^2 - \Omega^2 L^2}}{\Omega^2}. \]

There are two real roots when $E^2 > \Omega^2 L^2$. The above elliptical orbit moves between between $r_1$ and $r_2$. From the solution being an ellipse, we can also get that $T_r = T_\theta$.

**Homework.**

1. Show that the trajectory defined by (5.25) is an ellipse.

2. Find the integral
\[ \Delta \theta := \int_{r_1}^{r_2} \frac{2L}{r^2} \frac{dr}{\sqrt{2E - \Omega^2 r^2 - \frac{L^2}{r^2}}}. \]
**Kepler potential**

The Kepler force is \( F(r) = -GM/r^2 \), where \( M \) is the center mass, \( G \) the gravitational constant. The potential is \( \Phi(r) = -GM/r \). From (5.23),

\[
\frac{d^2u}{d\theta^2} + u = \frac{GM}{L^2}.
\]

This yields

\[
u = C \cos(\theta - \theta_0) + \frac{GM}{L^2},
\]

where \( C \) and \( \theta_0 \) are constants. By plugging this solution into the energy equation (5.24), we obtain

\[
\frac{1}{2} C^2 \sin^2(\theta - \theta_0) + \frac{1}{2} C^2 \cos^2(\theta - \theta_0) + C \cos(\theta - \theta_0) \cdot \frac{GM}{L^2} + \frac{G^2M^2}{2L^4} - \frac{GM}{L^2} C \cos(\theta - \theta_0) = \frac{E}{L^2}.
\]

This yields

\[
C = \sqrt{2E - G^2M^2/L^2}.
\]

We may assume \( \theta_0 = 0 \). Define

\[
e = \frac{CL^2}{GM}, \quad a = \frac{L^2}{GM(1 - e^2)},
\]

the eccentricity and the semi-major axis, respectively. The trajectory reads

\[
r = \frac{a(1 - e^2)}{1 + e \cos \theta}.
\]  

(5.26)

This is an ellipse. The pericenter distance \( r_1 = a(1 - e) \), whereas the apocenter distance \( r_2 = a(1 + e) \). The periods are

\[
T_r = T_\theta = 2\pi \sqrt{\frac{a^3}{GM}}.
\]  

(5.27)

**Homework.**

1. Prove (5.27).

**A perturbation of Kepler potential**

Let us consider the potential

\[
\Phi(r) = -GM \left( \frac{1}{r} + \frac{a}{r^2} \right).
\]

This potential can be viewed as a perturbation of the Kepler potential. The far field is dominated by the Kepler potential. However, in the near field, the force is attractive (but stronger) when \( a > 0 \) and becomes repulsive when \( a < 0 \).
The equation for this potential in the $r$-$\theta$ plane is
\[
d\frac{d^2u}{d\theta^2} + \left(1 - \frac{2GMa}{L^2}\right)u = \frac{GM}{L^2},
\]
where $u = 1/r$. Its general solution is
\[
\frac{1}{r} = u = C \cos \left(\frac{\theta - \theta_0}{K}\right) + \frac{GMK^2}{L^2},
\]
where
\[
K = \left(1 - \frac{2GMa}{L^2}\right)^{-1/2}.
\]
The constant $K > 1$ for $a > 0$ and $0 < K < 1$ for $a < 0$. The constant $C$ is related to the energy $E$ by
\[
E = \frac{1}{2} \frac{C^2L^2}{K^2} - \frac{1}{2} \left(\frac{GMK}{L}\right)^2.
\]
The pericenter and apocenter distances are respectively
\[
r_1 = \left(\frac{GMK^2}{L^2} + C\right)^{-1}, \quad r_2 = \left(\frac{GMK^2}{L^2} - C\right)^{-1}.
\]
The trajectory in $u$-$\theta$ plane is
\[
u = \frac{u_1 + u_2}{2} + \left(\frac{u_1 - u_2}{2}\right) \cos \left(\frac{\theta - \theta_0}{K}\right).
\]
Here, $u_1 = 1/r_1$ and $u_2 = 1/r_2$. To plot the trajectory on $u$-$\theta$ plane, we may assume $\theta_0 = 0$. If $K$ is rational, then the orbit is closed. For instance, when $K = 1$, the trajectory is an ellipse. When $K = 3/2$, the particle starts from $(u_1, 0)$, travels to $(u_2, 3/2\pi)$, then back to $(u_1, 3\pi)$, then to $(u_2, (3 + 3/2)\pi)$, finally return to $(r_1, 6\pi)$.


Homeworks
1. Consider the Duffing’s equation
\[
\ddot{s} = -y'(s), \quad y(s) = -\delta s^2/2 + s^4/4.
\]
   (a) Find the equilibria.
   (b) Plot the level curve of the energy $E$ on the phase plane $s$-$s'$.
   (c) Find the period $T$ as a function of $E$ and $\delta$.
   (d) Analyze the stability of the equilibria.
2. Consider the equation
\[ \ddot{x} = -V'(x), \quad V(x) = -\frac{x^2}{2} + \frac{x^3}{3}. \]

(a) Find the equilibria.
(b) Plot the level curve of the energy \( E \) on the phase plane \( s-s' \).
(c) Find the period \( T \) as a function of \( E \).
(d) Analyze the stability of the equilibria.
(e) There is a special orbit, called the homoclinic orbit, which starts from the origin, goes around a circle, then comes back to the origin. Find this orbit on the phase plane and try to find its analytic form.

3. Consider the Kepler problem.

(a) Plot the level curve of \( E \) on the phase plane \( r-\dot{r} \).
(b) Plot the level curve of \( E \) on the \( r-r' \) plane, where \( r' \) denotes for \( dr/d\theta \).

5.5 Damping

In this section, we consider dissipative nonlinear oscillators. The dissipation is due to friction. The friction force is usually a function of velocity. Let us call it \( b(\dot{y}) \). In general, we consider
\[ \ddot{y} = F(y) + b(\dot{y}) \]  
(5.28)

The friction force has the property:
\[ b(\dot{y}) \cdot \dot{y} < 0 \text{ and } b(0) = 0. \]

This means that the direction of the frictional force is in the opposite direction of the velocity and the friction force is zero if the particle is at rest. Here are two concrete examples of damping.

Simple pendulum with damping

The equation for simple pendulum is
\[ ml\ddot{\theta} = -mg \sin \theta. \]

A simple damping force is proportional to the angular speed \( \beta \dot{\theta} \), provided the damping comes from the friction at the fixed point. Here, \( \beta > 0 \). Thus the model for simple pendulum with friction reads
\[ ml\ddot{\theta} = -\beta \dot{\theta} - mg \sin \theta. \]  
(5.29)
5.5. DAMPING

An active shock absorber

In the mass-spring model, the friction force may depend on the velocity nonlinear, say \( \beta(v) \), say \( \beta(v) = v^4 \). Then the corresponding oscillation is nonlinear:

\[
m\ddot{y} = -\beta(\dot{y})\dot{y} - ky, \quad \beta(v) = v^4,
\]

(5.30)

5.5.1 Stability and Lyapunov method

Let us go back to consider the general problem

\[
\ddot{y} = F(y) + b(\dot{y}), \quad \text{with } b(\dot{y}) \cdot \dot{y} < 0, b(0) = 0.
\]

(5.31)

Suppose \( F(\bar{y}) = 0 \). Then steady state \( y(t) \equiv \bar{y} \) is a solution. Here, we have used \( b(0) = 0 \). We call the state \((\bar{y},0)\) an equilibrium on the phase plane.

Recall in the case of linear spring with damping, we have seen in Chapter 2 that every solution tends to the zero state as \( t \) tends to infinity. This zero state is an equilibrium and it is globally stable (i.e. any solution tends to this equilibrium as \( t \to \infty \)). Let us study the same global stability problem for the damped nonlinear equation(5.31). Let us suppose that the potential corresponding to the force \( F(y) \) is \( V(y) \), i.e. \( V'(y) = -F(y) \). We consider the following damped system:

\[
\ddot{y} = -V'(y) + b(\dot{y}).
\]

(5.32)

The following theorem give a sufficient condition for global stability of the equilibrium.

**Theorem 5.6** Suppose \( V(y) \to \infty \) as \( |y| \to \infty \) and \( V(y) \) has only one minimum \( \bar{y} \). Then any solution \( y \) satisfies

\[
y(t) \to \bar{y} \text{ and } \dot{y}(t) \to 0 \text{ as } t \to \infty.
\]

**Proof.** For simplicity, we assume \( b(\dot{y}) = -\dot{y} \). It should be clear if this case is done, how it can be generalized to more general cases. Without loss of generality, we may also assume \( \bar{y} = 0 \) and \( V(0) = 0 \). Otherwise, we may just replace \( y \) by \( y - \bar{y} \) and \( V(y) \) by \( V(y) - V(\bar{y}) \), which does not alter \( F(y) \) in the original problem.

We use energy method: multiplying \( \dot{y} \) on both sides of (5.31), we obtain

\[
\dot{y}\ddot{y} = -V'(y)\dot{y} - \dot{y}^2
\]

\[
\frac{dE}{dt} = -\dot{y}^2,
\]

(5.33)

where

\[
E(y, \dot{y}) := \frac{\dot{y}^2}{2} + V(y).
\]

(5.34)

The strategy is to prove (i) \( E(t) \to 0 \), and (ii) \( E(y, \dot{y}) = 0 \) if and only if \((y, \dot{y}) = (0, 0)\), and (iii) \((y(t), \dot{y}(t)) \to (0, 0)\). We divide the proof into the following steps.
Step 1. From (5.33), $E(t) := E(y(t), \dot{y}(t))$ is a decreasing function along any trajectory $(y(t), \dot{y}(t))$. Further, it has lower bound, namely, $E(y, \dot{y}) \geq 0$. We get $E(t)$ decreases to a limit as $t \to \infty$. Let us call this limit $\alpha$.

Step 2. Let us call the limiting set of $(y(t), \dot{y}(t))$ by $\Omega^+$. That is

$$\Omega^+ = \{(y, \dot{y}) | \exists t_n, t_n \to \infty s.t. (y(t_n), \dot{y}(t_n)) \to (y, \dot{y})\}.$$ 

Such a set is called an $\omega$-limit set. We claim that any trajectory $(\tilde{y}(\cdot), \dot{\tilde{y}}(\cdot))$ with initial data $(\tilde{y}(0), \dot{\tilde{y}}(0)) \in \Omega^+$ lies on $\Omega^+$ forever. The proof of this claim relies on the continuity theorem on the initial data. Namely, the solution of an ODE depends on its initial data continuously. Let us accept this fact. Suppose $(\tilde{y}(0), \dot{\tilde{y}}(0)) \in \Omega^+$, we want to prove that for any fixed $s > 0, (\tilde{y}(s), \dot{\tilde{y}}(s)) \in \Omega^+$. Since $(\tilde{y}(0), \dot{\tilde{y}}(0)) \in \Omega^+$, there exist $t_n \to \infty$ such that $(y(t_n), \dot{y}(t_n)) \to (\tilde{y}(0), \dot{\tilde{y}}(0))$. For large $n$, let us consider two solutions, one has initial data $(y(t_n), \dot{y}(t_n))$, the other has initial data $(\tilde{y}(0), \dot{\tilde{y}}(0))$. The two initial data are very closed. By the continuity dependence of the initial data, we get $(y(t_n + s), \dot{y}(t_n + s))$ is also closed to $(\tilde{y}(s), \dot{\tilde{y}}(s))$. This yields that $(y(t_n + s), \dot{y}(t_n + s)) \to (\tilde{y}(s), \dot{\tilde{y}}(s))$ as $n \to \infty$. Thus, $(\tilde{y}(s), \dot{\tilde{y}}(s)) \in \Omega^+$.

Step 3. We claim that $E(\tilde{y}(s), \dot{\tilde{y}}(s)) = \alpha$ for any $s \geq 0$. This is because $(y(t_n + s), \dot{y}(t_n + s)) \to (\tilde{y}(s), \dot{\tilde{y}}(s))$ as $n \to \infty$ and the corresponding energy $E(y(t_n + s), \dot{y}(t_n + s)) \to \alpha$. This implies

$$\frac{d}{ds} E(\tilde{y}(s), \dot{\tilde{y}}(s)) = 0.$$ 

On the other hand, $\frac{d}{ds} E(\tilde{y}(s), \dot{\tilde{y}}(s)) = -\dot{\tilde{y}}^2(s)$. Hence, we get $\dot{\tilde{y}}(s) \equiv 0$. This again implies $\tilde{y}(s) \equiv \tilde{y}$ for some constant $\tilde{y}$. Thus, $(\tilde{y}, 0)$ is an equilibrium state of the damping oscillation system (5.32). However, the only equilibrium state for (5.32) is $(0, 0)$ because $V$ has a unique minimum and thus the only zero of $F := -V'$ is 0. This implies

$$E(\tilde{y}(s), \dot{\tilde{y}}(s)) = \alpha = 0.$$ 

We conclude that

$$E(y(t), \dot{y}(t)) \to \alpha = 0 \text{ as } t \to \infty.$$ 

Step 4. From step 3,

$$E(y(t), \dot{y}(t)) = \frac{1}{2} \dot{y}(t)^2 + V(y(t)) \to 0 \text{ as } t \to \infty.$$ 

and $V(y) \geq 0$, we get

$$\dot{y}(t) \to 0 \text{ and } V(y(t)) \to 0, \text{ as } t \to \infty.$$ 

Since 0 is the unique minimum of $V$, we get that $V(y) \to 0$ forces $y \to 0$.  

The above method to show global stability is called the Lyapunov method. The energy function $E$ above is called a Lyapunov function. Thus, the effect of damping (dissipation) is a loss of energy.
In the active shock absorber:

\[ m\ddot{y} = -\beta(\dot{y})\dot{y} - ky, \quad \beta(v) = v^4, \]

the equilibrium state is \((0, 0)\). From Lyapunov method, we see that this equilibrium is globally stable.

For the simple pendulum, we see that \( V(\theta) = -g/l \cos \theta \) has infinite many minima: \( \theta = 2n\pi \). The function \( E(y, \dot{y}) \) has local minima \((2n\pi, 0)\). The local minimum \((2n\pi, 0)\) sits inside the basin

\[ B_n = \{ (y, \dot{y}) \mid E(y, \dot{y}) < g/l \}. \]

The equilibrium \((2n\pi, 0)\) is the only minimum of \( E \) in the basin \( B_n \). Suppose a solution starts from a state \((y(0), \dot{y}(0)) \in B_n\), then by using the Lyapunov method, we see that \((y(t), \dot{y}(t)) \rightarrow (2n\pi, 0)\) as \( t \rightarrow \infty \).

What will happen if \( E(0) \geq g/l \) initially? From the loss of energy we have \( E(t) \) will eventually go below \( g/l \). Thus, the trajectory will fall into some basin \( B_n \) for some \( n \) and finally goes to \((2n\pi, 0)\) as \( t \rightarrow \infty \).

**Homeworks**

1. Plot the phase portrait for the damped simple pendulum (5.29).

2. Consider a simple pendulum of length \( l \) with mass \( m \) at one end and the other end is attached to a vibrator. The motion of the vibrator is given by \((x_0(t), y_0(t))\). Let the angle of the pendulum to the verticle axis (in counterclockwise direction) is \( \theta(t) \).
   
   (a) Show that the position of the mass \( m \) at time \( t \) is \( (x(t), y(t)) = (x_0(t) + l \sin \theta(t), y_0(t) - \cos \theta(t)) \).

   (b) Find the velocity and acceleration of \( m \).

   (c) Suppose the mass is in the uniform gravitational field \((0, -mg)\). Use the Newton’s law to derive the equation of motion of \( m \).

   (d) Suppose \((x_0(t), y_0(t))\) is given by \((0, \alpha \sin(\omega_0 t))\). Can you solve this equation?

Chapter 6

Nonlinear systems in two dimensions

6.1 Biological models

6.1.1 Lotka-Volterra system

Predator-prey model

The populations of a predator and prey exhibit interesting periodic phenomenon. A simple example is the fox-rabbit system. Let $R(t)$ be the population of rabbit and $F(t)$ the population of fox. The model proposed by Lotka-Volterra reads

$$\dot{R} = \alpha R - \beta RF$$
$$\dot{F} = -\gamma F + \delta RF.$$

Here,

- $\alpha$ the growth rate of rabbits,
- $\gamma$ death rate of foxes,
- $RF$ the interaction rate of rabbits and foxes
- $\beta RF$ the amount of rabbits being eaten
- $\delta RF$ the amount of foxes increase from eating rabbits

Examples of numerical values of the parameters are: $\alpha = 2$, $\beta = 1.2$, $\gamma = 1$, $\delta = 0.9$.

If we take the environmental constraint into account, the model for the rabbits should be changed to

$$\dot{R} = \alpha R \left(1 - \frac{R}{K}\right) - \beta RF.$$
An epidemic model

Consider the spread of a viral epidemic through an isolated population. Let \( x(t) \) denote the number of susceptible people at time \( t \), \( y(t) \) the number of infected people. The epidemic model reads

\[
\begin{align*}
\dot{x} &= 0.0003x - 0.005xy \\
\dot{y} &= -0.1y + 0.005xy
\end{align*}
\]

The first equation means that the birth rate of susceptible people is 0.0003. Susceptible people are infected through interaction and the infected rate is proportional to \( xy \). The second equation means that the death rate of infected people is 0.1. The infected rate is the same as that in the first equation.

Competition model for two species

Let \( x_1 \) and \( x_2 \) are the populations of two species that compete same resources. The model for each species follows the logistic equation. The competing model reads

\[
\begin{align*}
\dot{x}_1 &= r_1x_1 \left( 1 - \frac{x_1}{K_1} \right) - \alpha_1x_1x_2 \\
\dot{x}_2 &= r_2x_2 \left( 1 - \frac{x_2}{K_2} \right) - \alpha_2x_1x_2
\end{align*}
\]

The quantity \( x_1x_2 \) is the interaction rate. It causes decreasing of population of each species due to competition. These decreasing rates are \( \alpha_1x_1x_2 \) and \( \alpha_2x_1x_2 \), respectively. Here \( \alpha_1 > 0, \alpha_2 > 0 \). As an example, we see two types of snails, the left-curling and the right-curling, compete the same resource. Because they are the same kind of snail, they have the same growth rate and carrying constant. Let us take \( r_1 = r_2 = 1 \) and \( K_1 = K_2 = 1 \). We take \( \alpha_1 = \alpha_2 = a \). We will see later that the structure of the solutions is very different between \( a < 1 \) and \( a > 1 \).

### 6.2 Autonomous systems

We consider general system of the form

\[
\begin{align*}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y)
\end{align*}
\]

We shall study the initial value problem for this system with initial data \((x(t_0), y(t_0)) = (x_0, y_0)\), where \( t_0 \) is the starting time. We may write this problem in vector form

\[
\begin{align*}
\dot{y} &= f(y) \\
y(t_0) &= y_0
\end{align*}
\]

First, we have the standard existence and uniqueness theorems.
Theorem 6.7 If \( f \) is continuously differentiable, then the initial value problem (6.2) and (6.3) has a unique solution for \( t \) in some small interval \( (t_0 - \delta, t_0 + \delta) \).

Notice that the vector field \( f(y) \) we consider here is independent of \( t \) explicitly. Such systems are called autonomous systems. For autonomous systems, we notice the following things.

- It is enough to study the initial value problems with \( t_0 = 0 \). For if \( y(t) \) is the solution with \( y(t_0) = y_0 \), then \( z(t) := y(t - t_0) \) is the solution with \( z(0) = y_0 \), and \( y(\cdot) \) and \( z(\cdot) \) trace the same trajectory on the plane. We call such trajectories the orbits, the \( y \)-plane the phase plane.
- Two orbits cannot intersect on the phase plane. This follows from the uniqueness theorem.
- An orbit cannot end in finite region. This means that it is not possible to find a finite time \( T \) such that (i) \( y(\cdot) \) exists in \( [0, T) \), (ii) \( y(\cdot) \) can not be extended beyond \( T \), and \( \{y(t) | t \in [0, T)\} \) stays in finite region. For the limit \( \lim_{t \to T^-} y(t) \) must exist and the existence theorem allows us to extend the solution beyond \( T \). Therefore, we can only have either \( \lim_{t \to T^-} |y(t)| = \infty \) or \( T = \infty \).

Our goal is to characterize the orbital structure on the phase plane. There are some special orbits which play important roles in the characterization of the whole orbital structure. They are (i) equilibria, (ii) periodic orbits, (iii) equilibria-connecting orbits.

6.3 Equilibria and linearization

Definition 3.1 A state \( \bar{y} \) is called an equilibrium of (6.2) if \( f(\bar{y}) = 0 \).

The constant function \( y(t) \equiv \bar{y} \) is a solution. We want to study the behaviors of solutions of (6.2) which take values near \( \bar{y} \). It is natural to take Taylor expansion of \( y \) about \( \bar{y} \). We have

\[
\dot{y} = f(y) = f(\bar{y}) + \frac{\partial f}{\partial y} (\bar{y}) (y - \bar{y}) + O(|y - \bar{y}|^2).
\]

Let \( u = y - \bar{y} \). Then \( u(t) \) satisfies

\[
\dot{u} = Au + g(u),
\]

where

\[
A := \frac{\partial f}{\partial y} (\bar{y}), \quad g(u) := f(\bar{y} + u) - \frac{\partial f}{\partial y} (\bar{y}) u = O(|u|^2).
\]

System (6.4) is called the linearized equation (or the perturbed equation) of (6.2) about \( \bar{y} \). We have already known the structure of the linear equation

\[
\dot{v} = Av.
\]

Do the orbits of (6.4) and (6.5) look “similar”? 
6.3.1 Hyperbolic equilibria

Before answering the question in the last part of the above subsection, let us first study the following linear perturbation problem. We consider the following system

$$
\dot{v}_1 = A_1 v_1, \tag{6.6}
$$

with $A_1 \sim A$. We ask when do the solutions of (6.6) and (6.5) look similar? The quantitative behaviors of solutions of (6.5) are determined by the eigenvalues of $A$. Namely,

$$
\lambda_1 = \frac{1}{2} \left( T + \sqrt{T^2 - 4D} \right), \lambda_2 = \frac{1}{2} \left( T - \sqrt{T^2 - 4D} \right),
$$

where $T = a + d$ and $D = ad - bc$. It is clear that $\lambda_i$ are continuous in $T$ and $D$, hence in $a, b, c, d$, or hence in $A$. Thus, if we vary $A$ slightly, then the change of $\lambda_i$ is also small on the complex plane.

Now suppose

$$
\text{Re} \lambda_i(A) \neq 0, i = 1, 2. \tag{6.7}
$$

Then this property is still satisfied for those $A_1$ sufficiently closed to $A$. The property (6.7) corresponds to that the zero state is a (spiral) source, a (spiral) sink, or a saddle. We conclude that sink, source and saddle are persistent under small linear perturbation.

Homework.

1. Suppose $\text{Re} \lambda_i(A) \neq 0, i = 1, 2$. Let

$$
A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.
$$

be a perturbation of $A$. Find the condition on $A_1$ so that

$$
\text{Re} \lambda_i(A_1) \neq 0, i = 1, 2.
$$

The same result is still valid for nonlinear perturbation. We use the following example to explain the picture. Consider

$$
\begin{cases}
\dot{x} = r_1 x \\
\dot{y} = r_2 y + \beta x^2.
\end{cases} \tag{6.8}
$$

The solution for $x(t)$ is

$$
x(t) = x_0 e^{r_1 t}. \tag{6.9}
$$

Let us assume $r_2 \neq 2r_1$ for simplicity. Then the general solution for $y(t)$ is

$$
y(t) = Ae^{r_2 t} + Be^{2r_1 t}.
$$

We plug this into the $y$-equation and obtain

$$
Ar_2 e^{r_2 t} + 2r_1 B e^{2r_1 t} = r_2 (Ae^{r_2 t} + Be^{2r_1 t}) + \beta x_0^2 e^{2r_1 t}.
$$
This yields

\[ 2r_1 B = r_2 B + \beta x_0^2. \]

Thus, general solutions of \( y(t) \) reads

\[ y(t) = Ae^{r_2 t} + \frac{\beta x_0^2}{2r_1 - r_2} e^{r_1 t}. \]  \hspace{1cm} (6.10)

We see that the asymptotical behavior of \( (x(t), y(t)) \) is

- When \( r_1 < 0 \) and \( r_2 < 0 \), then \( (x(t), y(t)) \to (0, 0) \) as \( t \to \infty \). We call \( (0, 0) \) a sink.

**Remark.** In the case \( r_1 < 0 \) and \( r_2 < 0 \), even in the resonant case, i.e. \( 2r_1 = r_2 \), we still have \( y(t) \to 0 \) as \( t \to \infty \).

- When \( r_1 > 0 \) and \( r_2 > 0 \), then \( (x(t), y(t)) \to (0, 0) \) as \( t \to -\infty \). We call \( (0, 0) \) a source.

- When \( r_1 > 0 \) and \( r_2 < 0 \), we have two subcases:
  - when \( x_0 = 0 \), then \( (x(t), y(t)) \to (0, 0) \) as \( t \to \infty \),
  - when \( A = 0 \), then \( (x(t), y(t)) \to (0, 0) \) as \( t \to -\infty \),

The orbit with \( x_0 = 0 \) is called a stable manifold passing \( (0, 0) \), while the orbit with \( A = 0 \) a unstable manifold. We denote the former one by \( M_s \) and the latter one by \( M_u \). We call \( (0, 0) \) a saddle point. By eliminate \( t \) from (6.9) and (6.10), we can obtain the equations of \( M_s \) and \( M_u \) as the follows.

\[
\begin{align*}
M_s & : x = 0, \\
M_u & : y = \frac{\beta}{2r_1 - r_2} x^2.
\end{align*}
\]

- When \( r_1 < 0 \) and \( r_2 > 0 \), \( (0, 0) \) is a saddle point. The stable and unstable manifolds are

\[
\begin{align*}
M_u & : x = 0, \\
M_s & : y = \frac{\beta}{2r_1 - r_2} x^2.
\end{align*}
\]

Let us go back to the general formulation (6.2). We have the following definitions.

**Definition 3.2** An equilibrium \( \bar{y} \) of (6.2) is called hyperbolic if all eigenvalues of the variation matrix \( A := \frac{\partial F}{\partial y(\bar{y})} \) have only nonzero real parts.

**Definition 3.3** An equilibrium \( \bar{y} \) of (6.2) is called

- a sink if \( y(t) \to \bar{y} \) as \( t \to \infty \),
• a source if \( y(t) \to \bar{y} \) as \( t \to -\infty \),

where \( y(t) \) is any solution of (6.2) with \( y(0) \sim \bar{y} \).

**Definition 3.4**

1. A curve \( M_s(\bar{y}) \) is called a stable manifold passing through the equilibrium \( \bar{y} \) if \( y(t) \to \bar{y} \) as \( t \to \infty \) for any solution \( y(t) \) with \( y(0) \in M_s(\bar{y}) \).

2. A curve \( M_u(\bar{y}) \) is called an unstable manifold passing through the equilibrium \( \bar{y} \) if \( y(t) \to \bar{y} \) as \( t \to -\infty \) for any solution \( y(t) \) with \( y(0) \in M_u(\bar{y}) \).

3. An equilibrium \( \bar{y} \) which is the intersection of a stable manifold and an unstable manifold is called a saddle point.

**Theorem 6.8** Consider the autonomous system (6.2) and its linearization (6.5) about an equilibrium. Suppose \( \bar{y} \) is hyperbolic. Then

\[
\begin{align*}
\bar{y} \text{ is a } & \begin{cases} 
\text{source} \\
\text{sink} \\
\text{saddle}
\end{cases} & \text{of the nonlinear equation} \\
\text{if and only if} & \\
\bar{0} \text{ is a } & \begin{cases} 
\text{source} \\
\text{sink} \\
\text{saddle}
\end{cases} & \text{of the linearized equation}
\end{align*}
\]

In other word, hyperbolicity is persistent under small perturbation.

**Remark.** If an equilibrium \( \bar{y} \) is not hyperbolic, then the perturbation can break the local orbital structure. Let us see the following example. Consider

\[
\begin{align*}
\dot{x} &= y + \gamma \frac{x^2+y^2}{2} x \\
\dot{y} &= -x + \gamma \frac{x^2+y^2}{2} y
\end{align*}
\]

When \( \gamma = 0 \), the orbits are circles with center at the origin. To see the effect of perturbation, we multiply the first equation by \( x \) and the second equation by \( y \) then add them together. We obtain

\[
\dot{\rho} = \gamma \rho^2
\]

where \( \rho = x^2 + y^2 \). The solution of \( \rho(t) \) is

\[
\rho(t) = \frac{1}{\rho(0)^{-1} - \gamma t}
\]

When \( \gamma < 0 \), the solution tends to 0 as \( t \to \infty \). When \( \gamma > 0 \), the solution tends to zero as \( t \to -\infty \). Moreover, the solution \( \rho(t) \to \infty \) as \( t \to \rho(0)^{-1}/\gamma \). Thus, the center becomes a sink if \( \gamma < 0 \) and a source when \( \gamma > 0 \).
6.3.2 The equilibria in the competition model

The two-species competition model reads

\[
\begin{align*}
\dot{x}_1 &= r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 \\
\dot{x}_2 &= r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) - \alpha_2 x_1 x_2
\end{align*}
\]

The null line of the vector field in the x-direction are

\[
\begin{align*}
&\left(1 - \frac{x_1}{K_1} - \frac{x_2}{L_1}\right) = 0, \\
&L_1 = \frac{r_1}{\alpha_1}.
\end{align*}
\]

This yields

\[
\begin{align*}
x_1 &= 0, \quad 1 - \frac{x_1}{K_1} - \frac{x_2}{L_1} = 0.
\end{align*}
\]

They are called the x-nullclines. Similarly, the y-nullclines are

\[
\begin{align*}
x_2 &= 0, \quad 1 - \frac{x_2}{K_2} - \frac{x_1}{L_2} = 0,
\end{align*}
\]

where \(L_2 = \frac{r_2}{\alpha_2}\).

The quantity \(L_1 = \frac{r_1}{\alpha_1}\) measures the “competitive capacity” of species 1. The quantity \(L_1\) is large means that \(r_1\) is large (species 1 has large growth rate) or \(\alpha_1\) is small (it is less sensitive to competition). Let us define

\[
\begin{align*}
s_1 &= \frac{L_1}{K_2}, \quad s_2 = \frac{L_2}{K_1}.
\end{align*}
\]

The quantity \(s_1\) measures the competitive ratio of species 1 relative to the maximal population of species 2. \(s_1 > 1\) means that species 1 is more competitive relative to the maximal population of species 2.

The intersection of a x-nullcline and a y-nullcline is an equilibrium. We are only interested in those equilibria in the first quadrant because \(x_i\) is the population of the \(i\) species. There are four cases.

- Case 1: \(s_1 > 1\) and \(s_2 < 1\) (species 1 is more competitive)
- Case 2: \(s_1 < 1\) and \(s_2 > 1\) (species 2 is more competitive)
- Case 3: \(s_1 < 1\) and \(s_2 < 1\) (both species are not competitive)
- Case 4: \(s_1 > 1\) and \(s_2 > 1\) (both species are competitive)
In the first two cases, there are three equilibria in the first quadrant: \( E_0 = (0, 0), E_1 = (K_1, 0) \) and \( E_2 = (0, K_2) \). In the last two cases, there are four equilibria: \( E_0 = (0, 0), E_1 = (K_1, 0), E_2 = (0, K_2) \) and \( E^* = (x_1^*, x_2^*) \), where

\[
\begin{align*}
  x_1^* &= \frac{\frac{1}{K_2} - \frac{1}{L_1}}{\frac{1}{K_1 K_2} - \frac{1}{L_1 L_2}} = \frac{L_2(s_1 - 1)}{s_1 s_2 - 1}, \\
  x_2^* &= \frac{\frac{1}{K_1} + \frac{1}{L_2}}{\frac{1}{K_1 K_2} - \frac{1}{L_1 L_2}} = \frac{L_1(s_2 - 1)}{s_1 s_2 - 1}.
\end{align*}
\]

The variation matrix at \((x_1, x_2)\) reads

\[
\frac{\partial f}{\partial x}(x_1, x_2) = \begin{pmatrix} r_1 \left(1 - \frac{2 r_1}{K_1} - \frac{x_2}{L_1}\right) & -\frac{r_1 x_1}{L_1} \\ -\frac{r_2 x_2}{L_2} & r_2 \left(1 - \frac{2 r_2}{K_2} - \frac{x_1}{L_2}\right) \end{pmatrix}.
\]

We get

\[
\frac{\partial f}{\partial x}(0, 0) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad \frac{\partial f}{\partial x}(K_1, 0) = \begin{pmatrix} -r_1 & -\frac{K_1 r_1}{s_1} \\ 0 & r_2 \left(1 - \frac{s_1}{s_2}\right) \end{pmatrix}.
\]

In all cases, \( E_0 \) is an unstable node.

After some computation, we can draw the following conclusion.

**Theorem 6.9** In the two-species competition model, the equilibria and their stability are the follows.

- **Case 1:** \( s_1 > 1 \) and \( s_2 < 1 \): \( E_1 \) is a stable node. \( E_2 \) is unstable.
- **Case 2:** \( s_1 < 1 \) and \( s_2 > 1 \): \( E_2 \) is a stable node. \( E_1 \) is unstable.
- **Case 3:** \( s_1 < 1 \) and \( s_2 < 1 \): \( E_1 \) and \( E_2 \) are stable and \( E^* \) is a saddle.
- **Case 4:** \( s_1 > 1 \) and \( s_2 > 1 \): both \( E_1 \) and \( E_2 \) are saddle and \( E^* \) is a stable node.

Ecologically, this theorem says that co-existence of two competing species can occur only when both are competitive.

In the case of the competitive model for the left curling snails and right curling snails, both have the same parameters \( r, K \) and \( \alpha \). Thus, both have the same competitive ratio:

\[
s = \frac{r}{\alpha K}.
\]

If \( s > 1 \), both would be competitive and they would co-exist. But this is not the case we have found. Instead, we find only one kind exists now in nature. To give an explanation, we notice that the term \(-r/K x_1^2\) represents the self competition, while the term \(-\alpha x_1 x_2\) the cross competition. We should expect that these two competition terms are about the same magnitude. That is, \( r/K \sim \alpha \). In this case, \( s \sim 1 \). If the cross competition is slightly stronger than the self competition, we would have \( s < 1 \). This would yield that only one species can survive in long time.

**Ref.** Clifford Henry Taubes, Modeling Differential Equations in Biology, pp. 23, pp. 73, pp. 81.
Homeworks.
1. Compute the eigenvalues of the variation matrix at $E_1$ and $E_2$.
2. Compute the variation matrix at $(x^*,y^*)$ and its eigenvalues.
3. Justify the statements of this theorem.

6.4 Phase plane analysis

In this section, we shall use Maple to plot the vector field and to find orbits which connect nodes.

Include packages we type

```maple
> with(DEtools):
> with(plots):
```

Define the vector field $(f,g)$ for the competition model

```maple
> f := r[1]*x(t)*(1-x(t)/K[1])-alpha[1]*x(t)*y(t);  
> g := r[2]*y(t)*(1-y(t)/K[2])-alpha[2]*x(t)*y(t);
```

Define the following quantities.

```maple
> L[1] := r[1]/alpha[1]:
> s[1] := L[1]/K[2]:
```

The equilibria are those states where $(f,g) = (0,0)$. They are

```maple
E_0 = (0,0), E_1 = (K_1,0), E_2 = (0,K_2), E* = (xs,ys), where (xs,ys) are given by
> xs := L[2]*(s[1]-1)/(s[1]*s[2]-1):
> ys := L[1]*(s[2]-1)/(s[1]*s[2]-1):
```

We have four cases: Case 1: $s_1 > 1, s_2 < 1$:

```maple
> Case1 := {
    > r[1] = 3, K[1] = 1, alpha[1] = 1,
> evalf(subs(Case1, [s[1], s[2]]), 3);
```

Plot the the curves where $(f,g) = (0,0)$:
> fig1 :=
> implicitplot( 
> subs(Case1, x(t)=x1, y(t)=x2, f=0),
> subs(Case1, x(t)=x1, y(t)=x2, g=0),
> x1=-0.2..1.5, x2=-0.2..3,
> grid=[100,100], color=navy):
> display(fig1, axes=boxed);

> f1 := subs(Case1, x(t)=x1, y(t)=x2, f):
> g1 := subs(Case1, x(t)=x1, y(t)=x2, g):
> vsign := piecewise(
> 2, f1 > 0 and g1 > 0,
> -2, f1 > 0 and g1 < 0,
> -1, f1 < 0 and g1 < 0,
> 1, f1 < 0 and g1 > 0);
> plot3d(vsign, x1=-0.2..1.5, x2=-0.2..3, axes=frame, grid=[100,100],
> orientation =[-90,0], style=HIDDEN, shading=ZHUE);

\[
\begin{align*}
\text{vsig} & := \\
& \begin{cases} 
-2 & -3x_1(1-x_1) + x_1 x_2 < 0 \text{ and } -2x_2\left(1 - \frac{x_2^2}{2}\right) + 4 x_1 x_2 < 0 \\
-1 & -3x_1(1-x_1) + x_1 x_2 < 0 \text{ and } 2x_2\left(1 - \frac{x_2^2}{2}\right) - 4 x_1 x_2 < 0 \\
1 & 3x_1(1-x_1) - x_1 x_2 < 0 \text{ and } 2x_2\left(1 - \frac{x_2^2}{2}\right) - 4 x_1 x_2 < 0 \\
2 & 3x_1(1-x_1) - x_1 x_2 < 0 \text{ and } -2x_2\left(1 - \frac{x_2^2}{2}\right) + 4 x_1 x_2 < 0 \end{cases}
\end{align*}
\]
6.4. PHASE PLANE ANALYSIS

Plot the vector field \((f, g)\) for case 1:

```maple
> fig2 := DEplot( subs(Case1,
> [diff(x(t),t)=f,diff(y(t),t)=g]),[x(t),y(t)], t=0..20,
> x=-0.2..1.5, y=-0.2..3,
> arrows=small, title='Vector field',
> color=subs(Case1,[f/sqrt(f^2+g^2),g/sqrt(f^2+g^2),0.1]):
> display({fig1,fig2}, axes=boxed);
```

Find the separametrices. You need to try to find a proper initial data such that it generates a separametrix.
> fig3 := DEplot( subs(Case1, 
> [diff(x(t),t)=f,diff(y(t),t)=g]),[x(t),y(t)],t=0..20, 
> [[x(0)=0.01,y(0)=3]],stepsize=0.05,x=-0.2..1.5,y=- 0.2..3, 
> color=cyan,arrows=LARGE,dirgrid=[10,10],linecolor=red): 
> fig4 := DEplot( subs(Case1, 
> [diff(x(t),t)=f,diff(y(t),t)=g]),[x(t),y(t)], t=0..20, 
> [[x(0)=-0.01,y(0)=3]],stepsize=0.05,x=-0.2..1.5,y= -0.2..3, 
> color=cyan,arrows=LARGE,dirgrid=[10,10],linecolor=blue): 
> fig5 := DEplot( subs(Case1, 
> [diff(x(t),t)=f,diff(y(t),t)=g]),[x(t),y(t)], t=0..20, 
> [[x(0)=0.001,y(0)=1]],stepsize=0.05,x=-0.2..1.5,y=-0.2..3, 
> color=cyan,arrows=LARGE,dirgrid=[10,10],linecolor=orange): 
> fig6 := DEplot( subs(Case1, 
> [diff(x(t),t)=f,diff(y(t),t)=g]),[x(t),y(t)], t=0..20, 
> [[x(0)=-0.001,y(0)=1]],stepsize=0.05,x=-0.2..1.5,y=-0.2..3, 
> color=cyan,arrows=LARGE,dirgrid=[10,10],linecolor=black): 
> display({fig1,fig3,fig4,fig5,fig6},axes=boxed);

Homeworks.

1. B-D: pp. 525, 8, 9

2. B-D: pp. 527, 17

3. Plot phase portraits for the four cases in the competitive model in the last subsection.
6.5 Hamiltonian systems

6.5.1 Examples

A Hamiltonian system is a first order system of the form

\[
\begin{align*}
\dot{x} &= H_p(x, p) \\
\dot{p} &= -H_x(x, p)
\end{align*}
\] (6.11)

where \( H : \mathbb{R}^2 \to \mathbb{R} \) is called the Hamiltonian of the system.

**Example 1.** The equation of motion in Newton’s mechanics with a conservative force field is

\[ m\ddot{x} = -V'(x) \]

where \( V \) is the potential. Define the momentum \( p = mv \) and the total energy

\[ H(x, p) = \frac{p^2}{2m} + V(x), \]

Then the Newton’s mechanics can be repressed in the form of Hamilton’s mechanics (6.11).

**Example 2.** Relativistic particle with rest mass \( m \). The Hamiltonian is given by

\[ H = \sqrt{p^2c^2 + m^2c^4} + V(x) \]

**Example 3.** The motion of a charge particle in an electromagnetic field.

\[ m\ddot{x} = -e \nabla \phi + \frac{e}{c} \dot{x} \wedge B \]

where \((\phi, A)\) is the vector potential and \( B = \nabla \wedge A \). In Hamiltonian form:

\[ p = L_v = m \dot{x} + \frac{e}{c} A \]

and

\[ H(x, p) = \frac{1}{2m} |p - \frac{e}{c} A(x)|^2 + e\phi. \]

**Example 4.** In fluid mechanics, an elementary two-dimensional flow is so called the potential flow, which is steady (time independent), incompressible (constant density), inviscid (no viscosity) and irrotational. It is characterized by the velocity field \((u(x, y), v(x, y))\), which satisfies

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \quad \text{incompressibility} \\
\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 0, \quad \text{irrotational}
\end{align*}
\]

The first equation is called the divergence free condition for \((u, v)\). It yields that there exists a function called stream function \( \psi(x, y) \) such that

\[ u(x, y) = \psi_y(x, y), \quad v(x, y) = -\psi_x(x, y). \]

Indeed, from this divergence free condition, we can define the stream function \( \psi(x, y) \) by the line integral:

\[ \psi(x, y) = \int_{(x_0, y_0)}^{(x, y)} (-v(x, y) dx + u(x, y) dy) \]
The starting point is unimportant. We can choose any point as our starting point. The corresponding \( \psi \) is defined up to a constant. By the divergence theorem, the integral is independent of path in a simply connected domain. Hence, \( \psi \) is well-defined on simply connected domain. You can check that \( \psi_y = u \) and \( \psi_x = -v \). If the domain is not simply connected, the steam function may be a multiple valued function. We shall not study this case now.

The particle trajectory is governed by

\[
\begin{align*}
\dot{x} &= u(x, y) = \psi_y(x, y) \\
\dot{y} &= v(x, y) = -\psi_x(x, y)
\end{align*}
\]

which is a Hamiltonian flow with Hamiltonian \( \psi(x, y) \). The second equation for the velocity field yields that

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.
\]

Such a function is called a harmonic function. The theory of potential flow can be analyzed by complex analysis. You can learn this from textbooks of complex variable or elementary fluid mechanics.

Here are two examples for the potential flow: (1) \( \psi = \text{Im}(z^2) \), (2) \( \psi(z) = \text{Im}(z + 1/z) \), where \( z = x + iy \) and \( \text{Im} \) is the imaginary part. The first one represents a jet. The second is a flow passing a circle (or cylinder if you view in three dimensions).

**Example 5.** The magnetic field \( \mathbf{B} \) satisfies \( \text{div} \mathbf{B} = 0 \). For two-dimensional steady magnetic field \( \mathbf{B} = (u, v) \), this reads

\[
u_x + v_y = 0.
\]

The magnetic field lines are the curves which are tangent to \( \mathbf{B} \) at every points on this line. That is, it satisfies

\[
\begin{align*}
\dot{x} &= u(x, y) = \psi_y(x, y) \\
\dot{y} &= v(x, y) = -\psi_x(x, y)
\end{align*}
\]

where \( \psi \) is the stream function corresponding to the divergent free field \( \mathbf{B} \).

**Example 6.** Linear Hamiltonian flow. If we consider

\[
H(x, y) = \frac{ax^2}{2} + bxy + \frac{cy^2}{2}
\]

the corresponding Hamiltonian system is

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
b & c \\
-a & -b
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\]

(6.12)

### 6.5.2 Orbits and level sets of Hamiltonian

A conservative quantity is a function \( \phi(x, y) \) which remains unchanged along trajectories. That is,

\[
\frac{d}{dt} \phi(x(t), y(t)) = 0.
\]
A conservative quantity we can immediately get is the Hamiltonian itself. That is, along any trajectory \((x(t), y(t))\) of (6.11), we have

\[
\frac{d}{dt} H(x(t), y(t)) = H_x \dot{x} + H_y \dot{y} = H_x \dot{x} + H_y (-H_x) = 0.
\]

In two dimension, the orbits of a Hamiltonian system in the phase plane are the level sets of its Hamiltonian.

### 6.5.3 Equilibria of a Hamiltonian system

**Definition 5.5** A critical point \(\bar{x}, \bar{y}\) of \(H\) is said to be non-degenerate if the hessian of \(H\) at \(\bar{x}, \bar{y}\) (i.e. the matrix \(d^2 H(\bar{x}, \bar{y})\)) is non-singular.

Since \(H\) is usually convex in \(y\) variable in mechanical problems, we further assume that \(H_{yy} > 0\) at the equilibrium. Notice that this assumption eliminates the possibility of any local maximum of \(H\).

The Jacobian of the linearized system of (6.11) at an equilibrium \((\bar{x}, \bar{y})\) has the form

\[
A = \begin{pmatrix}
H_{yx} & H_{yy} \\
-H_{xx} & -H_{xy}
\end{pmatrix}
\]

Since the trace part \(T\) of \(A\) is zero, its eigenvalues are

\[
\lambda_i = \pm \frac{1}{2} \sqrt{H_{yx}^2 - H_{xx} H_{yy}(\bar{x}, \bar{y})}, \quad i = 1, 2.
\]

On the other hand, suppose \((\bar{x}, \bar{y})\) is a local minimum of \(H\). This is equivalent to \(H_{xx} H_{yy} - H_{xy}^2 > 0\) at \((\bar{x}, \bar{y})\) (recall that we have assumed \(H_{yy} > 0\) and non-degeneracy). Hence, we have pure imaginary eigenvalues \(\lambda_i\) and the equilibrium is a center. Similarly, \(H_{xx} H_{pp} - H_{xp}^2 > 0\) is equivalent to \((\bar{x}, \bar{p})\) being a saddle of \(H\). And it is also equivalent to that two eigenvalues are real and with opposite signs. Hence the equilibrium is a saddle.

We summarize it by the following theorem.

**Theorem 6.10** Assuming that \((\bar{x}, \bar{y})\) is a non-degenerate critical point of a Hamiltonian \(H\) and assuming \(H_{yy}(\bar{x}, \bar{y}) > 0\). Then

1. \((\bar{x}, \bar{y})\) is a local minimum of \(H\) iff \((\bar{x}, \bar{y})\) is a center of the corresponding Hamiltonian flow.
2. \((\bar{x}, \bar{y})\) is a saddle of \(H\), iff \((\bar{x}, \bar{y})\) is a saddle.

The examples we have seen are

1. Simple pendulum: \(H(x, p) = \frac{1}{2} p^2 - \frac{g}{l} \cos x\).
2. Duffing oscillator: \(H(x, p) = \frac{1}{2} p^2 - \frac{\delta}{2} x^2 + \frac{x^4}{4}\).
3. Cubic potential: \(H(x, p) = \frac{1}{2} (p^2 - x^2 + x^3)\).
In the case of simple pendulum, \((2n\pi, 0)\) are the centers, whereas \((2(n + 1)\pi, 0)\) are the saddles. In the case of Duffing oscillator, \((\pm \sqrt{\delta}, 0)\) are the centers, while \((0, 0)\) is the saddle. In the last example, the Hamiltonian system reads

\[
\begin{aligned}
\dot{x} &= p \\
\dot{p} &= x - \frac{3}{2}x^2.
\end{aligned}
\] (6.13)

The equilibrium occurs at \((0, 0)\) and \((3/2, 0)\), where the right-hand sides of the ODE are 0's.

Below, we use Maple to plot the contour curves the Hamiltonian. These contour curves are the orbits.

```maple
with(DEtools):
with(plots):
E := y^2/2 + x^3/3 - delta*x^2/2;
Plot the level set for the energy. Due to conservation of energy, these level sets are the orbits.
```

6.5.4 Stability and asymptotic stability

**Definition 5.6** An equilibrium \(\vec{y}\) of the ODE \(\dot{y} = f(y)\) is said to be stable if for any \(\epsilon > 0\), there exists a \(\delta > 0\) such that for any solution \(y(\cdot)\) with \(|y(0) - \vec{y}| < \delta\), we have \(|y(t) - \vec{y}| < \epsilon\).

**Definition 5.7** An equilibrium \(\vec{y}\) of the ODE \(\dot{y} = f(y)\) is said to be asymptotically stable if there exists a \(\delta > 0\) such that any solution \(y(\cdot)\) with \(|y(0) - \vec{y}| < \delta\) satisfies \(y(t) \to \vec{y}\) as \(t \to \infty\).
HAMILTONIAN SYSTEMS

Remarks.

• It is clear that asymptotical stability implies stability.

• For linear systems, centers are stable whereas sinks and spiral sinks are asymptotically stable.

• For Hamiltonian system, the minimum of a Hamiltonian $H$ is a stable center.

Now we shall perturb a Hamiltonian system in a special way, called dissipative perturbation. In this case, the center becomes an asymptotical stable equilibrium. We shall use the following example for demonstration.

Consider the Hamiltonian $H(x, p) = \frac{p^2}{2} + V(x)$. Let us assume

- $\lim_{|x| \to \infty} V(x) = \infty$
- $0$ is the unique minimum of $V$ with $V(0) = 0$.

Let us perturb this mechanical system by some damping. This means that we exerted a friction force $b(p)$, where $p = \dot{x}$ is the velocity. This term is a friction if $b(0) = 0$, $b(p)p < 0$.

The latter simply means that the force is in the opposite direction of the velocity. Thus, the system reads

$$
\begin{align*}
\dot{x} &= p \\
\dot{p} &= -V'(x) + b(p)
\end{align*}
$$

You can check that along any trajectory $(x(t), v(t))$,

$$
\frac{d}{dt} (H(x(t), p(t)) = H_x \dot{x} + H_p \dot{p} = -V'(x)p + p(-V'(x) - b) = bp < 0.
$$

Thus, the Hamiltonian $h$ decreases along any trajectories until $p = 0$. Such a perturbation is called a dissipative perturbation. As a result, we can see that $(0, 0)$ becomes asymptotic stable. Indeed, we shall show in the section of Liapunov function that $(x(t), p(t)) \to (0, 0)$ as $t \to \infty$ for any trajectories. Here, we just show that, from the linear analysis, the center becomes a spiral sink for a Hamiltonian system with dissipative perturbation. We shall assume $b'(0) \neq 0$. The variational matrix at $(0, 0)$ is

$$
\begin{pmatrix}
0 & 1 \\
-H_{xx} & -H_{xp} + b'(0)
\end{pmatrix} = 
\begin{pmatrix}
0 & 1 \\
-V''(0) & -b'(0)
\end{pmatrix}
$$

Its eigenvalues are

$$
\lambda_{\pm} = b'(0) \pm i\sqrt{V''(0)}
$$

Now, the force $b(p)$ is a friction which means that $b(p)p < 0$. But $b(0) = 0$. We get that

$$
0 > b(p)p \sim b'(0)p \cdot p
$$

Thus, if $b'(0) \neq 0$, then $b'(0) < 0$. Hence, $(0, 0)$ becomes a spiral sink.
6.5.5 Gradient Flows

In many applications, we look for a strategy to find a minimum of a Hamiltonian system. This minimal state is called the ground state. One efficient way is to start from any state then follow the negative gradient direction of the Hamiltonian. Such a method is called the steepest descent method. The corresponding flow is called a (negative) gradient flow. To be precise, let us consider a Hamiltonian \( \psi(x, y) \). We consider the ODE system:

\[
\begin{align*}
\dot{x} &= -\psi_x(x, y) \\
\dot{y} &= -\psi_y(x, y)
\end{align*}
\] (6.14)

Along any of such a flow \((x(t), y(t))\), we have

\[
\frac{d\psi}{dt}(x(t), y(t)) = \psi_x \dot{x} + \psi_y \dot{y} = -\left(\psi_x^2 + \psi_y^2\right) < 0,
\]

unless the flow reaches a minimum of \(\psi\).

The gradient flow of \(\psi\) is always orthogonal to the Hamiltonian flow of \(\psi\). For if

\[
\begin{align*}
\dot{x} &= \psi_y(x, y) \\
\dot{y} &= -\psi_x(x, y)
\end{align*}
\]

then

\[
\dot{x}(t) \cdot \dot{\xi}(t) + \dot{y}(t) \cdot \dot{\eta}(t) = 0.
\]

Thus, the two flows are orthogonal to each other. We have seen that \(\psi\) is an integral of the Hamiltonian. If \(\phi\) is an integral of the gradient flow (6.14). That is, the gradient flows are the level sets of \(\phi\). We recall that the level sets of \(\psi\) are the orbits of the Hamiltonian flows. We conclude that the level sets of \(\psi\) and \(\phi\) are orthogonal to each other.

**Example 1.** Let \(\psi = (x^2 - y^2)/2\). Then the gradient flow satisfies

\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= +y
\end{align*}
\]

Its solutions are given by \(x = x_0 e^{-t}\) and \(y = y_0 e^t\). We can eliminate \(t\) to obtain that the function \(\phi(x, y) := 2xy\) is an integral. If we view these functions on the complex plane: \(z = x + iy\), we see that \(\psi(z) + i\phi(z) = z^2\).

**Example 2.** Let \(\psi(x, y) = (x^2 + y^2)/2\). The gradient flows are given by

\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= -y
\end{align*}
\]

Its solutions are given by \(x = x_0 e^{-t}\) and \(y = y_0 e^{-t}\). An integral is \(\phi = \tan^{-1}(y/x)\). On the other hand, the Hamiltonian flow is given by

\[
\begin{align*}
\dot{x} &= \psi_y = y \\
\dot{y} &= -\psi_x = -x
\end{align*}
\]

Its solutions are given by \(x = A\sin(t + t_0), y = A\cos(t + t_0)\). The integral is \(\psi = (x^2 + y^2)/2\). In fact, \(\frac{1}{2}\ln(x^2 + y^2)\) is also an integral of the Hamiltonian flow.
Example 3. In general, the hamiltonian
\[ \psi(x, y) = \frac{ax^2}{2} + bxy + \frac{cy^2}{2} \]
the corresponding Hamiltonian system is
\[ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
The gradient flow is
\[ \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

Example 4. Let
\[ \psi(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}. \]
The gradient flow is
\[ \begin{cases} \dot{x} = \psi_x = x - x^3 \\ \dot{y} = \psi_y = -y \end{cases} \]
The trajectory satisfies
\[ \frac{dy}{dx} = \frac{\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{y}{-x + x^3}. \]
By the separation of variable
\[ \frac{dy}{y} = \frac{dx}{-x + x^3}, \]
we get
\[ \ln y = \int \frac{dx}{-x + x^3} = -\ln|x| + \frac{1}{2} \ln|1 - x| + \frac{1}{2} \ln|1 + x| + C. \]
Hence, the solutions are given by
\[ \phi(x, y) := \frac{x^2y^2}{1 - x^2} = C_1. \]

Remarks.

• We notice that if \( \psi \) is an integral of an ODE system, so is the composition function \( h(\psi(x, y)) \) for any function \( h \). This is because
\[ \frac{d}{dt} h(\psi(x(t), y(t))) = h'(\psi) \frac{d}{dt} \psi(x(t), y(t)) = 0. \]

• If \( (0, 0) \) is the center of \( \psi \), then \( (0, 0) \) is a sink of the corresponding gradient flow.

• If \( (0, 0) \) is a saddle of \( \psi \), it is also a saddle of \( \phi \).

The properties of a gradient system are shown the the next theorem.
Chapter 6. Nonlinear Systems in Two Dimensions

Theorem 6.11 Consider the gradient system

\[
\begin{align*}
\dot{x} &= -\psi_x(x, y) \\
\dot{y} &= -\psi_y(x, y)
\end{align*}
\]

Assume that the critical points of \(\psi\) are isolated and non-degenerate. Then the system has the following properties.

- The equilibrium is either a source, a sink, or a saddle. It is impossible to have spiral structure.
- If \((\bar{x}, \bar{y})\) is an isolated minimum of \(\psi\), then \((\bar{x}, \bar{y})\) is a sink.
- If \((\bar{x}, \bar{y})\) is an isolated maximum of \(\psi\), then \((\bar{x}, \bar{y})\) is a source.
- If \((\bar{x}, \bar{y})\) is an isolated saddle of \(\psi\), then \((\bar{x}, \bar{y})\) is a saddle.

To show these, we see that the Jacobian of the linearized equation at \((\bar{x}, \bar{y})\) is the Hessian of the function \(\psi\) at \((\bar{x}, \bar{y})\): is

\[
- \begin{pmatrix}
\psi_{xx} & \psi_{xy} \\
\psi_{xy} & \psi_{yy}
\end{pmatrix}
\]

Its eigenvalues \(\lambda_i, i = 1, 2\) are

\[
-\frac{1}{2} \left( T \pm \sqrt{T^2 - 4D} \right),
\]

where \(T = \psi_{xx} + \psi_{yy}, D = \psi_{xx}\psi_{yy} - \psi_{xy}^2\). From

\[
T^2 - 4D = (\psi_{xx} - \psi_{yy})^2 + 4\psi_{xy}^2 \geq 0
\]

we have that the imaginary part of the eigenvalues \(\lambda_i\) are 0. Hence the equilibrium can only be a sink, a source or a saddle.

Recall from Calculus that whether the critical point \((\bar{x}, \bar{y})\) of \(\psi\) is a local maximum, a local minimum, or a saddle, is completely determined by \(\lambda_1, \lambda_2 < 0\), \(\lambda_1, \lambda_2 > 0\), or \(\lambda_1\lambda_2 < 0\), respectively. On the other hand, whether the equilibrium \((\bar{x}, \bar{y})\) of (6.14) is a source, a sink, or a saddle, is also completely determined by the same conditions.

Remarks. The integral \(\phi\) of a gradient system \(\psi\) can also be a Hamiltonian and generates a Hamiltonian flow. Suppose \((\bar{x}, \bar{y})\) is a sink of such a gradient flow of \(\psi\). In the mean time this gradient flow can be viewed as a Hamiltonian flow of \(\phi\). We then would get that \((\bar{x}, \bar{y})\) is a sink of a Hamiltonian flow. This contradicts to the theorem in the last section which says that the non-degenerate equilibrium of a Hamiltonian system can only be saddles or centers. What’s wrong? The problem is that \(\phi\) is no longer a smooth function at \((\bar{x}, \bar{y})\) if the latter is a sink or source of a gradient flow of \(\psi\). The theorem of the last section can not be applied for this case.
6.5. HAMILTONIAN SYSTEMS

Homeworks.

1. Consider a linear ODE

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

(a) Show that the system is a hamiltonian system if and only if \( a + d = 0 \). Find the corresponding hamiltonian.

(b) Show that the system is a gradient system if and only if \( b = c \), i.e. the matrix is symmetric.

6.5.6 Homoclinic orbits

The orbit which starts from an saddle and ends at the same saddle is called a homoclinic orbit. It plays important role in the chaos theory. Below, we shall find the homoclinic orbit for (6.13). This orbit is given by

\[
H(x, p) = \frac{1}{2} (p^2 - x^2 + x^3) = 0.
\]

Since \((0, 0)\) is a saddle, this homoclinic orbit \((x(t), p(t))\) of (6.13) satisfies

\[
x(\pm \infty) = 0, p(\pm \infty) = 0.
\]

Using \( p = \dot{x} \) and separation of variable, we have

\[
\dot{x} = \pm \sqrt{x^2 - x^3}
\]

\[
\int \frac{dx}{x \sqrt{1-x}} = \pm (t + C)
\]

Since the system is autonomous, we may normalize \( C = 0 \). For plus sign, we use the substitution \( u = \sqrt{1-x} \), for minus, we use \( u = -\sqrt{1-x} \). We get

\[
\int \frac{2u \, du}{(1-u^2)u} = t
\]

\[
\int \left( \frac{1}{1+u} + \frac{1}{1-u} \right) = t.
\]

\[
\ln \left| \frac{1+u}{1-u} \right| = t.
\]

\[
\left| \frac{1+u}{1-u} \right| = e^t.
\]

When \((1+u)/(1-u) \geq 0\), we obtain

\[
u = \frac{e^t - 1}{e^t + 1} = \tanh \left( \frac{t}{2} \right).
\]
This yields
\[ x(t) = 1 - u^2 = \text{sech}^2 \left( \frac{t}{2} \right). \]

When \((1 + u)/(1 - u) < 0\), we have
\[ u = \frac{e^t + 1}{e^t - 1} = \text{coth} \left( \frac{t}{2} \right). \]

This yields
\[ x(t) = 1 - u^2 = -\text{csch}^2 \left( \frac{t}{2} \right). \]

This should be the solution on the left half plane in the phase plane. From
\[ x(t) = \sinh^{-3} \left( \frac{t}{2} \right) \cosh \left( \frac{t}{2} \right) = \begin{cases} > & \text{for } t > 0 \\ < & \text{for } t < 0 \end{cases} \]

Hence, the branch on the upper plane is the one with \( t \in (0, \infty) \), while the lower branch, \( t \in (-\infty) \).

Below, we use Maple to plot these orbits in the phase plane and the corresponding graphs of \( x(t) \).

```maple
with(DEtools):
with(plots):
E := y^2/2 + x^4/4 - delta*x^2/2;
```

Plot the level set for the energy. Due to conservation of energy, these level sets are the orbits.

```maple
contourplot(subs(delta=1,E),x=-2..2,y=-2..2,grid=[80,80],contours=
=[-0.3,-0.2,-0.1,0,0.1,0.2,0.3],scaling=CONSTRAINED,labels=['s','s'],title='delta=1');
```

![Plot of level sets for energy](image)
Homeworks.

1. Use the same method to find the homoclinic orbits for the Duffing equation.

6.6 Liapunov function and global stability

We recall that when the perturbation of a hamiltonian system is dissipative, we observe that the hamiltonian $H$ decreases along any trajectory and eventually reaches a minimum of $H$. If there is only one minimum of $H$, then this minimum must be globally asymptotically stable. That is, every trajectory tends to this minimum as $t \to \infty$. So, the key idea here is that the globally asymptotical stability of an equilibrium is resulted from the decreasing of $H$. This idea can be generalized to general systems. The dissipation is measured by so called the Liapunov function $\Phi$, which decreases along trajectories. More precisely, let consider the general system

$$
\begin{align*}
\dot{x} &= f(x,y) \\
\dot{y} &= g(x,y)
\end{align*}
$$

Suppose $(0,0)$ is an equilibrium of this system. We have the following definition.

**Definition 6.8** A smooth function $\Phi(x,y)$ is called a Liapunov function for (6.15) if

(i) $\Phi(0,0) = 0$, $\Phi(x,y) > 0$ for $(x,y) \neq (0,0)$.

(ii) $\Phi(x,y) \to \infty$ as $|(x,y)| \to \infty$.

(iii) $\Phi_x(x,y)f(x,y) + \Phi_y(x,y)g(x,y) < 0$

- Condition (i) says that $(0,0)$ is the only isolated minimum of $\Phi$.
- Condition (ii) says that the region $\Phi(x,y) \leq E$ is always bounded.
- Condition (iii) implies that along any trajectory

$$
\frac{d\Phi(x(t), y(t))}{dt} < 0.
$$

Thus, $\Phi(x(t), y(t))$ is a decreasing function.

**Theorem 6.12** Consider the system (6.15). Suppose $(0,0)$ is its equilibrium. Suppose the system possesses a Liapunov function $\Phi$, then $(0,0)$ is globally and asymptotically stable. That is, for any trajectory, we have

$$
\lim_{t \to \infty} (x(t), y(t)) = (0,0).
$$

**Proof.** We shall use the extremal value theorem to prove this theorem. The extremal value theorem states that 
a continuous function in a bounded and closed domain in $\mathbb{R}^n$ attains its extremal value.  
Along any trajectory $(x(t), y(t))$, we have that $\Phi(x(t), y(t))$ is decreasing (condition (iii))
and bounded below (condition (i)). Hence it has a limit as \( t \) tends to infinity. Suppose 
\[ \lim_{t \to \infty} \Phi(x(t), y(t)) = m > 0. \]
Then the orbit \((x(t), y(t))\), \( t \in (0, \infty) \) is confined in the region \( S := \{(x, y)| m \leq \Phi(x, y) \leq \Phi(x(0), y(0))\} \). From condition (ii), this region is bounded and closed. Hence \( \frac{d\Phi(x(t), y(t))}{dt} \) can attain a maximum in this region (by the extremal value theorem). Let us call it \( \alpha \). From (6.16), we have \( \alpha < 0 \). But this implies
\[
\Phi(x(t), y(t)) = \int_0^t \frac{d\Phi(x(t), y(t))}{dt} dt \leq \alpha t \to -\infty \text{ as } t \to \infty.
\]
This is a contradiction. Hence \( \lim_{t \to \infty} \Phi(x(t), y(t)) = 0 \).

Next, we show \((x(t), y(t)) \to (0, 0)\) as \( t \to \infty \). Let \( \rho(t) = x(t)^2 + y(t)^2 \). Suppose \( \rho(t) \) does not tend to 0. This means that there exists a sequence \( t_n \) with \( t_n \to \infty \) such that \( \rho(t_n) \geq \rho_0 > 0 \). Then the region
\[
R := \{(x, y)| x^2 + y^2 \geq \rho_0 \text{ and } \Phi(x, y) \leq \Phi(x(0), y(0))\}
\]
is bounded and closed. Hence, by the extremal value theorem again that \( \Phi \) attains a minimum in this region. Since \( \Phi > 0 \) in this region, we have
\[
\min_\mathbb{R} \Phi(x, y) \geq \beta > 0.
\]
and because \((x(t_n), y(t_n)) \in R\), we obtain
\[
\min_{t_n} \Phi(x(t_n), y(t_n)) \geq \beta > 0.
\]
This contradicts to \( \lim_{t \to \infty} \Phi(x(t), y(t)) = 0 \). Hence, \( x^2(t) + y^2(t) \to 0 \) as \( t \to \infty \). Thus, we obtain that the global minimum \((0, 0)\) is asymptotically stable. \( \blacksquare \)

**Remark.** We give another intuitive proof to show that \( \Phi(x(t), y(t)) \to 0 \) as \( t \to \infty \) implies \((x(t), y(t)) \to (0, 0)\), at least locally. Since \( \Phi(x, y) \) has minimum at \((0, 0)\), it implies that there exists \( c > 0 \) such that
\[
\Phi(x, y) \geq c(x^2 + y^2)
\]
for \((x, y) \sim (0, 0)\). From \( \Phi(x(t), y(t)) \to 0 \) as \( t \to \infty \), we obtain
\[
c(x(t)^2 + y(t)^2) \leq \Phi(x(t), y(t)) \to 0, \text{ as } t \to \infty.
\]
for orbits with initial data \((x(0), y(0)) \sim (0, 0)\).

**Example.**

1. Damped simple pendulum.
\[
\ddot{\theta} = \frac{g}{l} \sin \theta - b \dot{\theta}
\]
Here, \( b > 0 \) is the damping coefficient. In the form of first order equation, it reads
\[
\begin{cases}
\dot{x} = y \\
\dot{y} = \frac{g}{l} \sin x - by
\end{cases}
\]
We take
\[ \Phi(x, y) = \frac{1}{2} y^2 + \frac{g}{l} (1 - \cos x). \]

Then
\[ \Phi_x f + \Phi_y g = \frac{g}{l} \sin(x) y + y \left( \frac{g}{l} \sin x - by \right) = -by^2 < 0. \]

Although the function \( \Phi \) does not satisfy (ii) in the definition of Liapunov function, it satisfies
\[ \{(x, y) | \Phi(x, y) \leq \frac{g}{l} - \epsilon \} \]
Chapter 7

Existence, Uniqueness Theorems

7.1 Existence

In this section, we study the existence, uniqueness and numerical schemes to construct solutions for the initial value problem

\[ y'(t) = f(t, y(t)), \quad (7.1) \]

\[ y(t_0) = y_0. \quad (7.2) \]

**Theorem 7.13 (Local Existence, Cauchy-Peano theory)** Consider the initial value problem \((7.1), (7.2)\). Suppose \(f\) and \(\partial f / \partial y\) are continuous in a neighborhood of \((t_0, y_0)\), then the initial value problem \((7.1)\) and \((7.2)\) has a solution \(y(\cdot)\) in \([t_0 - \tau, t_0 + \tau]\) for some \(\tau > 0\).

We partition the existence theory into following steps.

1. Convert to an equivalent integral equation. We can integrate \((7.1)\) in \(t\) and obtain

\[ y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds \quad (7.3) \]

This is an integral equation for \(y(\cdot)\). We claim that the initial value problem \((7.1)\) \((7.2)\) is equivalent to the integral equation \((7.3)\).

We have seen the derivation from \((7.1)\) and \((7.2)\) to \((7.3)\). Conversely, if \(y(\cdot)\) is continuous and satisfies \((7.3)\), then \(f(\cdot, y(\cdot))\) is continuous. Hence, \(\int_{t_0}^{t} f(s, y(s)) \, ds\) is differentiable. By the fundamental theorem of Calculus, \(y'(t) = f(t, y(t))\). Hence, \(y(\cdot)\) satisfies \((7.1)\). As \(t = t_0\), the integral part of \((7.3)\) is zero. Hence \(y(t_0) = y_0\).

2. Function space \(C[I]\) and function on function space. The integral equation can be viewed as a fixed point equation in a function space \(C[I]\) as the follows. First, let us denote the function \(y_0 + \int_{t_0}^{t} y(s, y(s)) \, ds\) by \(z(t)\). The mapping \(y(\cdot) \mapsto z(\cdot)\), denote by \(\Phi(y)\). maps a function to a function. The domain of \(\Phi\) consists of all continuous functions \(y\) defined in the interval \(I = [t_0 - \tau, t_0 + \tau]\), that is

\[ C[I] := \{ y \mid y : I \to \mathbb{R}^n \text{ is continuous} \} \]
The space \( C[I] \) depends on \( \tau \), and \( \tau > 0 \) is to be chosen later. We find that \( \Phi \) maps \( C[I] \) into itself. The integral equation (7.3) is equivalent to the fixed point equation

\[
y = \Phi(y)
\]

(7.4)
in the function space \( C[I] \).

3. Picard iteration to generate approximate solutions. Define

\[
y^0(t) \equiv y_0
\]

\[
y^{n+1}(t) = \Phi(y^n)(t) := y_0 + \int_{t_0}^t f(s, y^n(s)) \, ds, \quad n \geq 1.
\]

(7.5)

4. \( C[I] \) is a complete normed function space. In order to show the limit of \( y^n \) stays in \( C[I] \), we need to define a norm to measure distance between two functions. We define

\[
\|y\| = \max_{t \in I} |y(t)|
\]

It is called the norm of \( y \). The quantity \( \|y_1 - y_2\| \) is the maximal distance of \( y_1(t) \) and \( y_2(t) \) in the region \( I \). An important property of the function space \( C(I) \) is that all Cauchy sequence \( \{y^n\} \) has a limit in \( C(I) \). This property is called completeness. It allows us to take limit in \( C(I) \).

Remark. A sequence \( \{y^n\} \) is called a Cauchy sequence if for any \( \epsilon > 0 \), there exists an \( N \) such that for any \( m, n \geq N \), we have

\[
\|y^n - y^m\| < \epsilon.
\]

The definition of Cauchy sequence allows us to define the concept of potentially convergent sequence without knowing its limit.

5. The sequence \( \{y^n\} \) is a Cauchy sequence in \( C(I) \) if \( \tau \) is small enough. From (7.5), we have

\[
\|y^{n+1} - y^n\| = \|\Phi(y^n) - \Phi(y^{n-1})\| \leq \int_{t_0}^t |f(s, y^n(s)) - f(s, y^{n-1}(s))| \, ds
\]

\[
\leq \int_{t_0}^t L |y^n(s) - y^{n-1}(s)| \, ds \leq \tau L \|y^n - y^{n-1}\|
\]

Here, \( L = \max |\partial f(s, y)/\partial y| \). We choose \( \tau \) small enough so that \( \tau L = \rho < 1 \). With this,

\[
\|y^m - y^n\| \leq \sum_{k=n}^{m-1} \|y^{k+1} - y^k\| \leq \sum_{n}^{m-1} \rho^k < \epsilon
\]

provided \( n < m \) are large enough. Thus, \( \{y^n\} \) is a Cauchy sequence in \( C(I) \) if \( \tau \) is small enough. By the completeness of \( C(I) \), \( y^n \) converges to a function \( y \in C(I) \). This convergence is called uniform convergence. In particular, it implies that
7.2. **UNIQUENESS**

\( y_n(s) \to y(s) \) for all \( s \in I \). This convergence is called pointwise convergence. This also yields \( \lim f(s, y^n(s)) = f(s, y(s)) \) for all \( s \in I \) because \( f \) is continuous in \( y \). By the continuity of integration, we then get

\[
\int_{t_0}^{t} f(s, y^n(s)) \, ds \to \int_{t_0}^{t} f(s, y(s)) \, ds
\]

By taking limit \( n \to \infty \) in (7.5), we get that \( y(\cdot) \) satisfies the integral equation (7.3).

6. \( y(\cdot) \) is differentiable. \( y(\cdot) \) satisfies the integral equation and the right-hand side of the integral equation is an integral with continuous integrand. By the fundamental theorem of calculus, \( \int_{t_0}^{t} f(s, y(s)) \, ds \) is differentiable and its derivative at \( t \) is \( f(t, y(t)) \).

### 7.2 Uniqueness

**Definition 2.9** We say that \( f(s, y) \) is Lipschitz continuous in \( y \) if there exists a constant \( L \) such that

\[
|f(s, y_1) - f(s, y_2)| \leq L|y_1 - y_2|.
\]

for any \( y_1 \) and \( y_2 \).

If \( f(s, y) \) is continuously differentiable in \( y \), then by the mean value theorem, it is also Lipschitz in \( y \).

**Theorem 7.14** If \( f(s, y) \) is Lipschitz in \( y \) in a neighbor of \((t_0, y_0)\), then the initial value problem

\[
y'(t) = f(t, y(t)), \quad y(0) = y_0
\]

has a unique solution.

**Proof.** Suppose \( y_1(\cdot) \) and \( y_2(\cdot) \) are two solutions. Then Let \( \eta(t) := |y_2(t) - y_1(t)| \). We have

\[
\eta'(t) \leq |(y_2(t) - y_1(t))'| \leq |f(t, y_2(t)) - f(t, y_1(t))| \\
\leq L|y_2(t) - y_1(t)| = L\eta(t)
\]

We get

\[
\eta'(t) - L\eta(t) \leq 0.
\]

Multiplying \( e^{-Lt} \) on both sides, we get

\[
(e^{-Lt}\eta(t))' \leq 0.
\]

Hence

\[ e^{-Lt}\eta(t) \leq \eta(0). \]

But \( \eta(0) = 0 \) (because \( y_1(0) = y_2(0) = y_0 \) and \( \eta(t) = |y_1(t) - y_2(t)| \geq 0 \), we conclude that \( \eta(t) \equiv 0 \). \( \blacksquare \)
If $f$ does not satisfy the Lipschitz condition, then a counter example does exist. Typical counter example is

$$y'(t) = 2\sqrt{y}, \ y(0) = 0.$$ 

Any function has the form

$$y(t) = \begin{cases} 0 & t < c \\ (t - c)^2 & t \geq c \end{cases}$$

with arbitrary $c \geq 0$ is a solution.

### 7.3 Continuous dependence on initial data

### 7.4 Global existence

A vector field $f(t, y)$ is said to grow at most linearly as $|y| \to \infty$ if

$$|f(t, y)| \leq a|y| + b$$

for some positive constants $a, b$.

**Theorem 7.15** If $f(t, y)$ is smooth and grows at most linearly as $|y| \to \infty$, then all solutions of ODE $y' = f(t, y)$ can be extended to $t = \infty$.

**Proof.** Suppose a solution exists in the interval $[0, T)$, we give an a priori estimate for this solution. From the grow condition of $f$, we have

$$|y(t)'| \leq |y'(t)| \leq a|y(t)| + b.$$

Multiplying $e^{-at}$ on both sides, we get

$$(e^{-at}|y(t)|)' \leq e^{-at}b.$$

Integrating $t$ from 0 to $T$, we obtain

$$e^{-aT}|y(T)| - |y(0)| \leq \int_0^T e^{-at}b \, dt = \frac{b}{a} (1 - e^{aT}).$$

Hence

$$|y(T)| \leq |y(0)|e^{aT} + \frac{b}{a} e^{aT}.$$

Such an estimate is called a *priori estimate* of solutions. It means that as long as solution exist, it satisfies the above estimate.

Now suppose our solution exists in the interval $[0, T)$ and cannot be extended. From the above estimate, the limit $y(T^-)$ exists. This is because $y(\cdot)$ is bounded, hence $f(t, y(t))$ is bounded and hence $y'(t)$ is bounded for $t \in [0, T)$. Hence we can extend $y(\cdot)$ from $T$ with the $y(T^+) = y(T^-)$. By the local existence theorem, the solution can be extended for a short
time. Now, we have a solution on two sides of $T$ with the same data $y(T^-)$, we still need to show that it satisfies the equation at $t = T$. To see this, on the right-hand side

$$\lim_{t \to T^+} y'(t) = \lim_{t \to T^+} f(t, y(t)) = f(T, y(T^-)).$$

On the left-hand side, we also have

$$\lim_{t \to T^-} y'(t) = \lim_{t \to T^-} f(t, y(t)) = f(T, y(T^-)).$$

Therefore $y'(t)$ is continuous at $T$ and $y'(T) = f(T, y(T))$. Hence we get the extended solution also satisfies the equation at $T$. This is a contradiction. □

Remarks.

1. We can replace the growth condition by

$$|f(t, y)| \leq a(t)|y| + b(t) \quad (7.7)$$

where $a(t)$ and $b(t)$ are two positive functions and locally integrable, which means

$$\int_I a(t) \, dt, \int_I b(t) \, dt < \infty$$

for any bounded interval $I$.

2. In the proofs of the uniqueness theorem and the global existence theorem, we use so called the Gronwall inequality, which is important in the estimate of solutions of ODE.

Lemma 7.1 (Gronwall inequality) If

$$\eta'(t) \leq a(t)\eta(t) + b(t)$$

then

$$\eta(t) \leq e^{\int_0^t a(s) \, ds} \eta(0) + \int_0^t e^{\int_s^t a(\tau) \, d\tau} b(s) \, ds \quad (7.8)$$

Gronwall inequality can be used to show that the continuous dependence of solution to its initial data.

7.5 Supplementary

7.5.1 Uniform continuity

Pointwise continuity. The concept of continuity is a local concept. Namely, $y$ is continuous at $t_0$ means that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|y(t) - y(t_0)| < \epsilon$ as $|t - t_0| < \delta$. The continuity property of $y$ at $t_0$ is measured by the relation $\delta(\epsilon)$. The locality here means that $\delta$ also depends on $t_0$. This can be read by the example $y = 1/t$ for $t_0 \sim 0$. For any $\epsilon$, in order to have $|1/t - 1/t_0| < \epsilon$, we can choose $\delta \approx c t_0^2$ (Check by yourself). Thus, the continuity property of $y(t)$ for $t_0$ near 0 and 1 is different. The ratio $\epsilon/\delta$ is of the same magnitude of $y'(t_0)$, in the case when $y(\cdot)$ is differentiable.
Uniform continuity

**Theorem 7.16** When a function \( y \) is continuous on a bounded closed interval \( I \), the above local continuity becomes uniform. Namely, for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that
\[
|y(t_1) - y(t_2)| < \epsilon \quad \text{whenever} \quad |t_1 - t_2| < \delta.
\]

**Proof.** For any \( \epsilon > 0 \), any \( s \in I \), there exists \( \delta(\epsilon, s) > 0 \) such that
\[
|y(t) - y(s)| < \epsilon \quad \text{whenever} \quad |t - s| < \delta(\epsilon, s).
\]
Let us consider the open intervals
\[
U(s, \delta(\epsilon, s)) := (s - \delta(\epsilon, s), s + \delta(\epsilon, s)).
\]
The union \( \bigcup_{s \in I} U(s, \delta(\epsilon, s)) \) contain \( I \). Since \( I \) is closed and bounded, by so called the finite covering lemma, there exist finite many \( U(s_i, \delta(\epsilon, s_i)) \), \( i = 1, \ldots, n \) such that \( I \subset \bigcup_{i=1}^{n} U(s_i, \delta(\epsilon, s_i)) \). Then we choose
\[
\delta := \min_{i=1}^{n} \delta(\epsilon, s_i)
\]
then the distances between any pair \( s_i \) and \( s_j \) must be less than \( \delta \). For any \( t_1, t_2 \in I \) with \( |t_1 - t_2| < \delta \), Suppose \( t_1 \in U(s_k, \delta(\epsilon, s_k)) \) and \( t_2 \in U(s_l, \delta(\epsilon, s_l)) \), then we must have \( |s_k - s_l| < \delta \).
\[
|y(t_1) - y(t_2)| \leq |y(t_1) - y(s_k)| + |y(s_k) - y(s_l)| + |y(s_l) - y(t_2)| < 3\epsilon.
\]
This completes the proof. \( \blacksquare \)

The key of the proof is the finite convering lemma. It says that a local property can be uniform through out the whole interval \( I \). This is a key step from local to global.

### 7.5.2 \( C(I) \) is a normed linear space

If this distance is zero, it implies \( y_1 \equiv y_2 \) in \( I \). Also,
\[
\|ay\| = |a|\|y\|
\]
for any scalar \( a \). Moreover, we have
\[
\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|.
\]
If we replace \( y_2 \) by \( -y_2 \), it says that the distance between the two function is less than \( \|y_1\| \) and \( \|y_2\| \). This is exactly the triangular inequality. To show this inequality, we notice that
\[
|y_1(t)| \leq \|y_1\|, \quad |y_2(t)| \leq \|y_2\|, \quad \text{for all} \quad t \in I.
\]
Hence,
\[
|y_1(t) + y_2(t)| \leq |y_1(t)| + |y_2(t)| \leq \|bf y_1\| + \|y_2\|.
\]
By taking maximal value on the L.H. side for \( t \in I \), we obtain
\[
\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|.
\]

The function space \( C[I] \) with the norm \( \| \cdot \| \) is called a normed vector space.
7.5.3 \( C(I) \) is a complete

Such a space is called a Banach space.

**Definition 5.10** A sequence \( \{y^n\} \) is called a Cauchy sequence if for any \( \epsilon > 0 \), there exists an \( N \) such that for any \( m, n \geq N \), we have

\[
\|y^n - y^m\| < \epsilon.
\]

**Theorem 7.17** Let \( \{y^n\} \) be a Cauchy sequence in \( C(I) \). Then there exist \( y \in C(I) \) such that

\[
\|y^n - y\| \to 0 \text{ as } n \to \infty.
\]

To prove this theorem, we notice that for each \( t \in I \), \( \{y^n(t)\} \) is a Cauchy sequence in \( \mathbb{R} \). Hence, the limit \( \lim_{n \to \infty} y^n(t) \) exists. We define

\[
y(t) = \lim_{n \to \infty} y^n(t) \text{ for each } t \in I.
\]

We need to show that \( y \) is continuous and \( \|y^n - y\| \to 0 \). To see \( y \) is continuous, let \( t_1, t_2 \in I \). At these two points, \( \lim_{n \to \infty} y^n(t_i) = y(t_i), i = 1, 2 \). This means that for any \( \epsilon > 0 \), there exists an \( N > 0 \) such that

\[
|y^n(t_i) - y(t_i)| < \epsilon, \ i = 1, 2, \text{ for all } n \geq N.
\]

With this, we can estimate \( |y(t_1) - y(t_2)| \) through the help of \( y^n \) with \( n \geq N \). Namely,

\[
|y(t_1) - y(t_2)| \leq |y(t_1) - y^n(t_1)| + |y^n(t_1) - y^n(t_2)| + |y^n(t_2) - y(t_2)|
\]

\[
\leq 2\epsilon + |y^n(t_1) - y^n(t_2)| \leq 3\epsilon
\]

In the last step, we have used the uniform continuity of \( y^n \) on \( I \). Hence, \( y \) is continuous in \( I \).

Also, from the Cauchy property of \( y^n \) in \( C(I) \), we have for any \( \epsilon > 0 \), there exists an \( N > 0 \) such that for all \( n, m > N \), we have

\[
\|y^n - y^m\| < \epsilon
\]

But this implies that for all \( t \in I \), we have

\[
|y^n(t) - y^m(t)| < \epsilon
\]

Now, we fix \( n \) and let \( m \to \infty \). This yields

\[
|y^n(t) - y(t)| \leq \epsilon
\]

and this holds for \( n > N \). Now we take maximum in \( t \in I \). This yields

\[
\|y^n - y\| \leq \epsilon
\]

Thus, we have shown \( \lim y^n = y \) in \( C(I) \).
Chapter 8

Numerical Methods for Ordinary Differential Equations

8.1 Two simple schemes

We solve the initial value problem

\[ y' = f(t, y), \quad y(0) = y_0. \tag{8.1} \]

Numerical method is to approximate the solution \( y(\cdot) \) by \( y^n \sim y(t^n) \), where \( t^0 = 0 < t^1 < \cdots t^n \) are the discretized time steps. For simplicity, we take uniform step size \( h \). We define \( t^k = kh \). We want to find a procedure to construct \( y^{n+1} \) from the knowledge of \( y^n \). By integrating the ODE from \( t^n \) to \( t^{n+1} \), we get

\[ y(t^{n+1}) = y(t^n) + \int_{t^n}^{t^{n+1}} f(t, y(t)) \, dt \]

So the strategy is to approximate the integral by numerical integral \( hF_h(t^n, y^n) \).

Below, we give two popular methods

1. **Forward Euler method**

   \[ y^{n+1} = y^n + hf(t^n, y^n) \]

2. **Second-order Runge-Kutta method (RK2)**

   \[
   \begin{align*}
   y_1 &= y^n + hf(t^n, y^n), \\
   y^{n+1} &= y^n + \frac{1}{2}h(f(t^n, y^n) + f(t^{n+1}, y_1)) \\
   &= \frac{1}{2}(y_1 + (y^n + hf(t^{n+1}, y_1)))
   \end{align*}
   \]

8.2 Truncation error and orders of accuracy

In the forward Euler method, we can plug a true solution \( y(t) \) into the finite difference equation, by Taylor expansion, we get

\[ y(t^{n+1}) = y(t^n) + hf(t^n, y(t^n)) + \epsilon(h) \tag{8.2} \]
where the error term \( \epsilon(h) \) is obtained by

\[
\epsilon(h) = h \int_{t^n}^{t^{n+1}} f(t, y(t)) \, dt - h f(t^n, y(t^n)) = h \int_{t^n}^{t^{n+1}} (f(t, y(t)) - f(t^n, y(t^n))) \, dt = O(h^2).
\]

The error term \( \epsilon(h) \) is called the truncation error. You may view the forward Euler method is a rectangle method for numerical integration for \( \int_{t^n}^{t^{n+1}} f(s, y(s)) \, ds \). Similarly, we may use trapezoidal rule

\[
\int_{t^n}^{t^{n+1}} f(s, y(s)) \, ds = \frac{1}{2} h (f(t^n, y(t^n)) + f(t^{n+1}, y(t^{n+1})) + O(h^3).
\]

We do not have \( y(t^{n+1}) \), yet we can use \( y_1 \) obtained by the forward Euler to approximate \( y(t^{n+1}) \). From (8.2),

\[
f(t^{n+1}, y_1) = f(t^{n+1}, y(t^{n+1})) + O(h^2).
\]

This yields

\[
y(t^{n+1}) = y(t^n) + \frac{1}{2} h (f(t^n, y^n) + f(t^{n+1}, y_1)) + O(h^3),
\]

where \( y_1 = y(t^n) + hf(t^n, y(t^n)) \). In general, we can write our numerical scheme as

\[
y^{n+1} = y^n + hF_h(t^n, y^n) \tag{8.3}
\]

For instance, for the forward method

\[
F_h(t, y) = f(t, y)
\]

For the RK2,

\[
F_h(t, y) = \frac{1}{2} (f(t, y) + f(t+h, y+hf(t, y))).
\]

The function \( F \) is called a numerical vector field.

**Definition 2.11** The numerical scheme (8.3) for (8.1) is said of order \( p \) if any smooth solution \( y(\cdot) \) of (8.1) satisfies

\[
y(t^{n+1}) = y(t^n) + hF_h(t^n, y(t^n)) + O(h^{p+1}). \tag{8.4}
\]

Thus, forward Euler is first order while RK2 is second order. The quantity

\[
e^n(h) := y(t^{n+1}) - y(t^n) - hF_h(t^n, y(t^n))
\]

is called the truncation error of the scheme (8.3).

We can estimate the true error \( |y(t^n) - y^n| \) in terms of truncation errors. From

\[
y(t^{n+1}) = y(t^n) + hF_h(t^n, y(t^n)) + e^n
\]

\[
y^{n+1} = y^n + hF_h(t^n, y^n)
\]
Subtracting two equations, we get
\[ y(t^{n+1}) - y^{n+1} = (y(t^n) - y^n) + h(F(t^n, y(t^n)) - F(t^n, y^n)) + e^n \]

Let us denote the true error by \( e^n := |y(t^n) - y^n| \) It satisfies
\[ e^{n+1} \leq e^n + hLe^n + |e^n| \leq e^n + hLe^n + Mh^{p+1}. \]

Here we have used the assumption
\[ |e^n| \leq Mh^{p+1} \]

for order \( p \) schemes. This is a finite difference inequality. We can derive a discrete Gronwall inequality as below. We have
\[
\begin{align*}
    e^n &\leq (1 + hL)e^{n-1} + Mh^{p+1} \\
    &\leq (1 + hL)^2e^{n-2} + ((1 + hL) + 1)Mh^{p+1} \\
    &\vdots \\
    &\leq (1 + hL)^ne^0 + \left( \sum_{k=0}^{n-1} (1 + hL)^{k} \right) Mh^{p+1} \\
    &\leq (1 + hL)^ne^0 + \frac{(1 + hL)^n}{hL} Mh^{p+1} \\
    &\leq (1 + hL)^ne^0 + \frac{(1 + hL)^n}{L} Mh^p
\end{align*}
\]

Now, we fix \( nh = t \), this means that we want to find the true error at \( t \) as \( h \to 0 \). With \( t \) fixed, we have
\[
(1 + nh)^n = \left( (1 + hL)^{1/hL} \right)^{Lt} \leq e^{Lt}.
\]

Since the initial error \( e^0 = 0 \), the true error at \( t \) is
\[ e^n \leq Me^{Lt}h^p. \]

We conclude this analysis by the following theorem.

**Theorem 8.18** If the numerical scheme (8.3) is of order \( p \), then the true error at a fixed time is of order \( O(h^p) \).

### 8.3 High-order schemes

We list a fourth order Runge-Kutta method (RK4). Basically, we use Simpson rule for integration
\[
\int_{t^n}^{t^{n+1}} f(t, y(t)) \, dt \approx h \left( f(t^n, y(t^n)) + 4f(t^{n+1/2}, y(t^{n+1/2})) + f(t^{n+1}, y(t^{n+1})) \right).
\]
The RK4 can be expressed as

\begin{align*}
    k_1 &= f(t, y) \\
    k_2 &= f(t + h/2, y + hk_1/2) \\
    k_3 &= f(t + h/2, y + hk_2/2) \\
    k_4 &= f(t + h, y + hk_3)
\end{align*}

and

\[ F(t, y) = \frac{k_1 + 2(k_2 + k_3) + k_4}{6}. \]

One can check that the truncation error by Taylor expansion is \( O(h^5) \). Hence the RK4 is a fourth order scheme.
Chapter 9

Introduction to Dynamical System

9.1 Periodic solutions

9.1.1 Predator-Prey system

Let \( x \) be the population of rabbits (prey) and \( y \) the population of fox (predator). The equation for this predator-prey system is

\[
\begin{align*}
\dot{x} &= ax - \alpha xy := f(x, y) \\
\dot{y} &= -by + \beta xy := g(x, y),
\end{align*}
\]

where the coefficients \( a, b, \alpha, \beta > 0 \). The equilibria are those points such that \( f(x, y) = 0 \) and \( g(x, y) = 0 \). There are two: \( E_0 = (0, 0) \) and \( E_* = \left( \frac{b}{\beta}, \frac{a}{\alpha} \right) \). At \( E_0 \) the linearized equation is

\[\dot{\delta y} = \left. \frac{\partial F}{\partial y} \right|_0 \delta y\]

The corresponding

\[\left. \frac{\partial F}{\partial y} \right|_0 = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}\]

Since one eigenvalue is positive and the other is negative, we get \( E_0 \) is a saddle point. At \( E_* \), the linearized matrix is

\[\left. \frac{\partial F}{\partial y} \right|_{E_*} = \begin{pmatrix} 0 & -\alpha b/\beta \\ \alpha b/\beta & 0 \end{pmatrix}\]

The eigenvalues are pure imaginary. So \( E_* \) is an elliptic equilibrium. Near \( E_* \), the solution is expected to be a closed trajectories (a periodic solution). In fact, we can integrate the predator-prey system as the follows. We notice that

\[
\frac{dy}{dx} = \frac{y(-b + \beta x)}{x(a - \alpha y)}
\]

is deparable. It has the solution:

\[
a \ln y - \alpha y + b \ln x - \beta x = C.
\]

When \( C \) is the integration constant. The trajectories are closed curves surrounding \( E_* \). Thus, the solutions are periodic solutions.
Homeworks.

1. * How does the period $T$ depend on the coefficients?

### 9.1.2 van der Pol oscillator

In electric circuit theory, van der Pol proposed a model for electric circuit with vacuum tube, where $I = \phi(V)$ is a cubic function. Let $x$ be the potential, the resulting equation is

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0.$$  

Through a Liénard transform:

$$y = x - \frac{x^3}{3} - \frac{\dot{x}}{\epsilon}$$

the van der Pol equation can be expressed as

$$\dot{x} = \epsilon(x - \frac{x^3}{3} - y)$$
$$\dot{y} = \frac{x}{\epsilon}$$

We can draw the nullclines: $f = 0$ and $g = 0$. From the direction field of $(f, g)$, we see that the field points inwards for large $(x, y)$ and outward for $(x, y)$ near $(0, 0)$. This means that there will be a limiting circle in between.

As $\epsilon \gg 1$, we can observe that the time scale on $x$ variable is fast whereas it is slow on the $y$-variable. That is,

$$\dot{x}(t) = O(\epsilon), \dot{y}(t) = O(1/\epsilon).$$

On the $x - y$ plane, consider the curve

$$y = x - \frac{x^3}{3}$$

The solution moves fast to the curve $y = x - \frac{x^3}{3}$. Once it is closed to this curve, it moves slowly along it until it moves to the critical points $(\pm 1, \pm \frac{2}{3})$. At which it moves away from the curve fast and move to the other side of the curve. The solution then periodically moves in this way.

**Reference.** You may google website on the Van der Pol oscillator on the web site of scholarpedia for more details.

### 9.2 Poincaré-Bendixson Theorem

We still focus on two-dimensional systems

$$y' = f(y), y(0) = y_0$$  \hspace{1cm} (9.1)
9.2. POINCARÉ-BENDIXSON THEOREM

where \( y \in \mathbb{R}^2 \). As we mentioned, our goal is to characterize the whole orbital structure. We have seen the basic solutions are the equilibria. The second class are the orbits connecting these equilibria. In particular, we introduce the separatrices and the homoclinic orbits. We have seen in the damped pendulum that solutions enclosed in separatrices go to a sink time asymptotically. In this section, we shall see the case that the solution may go to an periodic solution. In other words, the solution goes to another separatrix. The van de Pol oscillator and the predator-prey system are two important examples.

We first introduce some basic notions. We denote by \( \phi(t, y_0) \) the solution to the problem (9.1). The orbit \( \gamma^+(y) = \{ \phi(t, y) | t \geq 0 \} \) is the positive orbit through \( y \). Similarly, \( \gamma^-(y) = \{ \phi(t, y) | t \leq 0 \} \) and \( \gamma(y) = \{ \phi(t, y) | -\infty < t < \infty \} \) are the negative orbit and the orbit through \( y \). If \( \phi(T, y) = y \) and \( \phi(t, y) \neq y \) for all \( 0 < t < T \), we say \( \{ \phi(t, y) | 0 \leq t < T \} \) a periodic orbit. A point \( p \) is called an \( \omega \) (resp. \( \alpha \)) point of \( y \) if there exists a sequence \( \{ t_n \}, t_n \to \infty \) (resp. \( -\infty \)) such that \( p = \lim_{n \to \infty} \phi(t_n, y) \). The collection of all \( \omega \) (resp. \( \alpha \)) limit point of \( y \) is called the \( \omega \) (resp. \( \alpha \)) limit set of \( y \) and is denoted by \( \omega(y) \) (resp. \( \alpha(y) \)). One can show that

\[
\omega(y) = \bigcap_{t \geq 0} \bigcup_{s \geq t} \phi(s, y)
\]

Thus, \( \omega(y) \) represents where the positive \( \gamma^+(y) \) ends up. A set \( S \) is called positive (resp. negative) invariant under \( \phi \) if \( \phi(t, S) \subset S \) for all \( t \geq 0 \) (resp. \( t \leq 0 \)). A set \( S \) is called invariant if \( S \) is both positive invariant and negative invariant. It is easy to see that equilibria and periodic orbits are invariant set. The closure of an invariant set is invariant. Further, we have the theorem.

**Theorem 9.19** \( \omega(y) \) and \( \alpha(y) \) are invariant.

**Proof.** The proof is based on the continuous dependence of the initial data. Suppose \( p \in \omega \). Thus, there exists \( t_n \to \infty \) such that \( p = \lim_{n \to \infty} \phi(t_n, y) \). Consider two solutions: \( \phi(s, p) \) and \( \phi(s + t_n, y) = \phi(s, \phi(t_n, y)) \), for any \( s > 0 \). The initial data are closed to each other when \( n \) is enough. Thus, by the continuous dependence of the initial data, we get \( \phi(s, p) \) is closed to \( \phi(s + t_n, y) \). \[\square\]

**Theorem 9.20 (Poincaré-Bendixson)** If \( \gamma^+(y) \) is contained in a bounded closed subset in \( \mathbb{R}^2 \) and \( \omega(y) \neq \phi \) and does not contain any critical points (i.e. where \( f(y) = 0 \)), then \( \omega(y) \) is a periodic orbit.