FUNCTIONS OF SEVERAL VARIABLES AND PARTIAL DIFFERENTIATION

1. Functions of Several Variables

A *function of two variables* is a rule that assigns a real number $f(x, y)$ to each ordered pair of real numbers (x, y) in the domain of the function. For a function f defined on the domain $D \subset \mathbb{R}^2$, we sometimes write $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ to indicate that f maps points in two dimensions to real numbers. You may think of such a function as a rule whose input is a pair of real numbers and whose output is a single real number.

Example 1.1. Functions $f(x, y) = xy^2$ and $g(x, y) = x^2 - e^y$ are both functions of the two variables x and y.

Likewise, a *function of three variables* is a rule that assigns a real number $f(x, y, z)$ to each ordered triple of real numbers (x, y, z) in the domain $D \subset \mathbb{R}^3$, we sometimes write $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ to indicate that f maps points in three dimensions to real numbers.

Example 1.2. Functions $f(x, y, z) = xy^2 \cos z$ and $g(x, y, z) = 3zx^2 - e^y$ are both functions of the three variables x, y and z.

We can similarly define functions of four (or five or more) variables. Our focus here is on functions of two and three variables, although most of our results can be easily extended to higher dimensions. Unless specifically stated otherwise, the domain of a function of several variables is taken to be the set of all values of the variables for which the given expression is defined.

Example 1.3. Find and sketch the domain for

(1) $f(x, y) = x \ln y$.
(2) $f(x, y) = \dfrac{2x}{y - x^2}$.
(3) $f(x, y) = \sqrt{4 - x^2 - y^2}$.
(4) $f(x, y) = \dfrac{1}{\sqrt{9 - x^2 - y^2}}$.
(5) $f(x, y) = \dfrac{xy}{x^2 - y}$.
(6) $f(x, y) = \ln(x + y)$.
(7) $f(x, y, z) = \dfrac{\cos(x + z)}{xy}$.
(8) $f(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2}$.
(9) $f(x, y, z) = xy \ln z$.

2. Limits and Continuity

Recall that for a function of a single variable, if we write $\lim_{x\to a} f(x) = L$, we mean that as x gets closer and closer to a, $f(x)$ gets closer and closer to the number L. Here, for functions of several variables, the idea is very similar. When we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

we mean that as (x, y) gets closer and closer to (a, b), $f(x, y)$ is getting closer and closer to the number L.

Theorem 2.1. *If $f(x, y)$ and $g(x, y)$ both have limits as (x, y) approaches (a, b), we have*

(i) $\displaystyle\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y),$

(ii) $\displaystyle\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right],$

(iii) $\displaystyle\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$, *provided that* $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) \neq 0.$

Example 2.1. Evaluate the limit whenever it exists.

(1) $\displaystyle\lim_{(x,y)\to(a,b)} x.$

(2) $\displaystyle\lim_{(x,y)\to(a,b)} y.$

(3) $\displaystyle\lim_{(x,y)\to(2,3)} (xy - 2).$

(4) $\displaystyle\lim_{(x,y)\to(-1,\pi)} (\sin xy - x^2 y).$

(5) $\displaystyle\lim_{(x,y)\to(\sqrt{3},-1)} \sqrt{x^2 - y^3}.$

(6) $\displaystyle\lim_{(x,y)\to(\infty,1)} \tan^{-1}\left(\frac{x}{y}\right).$

(7) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2 - 2xy + y^2}{x - y}.$

(8) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{2x^2y + 3xy}{5xy^2 + 3y}.$

(9) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$

Remark. If there is any way to approach the point (a, b) without the function values approaching the value L (for example, by virtue of the function values blowing up, oscillating or by approaching some other value), then the limit will not equal L. For the limit to equal L, the function has to approach L along every possible path. This gives us a simple method for determining that a limit does not exist.

Theorem 2.2. *If $f(x, y)$ approaches L_1 as (x, y) approaches (a, b) along a path γ_1 and $f(x, y)$ approaches $L_2 \neq L_1$ as (x, y) approaches (a, b) along a path γ_2, then* $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ *does not exist.*

Example 2.2. Evaluate the limit whenever it exists.

(1) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1}$.

(2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

(3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

(4) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

Theorem 2.3 (Sandwich Theorem)**.** *Suppose that* $|f(x,y)-L| \le g(x,y)$ *for all* (x,y) *in the interior of some circle centered at* (a,b)*, except possibly at* (a,b)*. If* $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = 0$*, then* $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = L$*.*

Example 2.3. Evaluate the limit whenever it exists.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$.

(2) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}$.

(3) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x-1)(y+2)}{(x-1)^2+(y+2)^2}$.

(4) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x^2+y^2-2x+1}{y^2-x^2+2x-1}$.

Remark. If you cannot find $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ using rectangular coordinates (x and y), you may change the problem into the polar coordinates (r and θ) using $x = r\cos\theta$ and $y = r\sin\theta$.

Example 2.4. Evaluate the limit whenever it exists.

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$.

(2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x-1)(y+2)}{(x-1)^2+(y+2)^2}$.

Suppose that $f(x,y)$ is defined in the interior of a circle centered at the point (a,b). We say that f is *continuous* at (a,b) if $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. If $f(x,y)$ is not continuous at (a,b), then we call (a,b) a *discontinuity* of f.

Example 2.5. Find all points where the given function is continuous

(1) $f(x,y) = \dfrac{x}{x^2-y}$.

(2) $f(x,y) = \begin{cases} \dfrac{x^4}{x(x^2+y^2)} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$

(3) $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = 0 \end{cases}$

(4) $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^{12}+y^4} & \text{if} \quad (x,y) \neq 0 \\ 0 & \text{if} \quad (x,y) = 0 \end{cases}$

(5) $f(x,y) = \begin{cases} \dfrac{xy^3}{x^4+y^4} & \text{if} \quad (x,y) \neq 0 \\ 0 & \text{if} \quad (x,y) = 0 \end{cases}$

(6) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if} \quad (x,y) \neq 0 \\ 0 & \text{if} \quad (x,y) = 0 \end{cases}$

(7) $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if} \quad (x,y) \neq 0 \\ 1 & \text{if} \quad (x,y) = 0 \end{cases}$

(8) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if} \quad (x,y) \neq 0 \\ 0 & \text{if} \quad (x,y) = 0 \end{cases}$

Theorem 2.4. *Suppose that $f(x,y)$ is continuous at (a,b) and $g(x)$ is continuous at the point $f(a,b)$. Then*

$$h(x,y) = (g \circ f)(x,y) = g(f(x,y))$$

is continuous at (a,b).

Example 2.6. Determine where $f(x,y) = e^{x^2y}$ is continuous.

All of the foregoing analysis is extended to functions of three (or more) variables in the obvious fashion.

Example 2.7. Evaluate $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2-z^2}{x^2+y^2+z^2}$.

Example 2.8. Find all points where $f(x,y,z) = \ln(9-x^2-y^2-z^2)$ is continuous.

3. Partial Derivatives

Let $f(x,y)$ be a multi-variable function. The *partial derivative of $f(x,y)$ with respect to x*, written $\dfrac{\partial f}{\partial x}$, is defined by

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h},$$

for any values of x and y for which the limit exists. The *partial derivative of $f(x,y)$ with respect to y*, written $\dfrac{\partial f}{\partial y}$, is defined by

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h},$$

for any values of x and y for which the limit exists.

To compute the partial derivative $\dfrac{\partial f}{\partial x}$, we simply take an ordinary derivative with respect to x, while treating y as a constant. Similarly, we can compute $\dfrac{\partial f}{\partial y}$ by taking an ordinary derivative with respect to y, while treating x as a constant.

Notation. If $z = f(x, y)$, then

$$\frac{\partial z}{\partial x} = z_x = f_x = f_1(x, y) = D_x(f(x, y)) = \frac{\partial f}{\partial x}(x, y),$$

and

$$\frac{\partial z}{\partial y} = z_y = f_y = f_2(x, y) = D_y(f(x, y)) = \frac{\partial f}{\partial y}(x, y).$$

Example 3.1. For $f(x, y) = 3x^2+x^3y+4y^2$, compute $\dfrac{\partial f}{\partial x}(x, y)$, $\dfrac{\partial f}{\partial y}(x, y)$, $f_x(1, 0)$ and $f_y(2, -1)$.

Example 3.2. Find all partial derivatives of f, if

(1) $f(x, y) = e^{xy} + \dfrac{x}{y}$,
(2) $f(x, y) = e^x \ln(x^2 + y^2 + 1)$,
(3) $f(x, y) = \sqrt{9 - x^2 - y^2}$,
(4) $f(x, y) = e^{x \ln y}$,
(5) $f(x, y) = \dfrac{x}{x^2 + y^2}$,
(6) $f(x, y) = \ln(\sec(xy) + \tan(xy))$,
(7) $f(x, y, z, w) = x^2 e^{2y+3z} \cos(4w)$,
(8) $f(x, y, z) = z \arcsin\left(\dfrac{y}{x}\right)$,
(9) $f(x, y, z) = x^{yz} + xz + xy + yz + \dfrac{x}{y} + \dfrac{y}{z} - \dfrac{z}{x} + \arcsin(xyz) + x^{\pi} + \pi^z$,
(10) $f(x, y, z) = (xy)^{\sin z}$,

Notice that the partial derivatives found in the preceding examples are themselves functions of two (or more) variables. We have seen that second- and higher-order derivatives of functions of a single variable provide much significant information. Not surprisingly, *higher-order partial derivatives* are also very important in applications.

For functions of two variables, there are four different second-order partial derivatives. The partial derivative with respect to x of $\dfrac{\partial f}{\partial x}$ is $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$, usually abbreviated as $\dfrac{\partial^2 f}{\partial x^2}$ or $(f_x)_x$ or f_{xx} or f_{11}. Similarly, taking two successive partial derivatives with respect to y gives us $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy} = f_{22}$. For *mixed second-order partial derivatives*, one derivative is taken with respect to each variable. If the first partial derivative is taken with respect to x, we have $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy} = f_{12}$. If the first partial derivative is taken with respect to y, we have $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx} = f_{21}$.

Example 3.3. Find all second-order partial derivatives of f, if

(1) $f(x, y) = x^2y - y^3 + \ln x$.
(2) $f(x, y) = x^4y^2 - 3x^2y^3 + 6xy$.
(3) $f(x, y) = x^3 \sin y + y^3 \cos x$.
(4) $f(x, y) = \arctan(xy)$.
(5) $f(x, y, z) = e^{xyz}$.
(6) $f(x, y, z) = xy \ln z$.

Theorem 3.1. *If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous on an open disk containing (a, b), then $f_{xy}(a, b) = f_{yx}(b, a)$.*

Remark. We can, of course, compute third-, fourth- or even higher-order partial derivatives. The theorem above can be extended to show that as long as the partial derivatives are all continuous in an open set, the order of differentiation does not matter. With higher-order partial derivatives, notations such as $\dfrac{\partial^3 f}{\partial x \partial y \partial x}$ become quite awkward and so, we usually use f_{xyx} instead.

Example 3.4. Compute

(1) f_{xyy} and f_{xyyy} for $f(x, y) = \cos(xy) - x^3 + y^4$.
(2) f_x, f_{xy} and f_{xyz} for $f(x, y, z) = \sqrt{xy^3z} + 4x^2y$.
(3) u_{xxy} for $u = 2^{xy}$.
(4) u_{xyx} for $u = \cos(x + \sin y)$.
(5) $\dfrac{\partial^5 u}{\partial x^3 \partial y^2}$ for $u = \sin^2 x \cos^2 y$.
(6) $\dfrac{\partial^3 u}{\partial x \partial y \partial z}$ for $u = \sin(xy) + \cos(xz) + \tan(yz)$.

4. The Chain Rule

Let us recall the chain rule for functions of one variable: If f and g are differentiable functions,we have

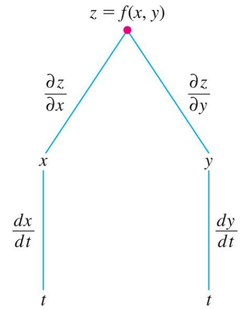
$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

We now extend the chain rule to functions of several variables.

Theorem 4.1 (Chain Rule)**.** *If $z = f(x(t), y(t))$, where $x(t)$ and $y(t)$ are differentiable and $f(x, y)$ is a differentiable function of x and y, then*

$$\frac{dz}{dt} = \frac{d}{dt}[f(x(t), y(t))] = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}.$$

As a convenient device for remembering the chain rule, we sometimes use a *tree diagram* like the one shown below.



Notice that if $z = f(x, y)$ and x and y are both functions of the variable t, then t is the independent variable. We consider x and y to be *intermediate variables*, since they both depend on t. In the tree diagram, we list the dependent variable z at the top, followed by each of the intermediate variables x and y, with the independent variable t at the bottom level, with each of the variables connected by a path. Next to each of the paths, we indicate the corresponding derivative. The chain rule then gives $\dfrac{dz}{dt}$ as the sum of all of the products of the derivatives along each path to t. That is,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 4.1. Find

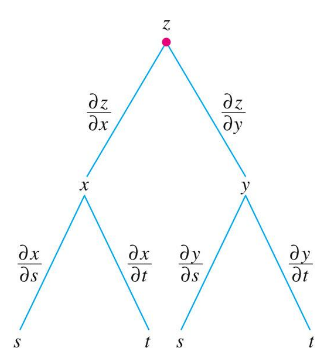
(1) $\dfrac{dz}{dt}$, if $z = \ln(3x^2 + y^3)$, $x = e^{2t}$, $y = t^{\frac{1}{3}}$.

(2) $\dfrac{dz}{dt}$, if $z = x^2 e^y$, $x = t^2 - 1$, $y = \sin t$.

Theorem 4.2 (Chain Rule)**.** *Suppose that $z = f(x, y)$, where f is a differentiable function of x and y where $x = x(s, t)$ and $y = y(s, t)$ both have first-order partial derivatives. Then we have the chain rules:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The tree diagram shown below as a convenient reminder of the chain rules indicated in the previous theorem, again by summing the products of the indicated partial derivatives along each path from z to s or t, respectively.



The chain rule is easily extended to functions of three or more variables.

Example 4.2. Find

(1) $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ if $z = e^{xy}$, $x(u,v) = 3u\sin v$ and $y(u,v) = 4v^2u$.
(2) $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ if $z = 16 - 4x^2 - y^2$, $x = u\sin v$ and $y = v\cos u$.
(3) $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ if $w = \sqrt{x^2+y^2+z^2}$, $x = e^r\cos s$, $y = e^r\sin s$, and $z = e^s$.

Example 4.3. For a differentiable function $f(x,y)$ with $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_r = f_x\cos\theta + f_y\sin\theta$ and $f_{rr} = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$.

Example 4.4. For a differentiable function $f(x,y)$ with $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_{xx} + f_{yy} = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}f_{\theta\theta}$.

Example 4.5. (1) Let $z = f(x-y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of the derivatives of f.

(2) Let $z = f(x^2y, x+2y)$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ in terms of the derivatives of f.
(3) Let $w = f(x^2-y^2, 2xy)$. Find w_{xy} in terms of the derivatives of f.
(4) Let $f(x,y) = h\left(x^3y, \dfrac{1}{xy^2}\right)$. Calculate f_x, f_y and f_{yy} in terms of the partial derivatives of h.

Example 4.6. Given $z = f(u,v)$ where $u = xy$, $v = x^2 - y^2$ and $f_u(1,0) = 2$, $f_v(1,0) = 3$, $f_{uu}(1,0) = 0$, $f_{uv}(1,0) = 1$, $f_{vu}(1,0) = 1$, $f_{vv}(1,0) = -2$. Find

(1) z_x,
(2) z_{xy}

at the point $x = 1$, $y = 1$.

5. Implicit Differentiation

Suppose that the equation $F(x,y) = 0$ defines y implicitly as a function of x, say $y = f(x)$. In MCS 155, we saw how to calculate $\dfrac{dy}{dx}$ in such a case. We can use the chain rule for functions of several variables to obtain an alternative method for calculating this.
We let $z = F(x,y)$, where $x = t$ and $y = f(t)$. From Theorem 4.1, we have

$$\frac{dz}{dt} = F_x\frac{dx}{dt} + F_y\frac{dy}{dt}.$$

But, since $z = F(x,y) = 0$, we have $\dfrac{dz}{dt} = 0$, too. Further, since $x = t$, we have $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = \dfrac{dy}{dx}$. This leaves us with

$$0 = F_x + F_y\frac{dy}{dx}.$$

Notice that we can solve this for $\dfrac{dy}{dx}$, provided $F_y \neq 0$. In this case, we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

We can extend this notion to functions of several variables defined implicitly, as follows. Suppose that the equation $F(x, y, z) = 0$ implicitly defines a function $z = f(x, y)$, where f is differentiable. Then, we can find the partial derivatives f_x and f_y using the chain rule, as follows. We first let $w = F(x, y, z)$. From the chain rule, we have

$$\frac{\partial w}{\partial x} = F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x}.$$

Notice that since $w = F(x, y, z) = 0$, $\dfrac{\partial w}{\partial x} = 0$. Also, $\dfrac{\partial x}{\partial x} = 1$ and $\dfrac{\partial y}{\partial x} = 0$, since x and y are independent variables. This gives us

$$0 = F_x + F_z \frac{\partial z}{\partial x}.$$

We can solve this for $\dfrac{\partial z}{\partial x}$, as long as $F_z \neq 0$, to obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

Likewise, differentiating w with respect to y leads us to

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z},$$

again, as long as $F_z \neq 0$.

Example 5.1. (1) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, given that $F(x, y, z) = xy^2 + z^3 + \sin(xyz) = 0$.

(2) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, given that $x - yz + \cos(xyz) = x^2 z^2 + 1$.

(3) Find $\dfrac{\partial y}{\partial x}$, given that $F(x, y) = x^2 + y^2 + \sin(xy^2) - 15 = 0$.

(4) Given $x^2 + y^2 z + z^3 = 25$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(5) Given $\dfrac{xz^2}{x + y} + y^2 = 10$, find $\dfrac{\partial z}{\partial x}$ at $P(-1, 2, 2)$.

(6) Find the value of $\dfrac{\partial x}{\partial z}$ and $\dfrac{\partial^2 x}{\partial z^2}$ at the point $A(1, -1, -3)$ if the equation

$$xz + y \ln x - x^2 + 4 = 0$$

defines x as a function of two independent variables y and z.

6. The Gradient and Directional Derivatives

The *directional derivative* of $f(x, y)$ at the point (a, b) and in the direction of the unit vector $\overrightarrow{u} =< u_1, u_2 >$ is given by

$$D_{\overrightarrow{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

provided the limit exists.
It turns out that any directional derivative can be calculated simply, in terms of the first partial derivatives, as we see in the following theorem.

Theorem 6.1. *Suppose that f is differentiable at (a, b) and $\overrightarrow{u} =< u_1, u_2 >$ is any unit vector. Then, we can write*

$$D_{\overrightarrow{u}} f(a, b) = f_x(a, b)u_1 + f_y(a, b)u_2.$$

Example 6.1. For $f(x, y) = x^2y - 4y^3$, compute $D_{\overrightarrow{u}} f(2, 1)$ for the directions

(1) $\overrightarrow{u} =< \frac{\sqrt{3}}{2}, \frac{1}{2} >$, and
(2) $\overrightarrow{u}$ in the direction from $(2, 1)$ to $(4, 0)$.

For convenience, we define the *gradient* of a function to be the vector-valued function whose components are the first-order partial derivatives of f. We denote the gradient of a function f by $\text{grad}\, f$ or ∇f. Namely, The *gradient* of $f(x, y)$ is the vector-valued function

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \frac{\partial f}{\partial x} \overrightarrow{i} + \frac{\partial f}{\partial y} \overrightarrow{j},$$

provided both partial derivatives exist.

Theorem 6.2. *If f is a differentiable function of x and y and $\overrightarrow{u}$ is any unit vector, then*

$$D_{\overrightarrow{u}} f(x, y) = \nabla f(x, y) \cdot \overrightarrow{u}.$$

Example 6.2. Let $f(x, y) = x^2 + y^2$.

(1) Find the directional derivative of f at the point $(1, 2)$ in the direction of $\overrightarrow{a} = \overrightarrow{i} + 2\overrightarrow{j}$.
(2) Find $D_{\overrightarrow{u}} f(1, -1)$ for
 (a) $\overrightarrow{u}$ in the direction of $\overrightarrow{v} =< -3, 4 >$ and
 (b) $\overrightarrow{u}$ in the direction of $\overrightarrow{v} =< 3, -4 >$.

Example 6.3. Let $x - yz + 2\cos(xyz) = \pi$. Find the directional derivative of z at the point $P(\pi, 1, 2)$ in the direction of $\overrightarrow{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$.

Keep in mind that a directional derivative gives the rate of change of a function in a given direction. So, its reasonable to ask in what direction a given function has its maximum or minimum rate of increase. First, recall that for any two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, we have $\overrightarrow{a} \cdot \overrightarrow{b} = \|\overrightarrow{a}\|\|\overrightarrow{b}\| \cos\theta$, where θ is the angle between the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$. Then, we have

$$\begin{aligned} D_{\overrightarrow{u}} f(a, b) &= \nabla f(a, b) \cdot \overrightarrow{u} \\ &= \|\nabla f(a, b)\|\|\overrightarrow{u}\| \cos\theta = \|\nabla f(a, b)\| \cos\theta, \end{aligned}$$

where θ is the angle between the gradient vector at (a,b) and the direction vector $\overrightarrow{u}$. Notice now that $\|\nabla f(a,b)\|\cos\theta$ has its maximum value when $\theta = 0$, so that $\cos\theta = 1$. The directional derivative is then $\|\nabla f(a,b)\|$. Further, observe that the angle $\theta = 0$ when $\nabla f(a,b)$ and $\overrightarrow{u}$ are in the *same* direction, so that $\overrightarrow{u} = \dfrac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$. Similarly, the minimum value of the directional derivative occurs when $\theta = \pi$, so that $\cos\theta = -1$. In this case, $\nabla f(a,b)$ and $\overrightarrow{u}$ have *opposite* directions, so that $\overrightarrow{u} = -\dfrac{\nabla f(a,b)}{\|\nabla f(a,b)\|}$. Finally, observe that when $\theta = \dfrac{\pi}{2}$, $\overrightarrow{u}$ is perpendicular to $\nabla f(a,b)$ and the directional derivative in this direction is zero. We summarize these observations in the following theorem

Theorem 6.3. *Suppose that f is a differentiable function of x and y at the point (a,b). Then*

(i) *the maximum rate of change of f at (a,b) is $\|\nabla f(a,b)\|$, occurring in the direction of the gradient;*

(ii) *the minimum rate of change of f at (a,b) is $-\|\nabla f(a,b)\|$, occurring in the direction opposite the gradient;*

(iii) *the rate of change of f at (a,b) is 0 in the directions orthogonal to $\nabla f(a,b)$.*

Remark. In using Theorem 6.3, remember that the directional derivative corresponds to the rate of change of the function $f(x,y)$ in the given direction.

Example 6.4. Suppose $f(x,y,z) = x^3y + 3x^2y^2z$. Find the directional derivative of f at $(1,1,-1)$ in the direction of the gradient.

Example 6.5. Find the maximum and minimum rates of change of the function $f(x,y) = x^2 + y^2$ at the point $(1,3)$.

Example 6.6. Let $f(x,y) = 3x^2 + 4y^2$ and $P_0(-1,1)$ be given.

(1) Find the direction in which f increases most rapidly and what is the directional derivative of f in this direction.

(2) Find the direction in which f decreases most rapidly and what is the directional derivative of f in this direction.

(3) Identify the directions in which the directional derivative of f is zero.

The *directional derivative* of $f(x,y,z)$ at the point (a,b,c) and in the direction of the unit vector $\overrightarrow{u} =< u_1, u_2, u_3 >$ is given by

$$D_{\overrightarrow{u}} f(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h},$$

provided the limit exists. The *gradient* of $f(x,y,z)$ is the vector-valued function

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k},$$

provided all the partial derivatives are defined.

Theorem 6.4. *If f is a differentiable function of x, y and z and $\overrightarrow{u}$ is any unit vector, then*

$$D_{\overrightarrow{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \overrightarrow{u}.$$

As in two dimensions, we have that

$$\begin{aligned} D_{\overrightarrow{u}} f(x,y,z) &= \nabla f(x,y,z) \cdot \overrightarrow{u} \\ &= \|\nabla f(x,y,z)\| \|\overrightarrow{u}\| \cos\theta \\ &= \|\nabla f(x,y,z)\| \cos\theta \end{aligned}$$

where θ is the angle between the vectors $\nabla f(x,y,z)$ and $\overrightarrow{u}$. For precisely the same reasons as in two dimensions, it follows that the direction of maximum increase at any given point is given by the gradient at that point.

Example 6.7. Find the gradient of $f(x,y,z) = x^3y^2z$ at $P(2,-1,2)$.

Example 6.8. Find the directional derivative of $f(x,y,z) = x^3y^2z$ at the point $P(2,-1,2)$ in the direction of $\overrightarrow{u} = 2\overrightarrow{i} - \overrightarrow{j} - 2\overrightarrow{k}$.

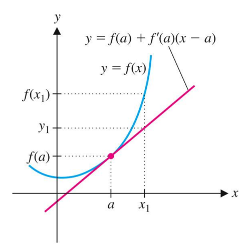
Example 6.9. If the temperature at point (x,y,z) is given by

$$T(x,y,z) = 85 + \left(1 - \frac{z}{100}\right) e^{-(x^2+y^2)},$$

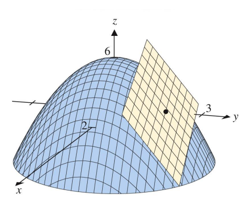
find the direction from the point $(2,0,99)$ in which the temperature increases most rapidly.

7. Tangent Planes and Linear Approximations

Recall that the tangent line to the curve $y = f(x)$ at $x = a$ stays close to the curve near the point of tangency. This enables us to use the tangent line to approximate values of the function close to the point of tangency.



The equation of the tangent line is given by $y = f(a) + f'(a)(x-a)$ We called this before the linear approximation to $f(x)$ at $x = a$. In much the same way, we can approximate the value of a function of two variables near a given point using the tangent plane to the surface at that point. For instance, the graph of $z = 6 - x^2 - y^2$ and its tangent plane at the point $(1,2,1)$ are shown in the figure below.



Notice that near the point $(1,2,1)$, the surface and the tangent plane are very close together.

Theorem 7.1. *Suppose that $f(x, y)$ has continuous first partial derivatives at (a, b). A normal vector to the tangent plane to $z = f(x, y)$ at (a, b) is then $< f_x(a, b), f_y(a, b), -1 >$. Further, an equation of the tangent plane is given by*

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

or

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

The line orthogonal to the tangent plane and passing through the point $(a, b, f(a, b))$ is then given by

$$x = a + f_x(a, b)t,\ y = b + f_y(a, b)t,\ z = f(a, b) - t.$$

This line is called the *normal line* to the surface at the point $(a, b, f(a, b))$.

Example 7.1. Find equations of the tangent plane and the normal line to

(1) $z = 6 - x^2 - y^2$ at the point $(1, 2, 1)$.
(2) $z = x^3 + y^3 + \dfrac{x^2}{y}$ at $(2, 1, 13)$.

The equation $f(x, y, z) = c$, where c is a constant also defines a surface in space. Now, suppose that $\overrightarrow{u}$ is any unit vector lying in the tangent plane to the surface $f(x, y, z) = c$ at a point (x_0, y_0, z_0) on the surface. Then, it follows that the rate of change of f in the direction of $\overrightarrow{u}$ at (x_0, y_0, z_0) [given by the directional derivative $D_{\overrightarrow{u}} f(x_0, y_0, z_0)$] is zero, since f is constant on a level surface. We now have that

$$0 = D_{\overrightarrow{u}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \overrightarrow{u}.$$

This occurs only when the vectors $\nabla f(x_0, y_0, z_0)$ and $\overrightarrow{u}$ are orthogonal. Since $\overrightarrow{u}$ was taken to be any vector lying in the tangent plane, we now have that $\nabla f(x_0, y_0, z_0)$ is orthogonal to every vector lying in the tangent plane at the point (x_0, y_0, z_0). Observe that this says that $\nabla f(x_0, y_0, z_0)$ is a normal vector to the tangent plane to the surface $f(x, y, z) = c$ at the point (x_0, y_0, z_0). Thus, we have

Theorem 7.2. *Suppose that $f(x, y, z)$ has continuous partial derivatives at the point (x_0, y_0, z_0) and $\nabla f(x_0, y_0, z_0) \neq 0$. Then, $\nabla f(x_0, y_0, z_0)$ is a normal vector to the tangent plane to the surface $f(x, y, z) = c$, at the point (x_0, y_0, z_0). Further, the equation of the tangent plane is*

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$

We refer to the line through (x_0, y_0, z_0) in the direction of $\nabla f(x_0, y_0, z_0)$ as the *normal line* to the surface at the point (x_0, y_0, z_0). Observe that this has parametric equations

$$x = x_0 + f_x(x_0, y_0, z_0)t, \quad y = y_0 + f_y(x_0, y_0, z_0)t, \quad z = z_0 + f_z(x_0, y_0, z_0)t.$$

Example 7.2. Find equations of the tangent plane and the normal line to

(1) $x^3y - y^2 + z^2 = 7$ at the point $P_0(1, 2, 3)$.
(2) $x^2 + xyz - z^3 = 1$ at the point $P_0(1, 1, 1)$.

Example 7.3. Two surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0$, $g(x, y, z) = x + z - 4 = 0$ intersect in a curve C. Find the line tangent to C at the point $P_0(1, 1, 3)$.

Before we found that a normal vector to the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is given by

$$\left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), -1 \right\rangle .$$

Note that this is simply a special case of the gradient formula of Theorem 7.2, as follows. First, observe that we can rewrite the equation $z = f(x, y)$ as $f(x, y) - z = 0$. We can then think of this surface as the surface $g(x, y, z) = f(x, y) - z = 0$, which at the point $(a, b, f(a, b))$ has normal vector

$$\nabla g(a, b, f(a, b)) = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b), -1 \right\rangle .$$

Example 7.4. Find an equation of the tangent plane to $z = \sin(x + y)$ at the point $(\pi, \pi, 0)$.

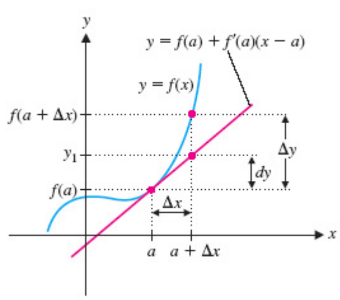
Example 7.5. Find equations for the tangent plane and normal line to the surface $z = f(x, y) = 9 - x^2 - y^2$ at the point $P_0(1, 1, 7)$.

The tangent planes stay close to the surface near the point of tangency. This says that the z-values on the tangent plane should be close to the corresponding z-values on the surface, which are given by the function values $f(x, y)$, at least for (x, y) close to the point of tangency. Further, the simple form of the equation for the tangent plane makes it ideal for approximating the value of complicated functions. We define the *linear approximation* $L(x, y)$ of $f(x, y)$ at the point (a, b) to be the function defining the z-values on the tangent plane, namely,

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

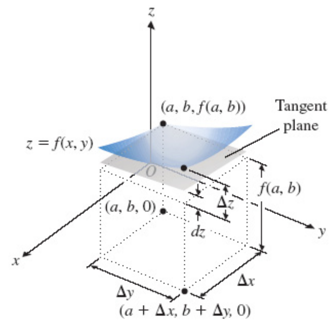
Example 7.6. Compute the linear approximation of $f(x, y) = 2x + e^{x^2-y}$ at $(0, 0)$.

Increments and Differentials. We defined before that the *increment* Δy of the function $f(x)$ at $x = a$ to be $\Delta y = f(a + \Delta x) - f(a)$. Referring to the figure below, notice that for Δx small, $\Delta y \approx dy = f'(a)\Delta x$, where we referred to dy as the *differential* of y.



For $z = f(x, y)$, we define the increment of f at (a, b) to be

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$



Theorem 7.3. *Suppose that $z = f(x,y)$ is defined on the rectangular region $R = \{(x,y)|\ x_0 < x < x_1, y_0 < y < y_1\}$ and f_x and f_y are defined on R and are continuous at $(a,b) \in R$. Then for $(a+\Delta x, b+\Delta y) \in R$,*

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy that both tend to zero, as $(\Delta x, \Delta y) \to (0,0)$.

Example 7.7. For $z = f(x,y) = x^2 - 5xy$, find Δz and write it in the form indicated in Theorem 7.3.

Look closely at the first two terms in the expansion of the increment Δz given in Theorem 7.3. If we take $\Delta x = x - a$ and $\Delta y = y - b$, then they correspond to the linear approximation of $f(x,y)$. In this context, we give this a special name. If we increment x by the amount $dx = \Delta x$ and increment y by $dy = \Delta y$, then we define the *differential* of z to be

$$\Delta z = f_x(x,y)dx + f_y(x,y)dy.$$

Corollary 7.4. *Suppose that $z = f(x,y)$ is defined on a region R and f_x and f_y are defined on R and are continuous at an interior point (a,b) of R. Then for $(x,y) \in R$ and close to (a,b),*

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b),$$

Example 7.8. Approximate

(1) $\sqrt{(1.06)^2 + (1.97)^3}$.
(2) $(0.98)^{1.05}$.
(3) $\sin(31°)\cos(58°)$.

The idea of a linear approximation extends easily to three or more dimensions. We lose the graphical interpretation of a tangent plane approximating a surface, but the definition should make sense.

Theorem 7.5. *Suppose that $w = f(x,y,z)$ is defined on a region R and f_x, f_y and f_z are defined on R and are continuous at an interior point (a,b,c) of R. Then for $(x,y,z) \in R$ and close to (a,b),*

$$f(x,y,z) \approx L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c).$$

Example 7.9. Approximate $(1.002)(2.003)^2(3.004)^3$

8. Extrema of Functions of Several Variables

Local Extrema. Let $f(x, y)$ be defined on a region R containing the point (x_0, y_0). Then

- $f(x_0, y_0)$ is a *local maximum* value of f if $f(x_0, y_0) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (x_0, y_0).
- $f(x_0, y_0)$ is a *local minimum* value of f if $f(x_0, y_0) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (x_0, y_0).
- $f(x_0, y_0)$ is a *local extremum* value of f if either $f(x_0, y_0)$ is a local maximum or a local minimum value of f.

The point (a, b) is a *critical point* of the function $f(x, y)$ if (a, b) is in the domain of f and either $\dfrac{\partial f}{\partial x}(a, b) = \dfrac{\partial f}{\partial y}(a, b) = 0$ or one or both of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial x}$ do not exist at (a, b).

Theorem 8.1 (First Derivative Test for Local Extreme Values)**.** *If $f(x, y)$ has a local extremum at (a, b), then (a, b) must be a critical point of f.*

Remark. Although local extrema can occur only at critical points, every critical point need not correspond to a local extremum. For this reason, we refer to critical points as *candidates* for local extrema.

Example 8.1. Find all critical points of $f(x, y) = xe^{-\frac{x^2}{2} - \frac{y^3}{3} + y}$.

A differentiable function $f(x, y)$ has a *saddle point* at a critical point (x_0, y_0) if in every open disk centered at (x_0, y_0) there are domain points (x, y) where $f(x, y) > f(x_0, y_0)$ and domain points (x, y) where $f(x, y) < f(x_0, y_0)$. The corresponding point $(x_0, y_0, f(x_0, y_0))$ on the surface $z = f(x, y)$ is called a saddle point of the surface.

Theorem 8.2 (Second Derivative Test for Local Extreme Values)**.** *Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (x_0, y_0) and that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Define the* discriminant *D for the point (x_0, y_0) by*

$$D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2 .$$

(i) *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$ (or, $f_{yy}(x_0, y_0) > 0$), then f has a local minimum at (x_0, y_0).*
(ii) *If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$ (or, $f_{yy}(x_0, y_0) < 0$), then f has a local maximum at (x_0, y_0).*
(iii) *If $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f. (In this case, f has neither maximum nor minimum at (x_0, y_0).)*
(iv) *If $D(x_0, y_0) = 0$, then no conclusion can be drawn.*

Example 8.2. Find and classify all the critical points of f, if

(1) $f(x, y) = 2x^2 - y^3 - 2xy$.
(2) $f(x, y) = x^3 - 2y^2 - 2y^4 + 3x^2 y$.
(3) $f(x, y) = 4xy - x^4 - y^4$.
(4) $f(x, y) = x^2 y - 2yx + 2y^2 - 15y$.
(5) $f(x, y) = x^3 + y^3 - 3xy$.
(6) $f(x, y) = x^3 + y^3 + 3x^2 y - 15y^2 + 2$.

(7) $f(x, y) = x^3 + 3x^2y + 3y^2 + 2y^3 + 10$.
(8) $f(x, y) = 2x^4 + y^4 - x^2 - 2y^2$.
(9) $f(x, y) = x(1 - x^2 - y^2)$.
(10) $f(x, y) = xy(9 - x - y)$.

Absolute Maxima and Minima on Closed Bounded Regions. We call $f(x_0, y_0)$ the *absolute maximum* of f on the region R if $f(x_0, y_0) \geq f(x, y)$ for all $(x, y) \in R$. Similarly, $f(x_0, y_0)$ is called the *absolute minimum* of f on R if $f(x_0, y_0) \leq f(x, y)$ for all $(x, y) \in R$. In either case, $f(x_0, y_0)$ is called an *absolute extremum* of f.

Theorem 8.3 (Extreme Value Theorem)**.** *Suppose that $f(x, y)$ is continuous on the closed and bounded region $R \subset \mathbb{R}^2$. Then f has both an absolute maximum and an absolute minimum on R. Further, an absolute extremum may only occur at a critical point in R or at a point on the boundary of R.*

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

(1) Find all critical points of f in the region R.
(2) Find the maximum and minimum values of f on the boundary of R.
(3) Compare the values of f at the critical points with the maximum and minimum values of f on the boundary of R.

Example 8.3. Find the absolute extrema of f on R, if

(1) $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$, R is the region bounded by the lines $y = 2$, $y = x$ and $y = -x$.
(2) $f(x, y) = 2xy + y^2 + 8x - 4y,\ \ R = \{(x, y) \mid 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$.
(3) $f(x, y) = xy(3 - x - y),\ \ R = \{(x, y) \mid 0 \leq x \leq 4,\ 0 \leq y \leq 4 - x\}$.
(4) $f(x, y) = x^2 - y^2,\ \ R = \{(x, y) \mid x^2 + y^2 \leq 1\}$.
(5) $f(x, y) = xy^2,\ \ R = \{(x, y) \mid x^2 + y^2 \leq 3\}$.

9. Constrained Optimization and Lagrange Multipliers

Theorem 9.1. *Suppose that $f(x, y, z)$ and $g(x, y, z)$ are functions with continuous first order partial derivatives and $\nabla g(x, y, z) \neq 0$ on the surface $g(x, y, z) = 0$. Suppose that either*

(i) *the minimum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0); or*
(ii) *the maximum value of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0).*

*Then $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$, for some constant λ (called a **Lagrange multiplier**).*

For functions of two independent variables, the condition is similar, but without the variable z.

Example 9.1. (1) Find the point on the line $y = 3 - 2x$ that is closest to the origin.
(2) Suppose that the temperature of a metal plate is given by $T(x, y) = x^2 + 2x + y^2$, for points (x, y) on the elliptical plate defined by $x^2 + 4y^2 \leq 24$. Find the maximum and minimum temperatures on the plate.

(3) For a business that produces three products, suppose that when producing x, y and z thousand units of the products, the profit of the company (in thousands of dollars) can be modeled by $P(x, y, z) = 4x+8y+6z$. Manufacturing constraints force $x^2+4y^2+2z^2 \leq 800$. Find the maximum profit for the company. Rework the problem with the constraint $x^2 + 4y^2 + 2z^2 \leq 801$ and use the result to interpret the meaning of λ.
(4) Find the maximum and minimum values of $f(x, y, z) = 2x - 3y + z$ on the sphere $x^2 + y^2 + z^2 = 14$.
(5) Find the extreme values of $f(x, y, z) = xy+yz$ subject to the constraint $x^2+y^2+z^2 = 8$.
(6) Find the extreme values of $f(x, y) = x^2y$ subject to the constraint $2x^2 + y^2 = 3$.
(7) Use the Lagrange multipliers rule to find the volume of the largest rectangular box with faces parallel to the coordinate planes that can be inscribed in the sphere $x^2+y^2+z^2 = 3$.

Lagrange Multipliers with Two Constraints. Suppose that $f(x, y, z)$, $g_1(x, y, z)$ and $g_2(x, y, z)$ are differentiable. To find the local maximum and minimum values of f subject to the constraints $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, find the values of x, y, z, λ, and μ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x, y, z) = 0 \quad \text{and} \quad g_2(x, y, z) = 0.$$

Example 9.2. (1) Find the minimum distance between the origin and a point on the intersection at a paraboloid $z = \frac{3}{2} - x^2 - y^2$ and the plane $x + 2y = 1$.
(2) The plane $x + y + z = 12$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the point on the ellipse that is closest to the origin.