15.3 Extrema of Multivariable Functions

Question 1: What is a relative extrema and saddle point?

Question 2: How do you find the relative extrema of a surface?

In an earlier chapter, you learned how to find relative maxima and minima on functions of one variable. In those sections, we used the first derivative to find critical numbers. These values are where a potential maximum or minimum might be. Then the second derivative is applied to determine whether the function is concave up (a relative minimum) or concave down (a relative maximum).

For instance, suppose we have the function $g(x) = x^3 - 9x^2 + 24x - 3$. Critical numbers for this function are where the derivative,

$$\begin{aligned} g'(x) &= 3x^2 - 18x + 24 \\ &= 3(x-2)(x-4) \end{aligned}$$

is equal to zero. The derivative is zero at $x = 2$ and $x = 4$.

If we substitute the critical numbers in the second derivative, $g''(x) = 6x - 18$, we get

$$\begin{aligned} g''(2) &= 6(2) - 18 = -6 \quad \rightarrow \quad \text{concave down at } x = 2 \\ g''(4) &= 6(4) - 18 = 6 \quad \rightarrow \quad \text{concave up at } x = 4 \end{aligned}$$

Since the function is concave at $x = 2$, the critical number corresponds to a relative maximum. At $x = 4$, the function is concave down so the critical number matches a relative minimum.

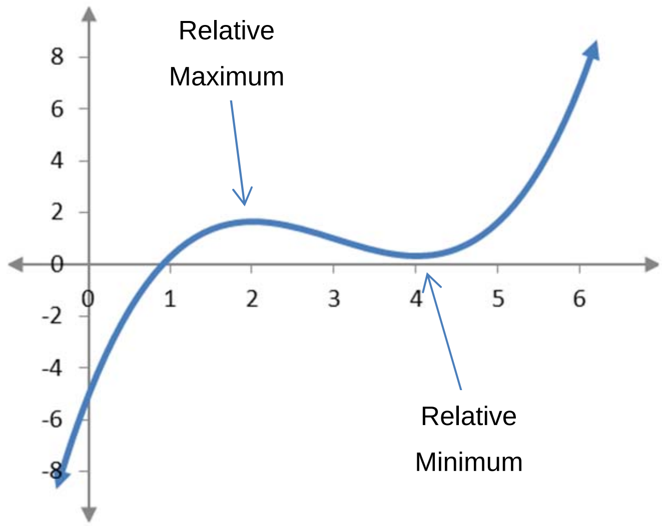

Figure 1 - The function $g(x) = x^3 - 9x^2 + 24x - 3$ and its relative extrema.

Multivariable functions also have high points and low points. In this section, the techniques developed in an earlier chapter will be extended to help you find these extrema. As you might expect, these techniques will utilized the first and second partial derivatives.

Question 1: What are relative extrema and a saddle points?

In an earlier chapter, we defined relative maxima and minima with respect to the points nearby. The relative extrema for functions of two variables are defined in a similar manner.

> A point $(c, d, f(c,d))$ is a relative maximum of a function f if there exists some region surrounding (c,d) for which $f(c,d) \geq f(x,y)$ for all (x,y) in the region.
>
> A point $(c, d, f(c,d))$ is a relative minimum of a function f if there exists some region surrounding (c,d) for which $f(c,d) \leq f(x,y)$ for all (x,y) in the region.

This definition says that a relative maximum on a surface is a point that is higher than the points nearby.

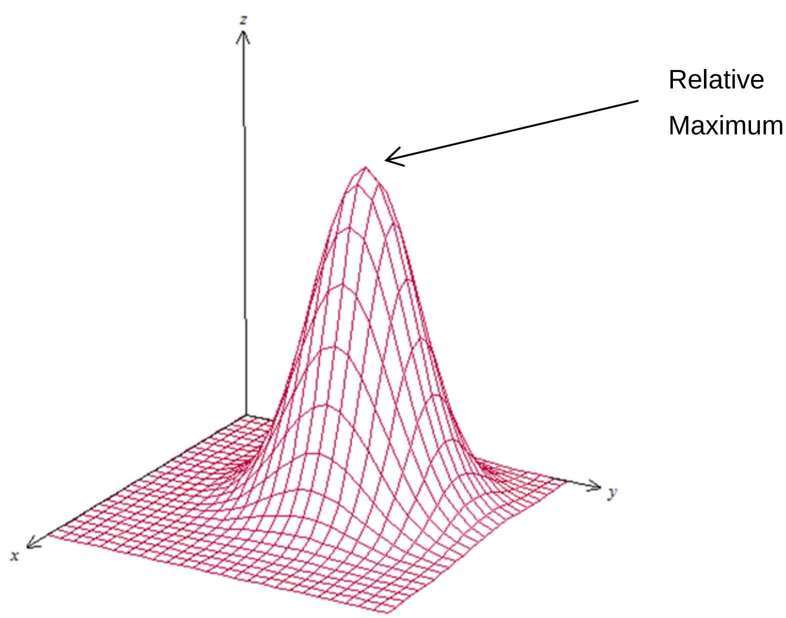

Figure 2 - A relative maximum is higher than the points in a region surrounding it.

A relative minimum is a point lower that all points nearby. .

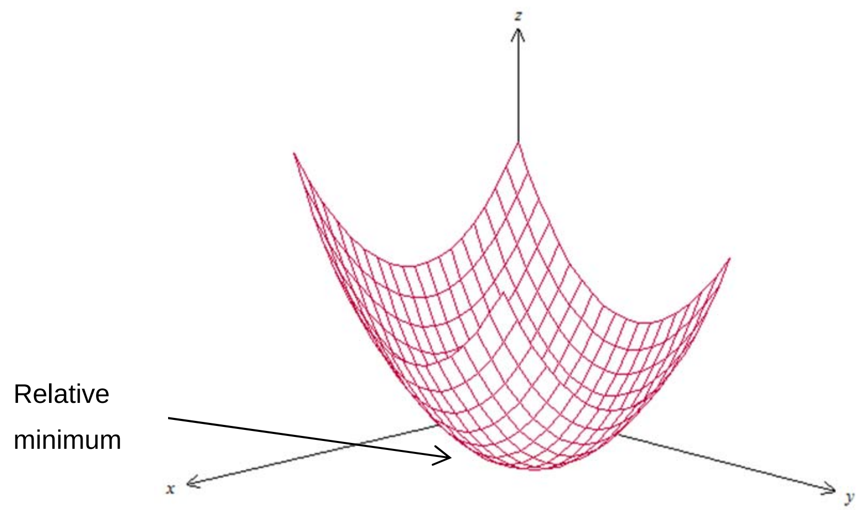

Figure 3 - A relative minimum is lower than the points in a region surrounding it.

Let's examine slices on these functions that pass through the relative extrema. The relative minimum on the function in Figure 3, $f(x,y) = x^2 - 10x + y^2 - 12y + 71$, is located at $(5,6,10)$. The slice located at $x = 5$ is

$$\begin{aligned} f(5,y) &= 5^2 - 10(5) + y^2 - 12y + 71 \\ &= y^2 - 12y + 46 \end{aligned}$$

The red graph in Figure 4 shows this slice and a point where the tangent line to the slice is horizontal at $y = 6$.

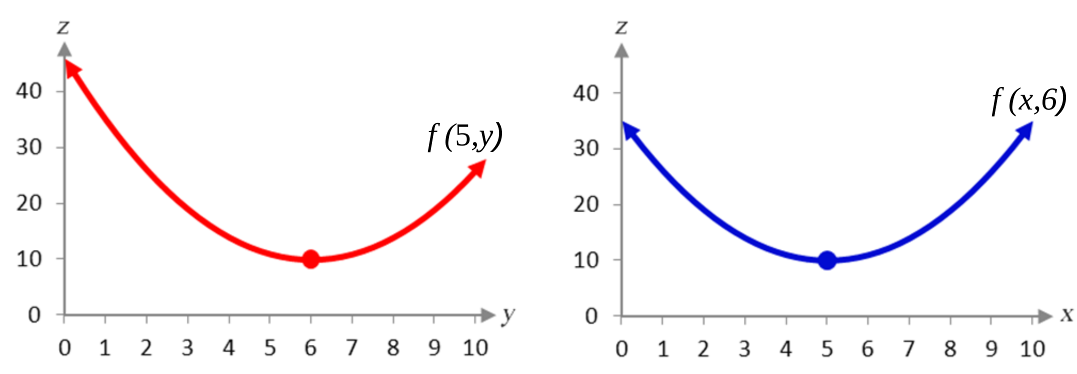

Figure 4 - The graphs of slices through the relative minima of $f(x,y) = x^2 - 10x + y^2 - 12y + 71$.

Similarly, the blue graph in Figure 4 represents the slice at $y = 6$,

$$f(x,6) = x^2 - 10x + 6^2 - 12(6) + 71$$
$$= x^2 - 10x + 35$$

A horizontal tangent line is also located on this graph at $x = 5$. A horizontal tangent line indicates that the partial derivative is equal to zero. This fact leads us to a relationship between relative extrema and partial derivatives.

Critical Points of a Function of Two Variables

A function of two variables f has a critical point at the ordered pair (c,d) if

$$f_x(c,d) = 0 \text{ and } f_y(c,d) = 0$$

If a function has a relative maximum or relative minimum, it will occur at a critical point.

Example 1 Critical Points

Use partial derivatives to find any critical points of

$$f(x,y) = x^2 - 10x + y^2 - 12y + 71$$

Solution We motivated the idea of the critical point with this function. Now we will use the partial derivatives to find them. The partial derivatives are

$$f_x(x,y) = 2x - 10$$
$$f_y(x,y) = 2y - 12$$

At the critical point, both partial derivatives should be zero. If we set each equal to zero and solve for the variable, we get

$$\begin{aligned} 2x-10&=0 \qquad & 2y-12&=0 \\ 2x&=10 & 2y&=12 \\ x&=5 & y&=6 \end{aligned}$$

The critical point is at $(5,6)$.

Example 2 Critical Points

Find all critical points of $h(x,y)=x^2-4x-y^2+2y+4$.

Solution The critical points are found by setting each partial derivative equal to zero. The partial derivatives are

$$h_x(x,y)=2x-4 \qquad h_y(x,y)=-2y+2$$

The critical point is found by solving the partial derivative equations,

$$\begin{aligned} 0&=2x-4 \qquad & 0&=-2y+2 \\ -2x&=-4 & 2y&=2 \\ x&=2 & y&=1 \end{aligned}$$

The critical point is at $(2,1)$.

For some functions, you may need to solve a system of equations to find the critical point. Although this complicates the problem slightly, it does not change the fact that we need to set the partial derivatives equal to zero to find the critical points.

Example 3 Critical Points

Find all critical points of $g(x,y) = x^3 - y^2 - xy + 1$

Solution The partial derivatives of the function are

$$g_x(x,y) = 3x^2 - y \qquad g_y(x,y) = -2y - x$$

To find the critical points, we must solve the system of equations

$$3x^2 - y = 0$$
$$-2y - x = 0$$

Solve the second equation for x to give $x = -2y$. This expression may be substituted in the first equation to yield

$$3(-2y)^2 - y = 0$$

This simplifies to $12y^2 - y = 0$. This may be solved by factoring,

$$y(12y-1) = 0$$
$$y = 0 \quad 12y - 1 = 0$$
$$y = \frac{1}{12}$$

Each of these values corresponds to an x value through $x = -2y$:

$$y = 0 \quad \to \quad x = -2(0) = 0$$
$$y = \frac{1}{12} \quad \to \quad x = -2\left(\frac{1}{12}\right) = -\frac{1}{6}$$

This function has critical points at $(0,0)$ and $\left(-\frac{1}{6}, \frac{1}{12}\right)$.

Many critical points correspond to relative maximums and relative minimums. However, some critical points correspond to saddle points. Saddle points are not relative extrema. For instance, the critical point in Example 2 is a saddle point. If we look at slices through the critical point, we see important features.

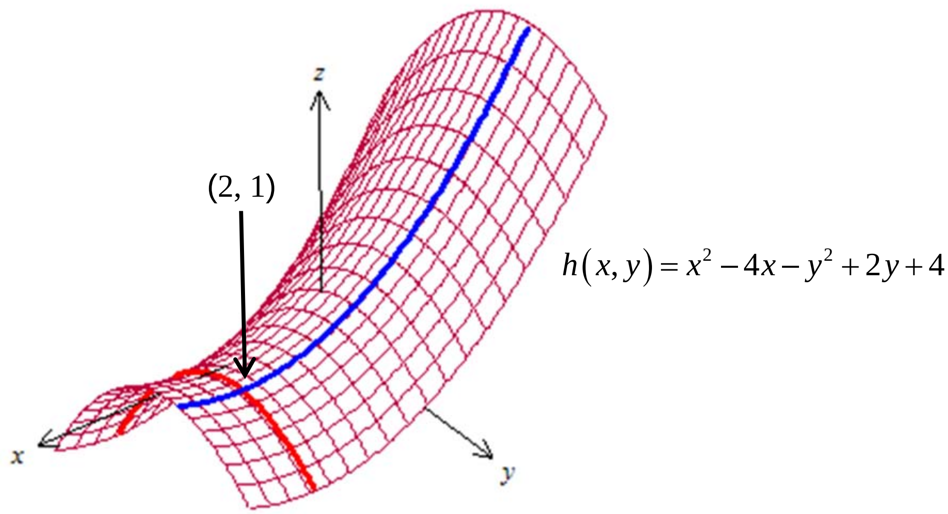

Figure 5 - The surface h(x,y) with two slices labled in blue (y = 1) and red (x = 2).

The blue slice has a minimum at $x = 2$. The red slice has a maximum at $y = 1$. This is what leads to the partial derivatives both being zero. However, the critical point is not a relative extrema since there is no region surrounding $(2,1)$ where the critical point is higher or lower than all of the points in the region.

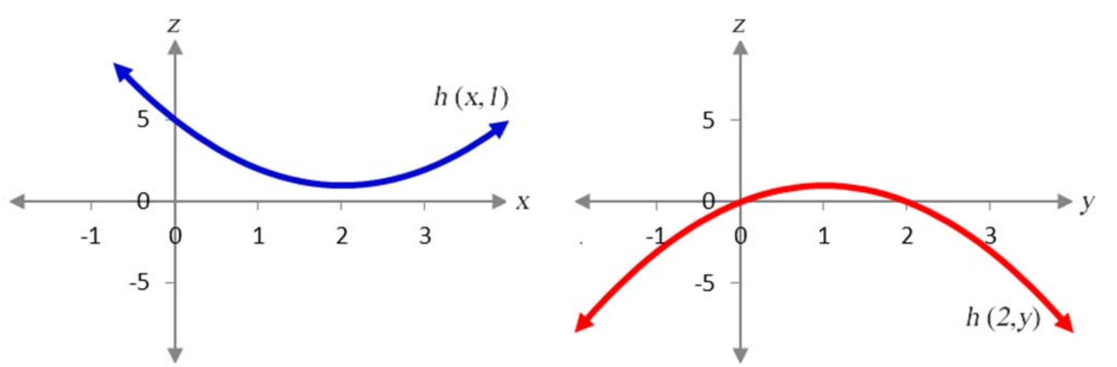

Figure 6 - Two slices through $h(x,y) = x^2 - 4x - y^2 + 2y + 4$. The blue slice is at $y = 1$ and parallel to the x axis. The red slice is through $x = 2$ and parallel to the y axis.

This kind of behavior is typical of a saddle point. In one direction the surface is at a high point. In the other direction the surface is at a low point. This resemblance to a saddle is what gives the point its name.

In the next question, we’ll use the second derivative to help distinguish between relative maximums, relative minimums, and saddle points.

Question 2: How do you find the relative extrema of a surface?

Recall that the second partial derivatives are related to how a surface is curved. We can use the second derivatives in a test to determine whether a critical point is a relative extrema or saddle point.

Test for Relative Extrema of a Function of Two Variables

Suppose the function $z = f(x,y)$ has a critical point at (c,d) and that the second partial derivatives all exist nearby the critical point. Define a number D,

$$D = f_{xx}(c,d)\cdot f_{yy}(c,d) - \left[f_{xy}(c,d)\right]^2$$

If $D > 0$ and $f_{xx}(c,d) < 0$, then $f(c,d)$ is a relative maximum.

If $D > 0$ and $f_{xx}(c,d) > 0$, then $f(c,d)$ is a relative minimum.

If $D < 0$, then $f(c,d)$ is a saddle point.

If $D = 0$, the test give no information and we must examine the critical point by some other means.

Example 4 Test for Relative Extrema

The critical point for $f(x,y) = x^2 - 10x + y^2 - 12y + 71$ is $(5,6)$. Determine if the critical point is a relative maximum, minimum, or saddle point.

Solution In Example 1, th first partial derivatives were calcuated to be

$$f_x(x,y)=2x-10 \qquad f_y(x,y)=2y-12$$

The second partial derivatives are

$$f_{xx}(x,y)=2 \qquad f_{yy}(x,y)=2 \qquad f_{xy}(x,y)=0$$

Since these functions are all constants, substituting the critical points yields the same constants. The value of D at $(5,6)$ is

$$\begin{aligned} D &= f_{xx}(5,6)\,f_{yy}(5,6)-\left[f_{xy}(5,6)\right]^2 \\ &= 2 \cdot 2 - 0 \\ &= 4 \end{aligned}$$

Since D and $f_{xx}(5,6)$ are positive, the critical point corresponds to a relative minimum. The z value of the relative minimum is

$$f(5,6)=5^2-10\cdot 5+6^2-12\cdot 6+71=10$$

Example 5 Test for Relative Extrema

In Example 3, we found the critical numbers for $g(x,y)=x^3-y^2-xy+1$ to be $(0,0)$ and $\left(-\frac{1}{6},\frac{1}{12}\right)$. Use the Test for Relative Extrema to decide whether each critical point corresponds to a relative maximum, relative minimum, or saddle point.

Solution The first derivatives are

$$g_x(x,y)=3x^2-y \qquad g_y(x,y)=-2y-x$$

The second derivatives are

$$g_{xx}(x,y)=6x \qquad g_{yy}(x,y)=-2 \qquad g_{xy}(x,y)=-1$$

Each second partial derivative may be evaluated at each critical point.

	$g_{xx}(x,y)=6x$	$g_{yy}(x,y)=-2$	$g_{xy}(x,y)=-1$	D	Type
$(0,0)$	0	-2	-1	-1	Saddle point
$\left(-\frac{1}{6},\frac{1}{12}\right)$	-1	-2	-1	1	Relative maximum

The z values at each point is

$$g(0,0)=0^3-0^2-0\cdot 0+1=1$$

$$g\left(-\tfrac{1}{6},\tfrac{1}{12}\right)=\left(-\tfrac{1}{6}\right)^3-\left(\tfrac{1}{12}\right)^2-\left(-\tfrac{1}{6}\right)\left(\tfrac{1}{12}\right)+1=\tfrac{433}{432}\approx 1.002$$

Notice that the relative maximum is only a tiny bit higher than the saddle point. On a graph, the relative maximum would be nearly impossible to see visually. However, the Test for Extrema confirms it is there.

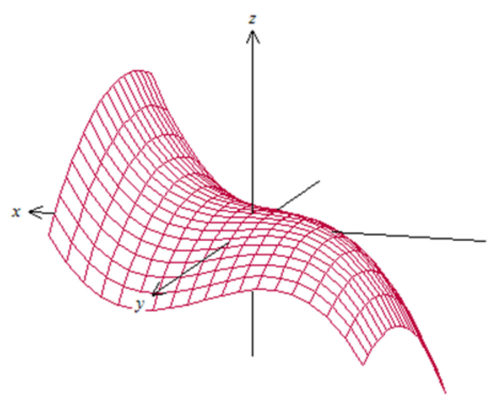

Figure 7 - The function in Example 5. The saddle point at $(0,0)$ is readily apparent. However, the relative maximum is nearly impossible to see at $\left(-\frac{1}{6},\frac{1}{12}\right)$.

When D is negative, the critical point is always a saddle point. When D is positive, the type of relative extrema depends on the sign of the g_{xx}

When you need to find the relative extrema of a function:

1. Find the critical points by setting the partial derivatives equal to zero. Solve these equations to get the x and y values of the critical point.
2. Evaluate f_{xx}, f_{yy}, and f_{xy} at the critical points.
3. Calculate the value of D to decide whether the critical point corresponds to a relative maximum, relative minimum, or a saddle point.

In the next example, we will follow these steps to identify all of the relative extrema and saddle points of a new function.

Example 6 Identify Critical Points

Use the Test for Relative Extrema to classify the critical points for $f(x,y) = y^4 - 32y + x^3 - x^2$ as relative maximum, relative minimum, or saddle points.

Solution To find the critical points, we need to compute the first partial derivatives of the function. The first partial derivatives are

$$f_x(x,y) = 3x^2 - 2x \qquad f_y(x,y) = 4y^3 - 32$$

Set each partial derivative equal to zero to find the critical points. Let's start with the partial derivative with respect to x:

$$\begin{aligned} 0 &= 3x^2 - 2x \\ 0 &= x(3x-2) \\ x = 0 \text{ or } 3x - 2 &= 0 \\ x &= \tfrac{2}{3} \end{aligned}$$

Factor the right side. Set each factor equal to zero to solve for x.

Now the partial derivative with respect to y:

$0 = 4y^3 - 32$

$0 = 4\left(y^3 - 8\right)$ Factor the right side. Set each factor equal to zero. Only the factor in parentheses can be zero.

$0 = y^3 - 8$

$8 = y^3$ Cube root both side to solve for y.

$2 = y$

Combining these together we get two critical points, $(0,2)$ and $\left(\frac{2}{3},2\right)$.

We'll need the second partial derivatives to apply the Test for Relative Extrema to each critical point.

$$f_{xx}(x,y) = 6x - 2 \qquad f_{yy}(x,y) = 12y^2 \qquad f_{xy}(x,y) = 0$$

Evaluate each second partial derivative at the critical points to find the value of D.

	$f_{xx}(x,y)=6x-2$	$f_{yy}(x,y)=12y^2$	$f_{xy}(x,y)=0$	D	Type
$(0,2)$	-2	48	0	-96	Saddle point
$\left(\frac{2}{3},2\right)$	2	48	0	96	Relative minimum

With each critical point classified, the corresponding z values are

$$f(0,2) = 2^4 - 32\cdot 2 + 0^3 - 0^2 = -48$$

$$f\left(\tfrac{2}{3},2\right) = 2^4 - 32\cdot 2 + \left(\tfrac{2}{3}\right)^3 - \left(\tfrac{2}{3}\right)^2 = -\tfrac{1300}{27} \approx -48.15$$

When a business produces several products, a multivariable profit function may be calculated to find the production levels that maximize profit.

Example 7 Maximize Profit

A small startup company produces speakers and subwoofers for computers that they sell through a website. After extensive research, the company has developed a revenue function,

$$R(x, y) = x(110 - 4.5x) + y(155 - 2y) \quad \text{thousand dollars}$$

where x is the number of subwoofers produced and sold in thousands and y is the number of speakers produced and sold in thousands. The corresponding cost function is

$$C(x, y) = 3x^2 + 3y^2 + 5xy - 5y + 50 \quad \text{thousand dollars}$$

Find the production levels that maximize revenue.

Solution By subtracting the cost from the revenue, we get the profit function,

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= x(110 - 4.5x) + y(155 - 2y) - \left(3x^2 + 3y^2 + 5xy - 5y + 50\right) \\ &= 110x - 4.5x^2 + 155y - 2y^2 - 3x^2 - 3y^2 - 5xy + 5y - 50 \\ &= -7.5x^2 - 5y^2 - 5xy + 110x + 150y - 50 \end{aligned}$$

The first partial derivatives are

$$\begin{aligned} P_x(x, y) &= -15x - 5y + 110 \\ P_y(x, y) &= -10y - 5x + 150 \end{aligned}$$

The critical point are found by setting the partial derivatives equal to zero. This results in a system of equations that is solved using the Elimination Method.

$$\begin{array}{l} -15x-5y+110=0 \\ -10y-5x+150=0 \end{array} \rightarrow \begin{array}{r} 15x+5y=110 \\ 5x+10y=150 \end{array}$$

Multiply the second equation by -3 and add it to the first equation to eliminate x:

$$\begin{array}{r} 15x+\ \ 5y=110 \\ \underline{-15x-30y=-450} \\ -25y=-340 \end{array} \quad \rightarrow \quad y=13.6$$

If we substitute this value in the second equation, we get a value for x,

$$\begin{aligned} 5x+10(13.6)&=150 \\ 5x+\quad 136\quad &=150 \\ 5x&=14 \\ x&=2.8 \end{aligned}$$

The critical point is at $(2.8, 13.6)$. This point could be a relative maximum, a relative minimum, or a saddle point. The Test for Relative Extrema helps us to distinguish whether the point is a relative maximum. The second partial derivatives are

$$P_{xx}(x,y)=-15 \qquad P_{yy}(x,y)=-10 \qquad P_{xy}(x,y)=-5$$

Each of these derivatives is a constant so any critical points have

$$D=\underbrace{(-15)}_{P_{xx}}\underbrace{(-10)}_{P_{yy}}-\underbrace{[-5]}_{P_{xy}}{}^2=125$$

Since $D>0$ and $P_{xx}<0$, the critical point is a relative maximum. The profit at these production levels is

$$\begin{aligned} P(2.8, 13.6) &= -7.5(2.8)^2 - 5(13.6)^2 - 5(2.8)(13.6) + 110(2.8) + 150(13.6) - 50 \\ &= -6326 \end{aligned}$$

At a production level of 2.8 thousand subwoofers and 13.6 thousand speakers, the company will lose 6326 thousand dollars. The company will need to reassess their business model in order to make positive profit. $\blacksquare$