Chapter 2

Metric and Topological Spaces

Topology begins where sets are implemented with some cohesive properties enabling one to define continuity.

SOLOMON LEFSCHETZ

In order to forge a language of continuity, we begin with familiar examples. Recall from single-variable calculus that a function \( f : \mathbb{R} \to \mathbb{R} \) is continuous at a point \( x_0 \in \mathbb{R} \) if for every \( \epsilon > 0 \), there is a \( \delta > 0 \) so that, whenever \( |x - x_0| < \delta \), we have \( |f(x) - f(x_0)| < \epsilon \). The route to generalization begins with the distance notion on the real line: the distance between the real numbers \( x \) and \( y \) is given by \( |x - y| \). The general properties of a distance are abstracted in the notion of a metric space, first introduced by MAURICE FRÉCHET (1878–1973) and named by Hausdorff.

**Definition 2.1.** A metric space is a set \( X \) together with a distance function \( d : X \times X \to \mathbb{R} \) satisfying

i) \( d(x, y) \geq 0 \) for all \( x, y \in X \) and \( d(x, y) = 0 \) if and only if \( x = y \).

ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \).
iii) The Triangle Inequality: \( d(x, y) + d(y, z) \geq d(x, z) \) for all \( x, y, z \in X \).

The **open ball** of radius \( \epsilon > 0 \) centered at a point \( x \) in a metric space \( (X, d) \) is given by

\[
B(x, \epsilon) = \{ y \in X \mid d(x, y) < \epsilon \},
\]

that is, the points in \( X \) within \( \epsilon \) in distance from \( x \).

The intuitive notion of 'near' can be made precise in a metric space: a point \( y \) is 'near' the point \( x \) if it is in \( B(x, \epsilon) \) for \( \epsilon \) suitably small.

**Examples.** (1) The most familiar example is \( \mathbb{R}^n \). If \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \), then the Euclidean metric is given by

\[
d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.
\]

In fact, one can endow \( \mathbb{R}^n \) with other metrics, for example,

\[
d_1(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.
\]

The nonnegative, nondegenerate, and symmetric conditions are clear for \( d_1 \). The triangle inequality follows in the same way as the proof in the next example.

Notice that an open ball with this metric is an 'open box' as pictured here in \( \mathbb{R}^2 \).

(2) Let \( X = \text{Bdd}([0, 1], \mathbb{R}) \) denote the set of *bounded functions* \( f : [0, 1] \to \mathbb{R} \), that is, functions \( f \) for which there is a real number
$M(f)$ such that $|f(t)| < M(f)$ for all $t \in [0, 1]$. Define the distance between two such functions to be

$$d(f, g) = \text{lub}_{t \in [0, 1]} \{|f(t) - g(t)|\}.$$ 

Certainly $d(f, g) \geq 0$ and $d(f, g) = 0$ if and only if $f = g$. Furthermore, $d(f, g) = d(g, f)$. The triangle inequality is more subtle:

$$d(f, h) = \text{lub}_{t \in [0, 1]} \{|f(t) - h(t)|\}$$

$$\leq \text{lub}_{t \in [0, 1]} \{|f(t) - g(t)| + |g(t) - h(t)|\}$$

$$\leq \text{lub}_{t \in [0, 1]} \{|f(t) - g(t)|\} + \text{lub}_{t \in [0, 1]} \{|g(t) - h(t)|\}$$

$$= d(f, g) + d(g, h).$$

An open ball in this metric space, $B(f, \epsilon)$, consists of all functions defined on $[0, 1]$ with graph in the stripe pictured:

(3) Let $X$ be any set and define

$$d(x, y) = \begin{cases} 
0, & \text{if } x = y, \\
1, & \text{if } x \neq y.
\end{cases}$$

This is a perfectly good distance function—open balls are funny, however—either they consist of one point or the whole space depending on whether $\epsilon \leq 1$ or $\epsilon > 1$. The resulting metric space is called the discrete metric space.

Using open balls, we can rewrite the definition of a continuous real-valued function $f: \mathbb{R} \to \mathbb{R}$ to say:

A function $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for any $\epsilon > 0$, there is a $\delta > 0$ so that $B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon))$. 
The step from this definition of continuity to a general definition of continuous mappings of metric spaces is clear.

**Definition 2.2.** Suppose \((X, d_X)\) and \((Y, d_Y)\) are two metric spaces and \(f: X \to Y\) is a function. Then \(f\) is **continuous at** \(x_0 \in X\) if for any \(\epsilon > 0\), there is a \(\delta > 0\) so that \(B(x_0, \delta) \subset f^{-1}(B(f(x_0), \epsilon))\). The function \(f\) is **continuous** if it is continuous at \(x_0\) for all \(x_0 \in X\).

For example, if \(X = Y = \mathbb{R}^n\) with the usual Euclidean metric \(d(x, y) = \|x - y\|\), then \(f: \mathbb{R}^n \to \mathbb{R}^n\) is continuous at \(x_0\) if for any \(\epsilon > 0\), there is a \(\delta > 0\) so that whenever \(x \in B(x_0, \delta)\), that is, \(\|x - x_0\| < \delta\), then \(x \in f^{-1}(B(f(x_0), \epsilon))\), which is to say, \(f(x) \in B(f(x_0), \epsilon)\), or \(\|f(x) - f(x_0)\| < \epsilon\). Thus we recover the \(\epsilon-\delta\) definition of continuity. We develop the generalization further.

**Definition 2.3.** A subset \(U\) of a metric space \((X, d)\) is **open** if for any \(u \in U\), there is an \(\epsilon > 0\) so that \(B(u, \epsilon) \subset U\).

We note the following properties of open subsets of metric spaces.

1) An open ball \(B(x, \epsilon)\) is an open set in \((X, d)\).
2) An arbitrary union of open subsets in a metric space is open.
3) The finite intersection of open subsets in a metric space is open.

Suppose \(y \in B(x, \epsilon)\). Let \(\delta = \epsilon - d(x, y) > 0\). Consider the open ball \(B(y, \delta)\). If \(z \in B(y, \delta)\), then \(d(z, y) < \delta = \epsilon - d(x, y)\), or \(d(z, y) + d(y, x) < \epsilon\). By the triangle inequality \(d(z, x) \leq d(z, y) + d(y, x)\) and so \(d(z, x) < \epsilon\) and \(B(y, \delta) \subset B(x, \epsilon)\). Thus \(B(x, \epsilon)\) is open.
Chapter 3

Geometric Notions

At the basis of the distance concept lies, for example, the concept of convergent point sequences and their defined limits, and one can, by choosing these ideas as those fundamental to point set theory, eliminate the notions of distance.

Felix Hausdorff

By choosing open sets as the basic notion we can generalize familiar analytic and geometric notions from Euclidean space to the new setting of topology. Two fundamental notions were introduced by Cantor in his work [13] on analysis. In the language of topology, these ideas have simple definitions.

Definition 3.1. Let \((X, T)\) be a topological space. A subset \(K\) of \(X\) is closed if its complement in \(X\) is open. If \(A \subseteq X\), where \(X\) is a topological space and \(x \in X\), then \(x\) is a limit point of \(A\), if, whenever \(U \subset X\) is open and \(x \in U\), there is some \(y \in U \cap A\), with \(y \neq x\).

Closed sets are the natural generalization of closed sets in \(\mathbb{R}^n\). Notice that an arbitrary subset of a topological space can be neither open nor closed, for example, \([a, b) \subset \mathbb{R}\) in the usual topology. A slogan to remember is that "a subset is not a door."
In a metric space the notion of a limit point $w$ of a subset $A$ is given by a sequence $\{x_i, i = 1, 2, \ldots\}$ with $x_i \in A$ for all $i$ and $\lim_{i \to \infty} x_i = w$. The limit is defined as usual: for any $\epsilon > 0$, there is an integer $N$ for which whenever $n \geq N$, we have $d(x_n, w) < \epsilon$. We distinguish two cases: If $w \in A$, then we can choose the constant sequence to converge to $w$. For $w \notin A$ to be a limit point, we want, for each $\epsilon > 0$, that there be some other point $a_\epsilon \in A$ with $a_\epsilon \neq w$ and $a_\epsilon \in B(w, \epsilon)$. When $w$ is a limit point of $A$, such points $a_\epsilon$ always exist. If we form the sequence $\{x_i = 1/i\}$, then $\lim_{i \to \infty} x_i = w$ follows. Conversely, if there is a sequence of infinitely many distinct points $x_i \in A$ with $\lim_{i \to \infty} x_i = w$, then $w$ is a limit point of $A$.

The limit points of a subset of a metric space are "near" the subset. In the most general topological spaces, the situation can be quite different. Consider $\mathbb{R}$ with the finite-complement topology and let $A = \mathbb{Z}$, the set of integers in $\mathbb{R}$. Choose any real number $r$ and suppose $U$ is an open set containing $r$. Then $U = \mathbb{R} - \{s_1, s_2, \ldots, s_k\}$ for some choices of real numbers $s_1, \ldots, s_k$. Since this set leaves out only finitely many points and $\mathbb{Z}$ is infinite, there are infinitely many integers in $U$ and certainly one not equal to $r$. Thus $r$ is a limit point of $\mathbb{Z}$. This is an extreme case—every point in the space is a limit point of a proper subset.

Closed sets and limit points are related.

**Proposition 3.2.** A subset $K$ of a topological space $(X, T)$ is closed if and only if it contains all of its limit points.

**Proof.** Suppose $K$ is closed, $x \in X$ is some point, and $x \notin K$. Then $x \in X - K$ and $X - K$ is open. So $x$ is contained in an open set that does not intersect $K$, and therefore, $x$ is not a limit point of $K$. Thus all limit points of $K$ must be in $K$.

Suppose $K$ contains all of its limit points. Let $x \in X - K$. Then $x$ is not a limit point and so there exists an open set $U^x$ with $x \in U^x$ and $U^x \cap K = \emptyset$, that is, $U^x \subset X - K$. Since we can find such an open set $U^x$ for all $x \in X - K$, we have

$$X - K \subset \bigcup_{x \in X - K} U^x \subset X - K.$$
We have written $X - K$ as a union of open sets. Hence $X - K$ is open and $K$ is closed.

Let $(X, T)$ be a topological space and $A$ an arbitrary subset of $X$. We associate to $A$ subsets definable with the open sets in the topology as follows:

**Definition 3.3.** The **interior** of $A$ is the largest open set contained in $A$, that is,

$$\text{int } A = \bigcup_{U \subseteq A, \text{open}} U.$$  

The **closure** of $A$ is the smallest closed set in $X$ containing $A$, that is,

$$\text{cls } A = \bigcap_{K \supseteq A, \text{closed}} K.$$  

These operations tell us something geometric about subsets, for example, the subset $Q \subset (\mathbb{R}, \text{usual})$ has empty interior and closure all of $\mathbb{R}$. To see this suppose $U \subset \mathbb{R}$ is open. Then there is an interval $(a, b) \subset U$ for some $a < b$. Since $(a, b)$ contains an irrational number, $(a, b) \cap \mathbb{R} - Q \neq \emptyset$, $U \not\subset Q$ and so $\text{int } Q = \emptyset$. If $Q \subset K$ is a closed subset of $\mathbb{R}$, then $\mathbb{R} - K$ is open and contains no rationals. It follows that it contains no interval because every nonempty interval of real numbers contains a rational number. Thus $\mathbb{R} - K = \emptyset$ and $\text{cls } Q = \mathbb{R}$.

The operation of closure ought to be a kind of ‘closing’ up of the set by putting in all the ‘ragged edges.’ We make this precise as follows:

**Proposition 3.4.** If $A \subset X$, where $X$ is a topological space, then $\text{cls } A = A \cup A'$, where

$$A' = \{\text{limit points of } A\}.$$  

$A'$ is called the **derived set** of $A$.

**Proof.** By definition, $\text{cls } A$ is closed and contains $A$ so $A \subset \text{cls } A$. It follows that if $x \not\in \text{cls } A$, then there exists an open set $U$ containing $x$ with $U \cap A = \emptyset$ and so $x \not\in A$ and $x \not\in A'$. This shows $A \cup A' \subset \text{cls } A$. To show the other containment, suppose $y \in \text{cls } A$ and $V$ is an open set containing $y$. If $V \cap A = \emptyset$, then $A \subset (X - V)$, which is a closed set, and so $\text{cls } A \subset (X - V)$. But then $y \not\in \text{cls } A$, a contradiction. If
$y \in \text{cls} A$ and $y \notin A$, then, for any open set $V$ with $y \in V$, we have $V \cap A \neq \emptyset$ and so $y$ is a limit point of $A$. Thus $\text{cls} A \subseteq A \cup A'$.  

For any subset $A \subseteq X$, we have the following sequence of subsets:

$$\text{int} A \subseteq A \subseteq \text{cls} A = A \cup A'.$$

We add another more refined distinction between points in the closure.

**Definition 3.5.** Let $A$ be a subset of $X$, a topological space. A point $x \in X$ is in the **boundary** of $A$, if for any open set $U \subset X$ with $x \in U$, we have $U \cap A \neq \emptyset$ and $U \cap (X - A) \neq \emptyset$. The set of points in the boundary of $A$ is denoted $\text{bdy} A$.

A boundary point of a subset is “on the edge” of the set. For example, suppose $A = (0,1] \cup \{2\}$ in $\mathbb{R}$ with the usual topology. The point $0$ is a boundary point and a point in the derived set, but not in $A$; $1$ is a boundary point, a point in the derived set, and a point in $A$; and $2$ is boundary point, not in the derived set, but in $A$.

The boundary points lie outside the interior of $A$. We next see how the boundary relates to the closure.

**Proposition 3.6.** $\text{cls} A = \text{int} A \cup \text{bdy} A$.

**Proof.** Suppose $x \in \text{bdy} A$ and $K \subset X$ is closed with $A \subseteq K$. If $x \notin K$, then the open set $V = X - K$ contains $x$. Since $x \in \text{bdy} A$, we have $V \cap A \neq \emptyset \neq V \cap (X - A)$. But $A \subseteq K$ implies $V \cap A = \emptyset$, a contradiction. Thus $\text{bdy} A \subseteq \text{cls} A$, and so $\text{bdy} A \cup \text{int} A \subseteq \text{cls} A$.

We have already shown that $A \cup A' = \text{cls} A$. If $x \in A - \text{int} A$, then for any open set $U$ containing $x$, $U \cap (X - A) \neq \emptyset$, otherwise $x$ would be in the interior of $A$. By virtue of $x \in A$, $U \cap A \neq \emptyset$, so $x \in \text{bdy} A$. Thus $\text{int} A \cup \text{bdy} A \supset A$. Consider $y \in A' \cap (X - A)$ and any open set $V$ containing $y$. Since $y \in A'$, $V \cap A \neq \emptyset$. Also $V \cap (X - A) \neq \emptyset$ since $y \notin A$. Thus $A'$ is a subset of $\text{bdy} A$ and $\text{cls} A \subseteq \text{int} A \cup \text{bdy} A$.  

In a metric space, the notion of limit point agrees with the natural idea of the limit of a sequence of points from the subset. We next generalize convergence to topological spaces.
Chapter 4

Building New Spaces from Old

The use of figures is, above all, then, for the purpose of making known certain relations between the objects that we study, and these relations are those which occupy the branch of geometry that we have called Analysis Situs, ...

HENRI POINCARÉ, 1895

Having introduced topologies on sets and continuous functions, it would be useful to know what new topological spaces can be formed from a given space or spaces using certain set-theoretic constructions. The principal examples are:

1) the formation of subsets,

2) the formation of products, and

3) the formation of quotients by equivalence relations.

In later chapters, we will also introduce function spaces. In all cases we will be guided by the need for naturally occurring functions to be continuous.
Subspaces

Many interesting mathematical objects are subsets of Euclidean space, which is a topological space—how are these subsets topological spaces? By restricting the metric to a subset, it becomes a metric space and so has a topology. However, this procedure does not generalize to all topological spaces. We need a more flexible approach.

For any subset $A$ of a set $X$, we associate the function $i: A \to X$ given by $i(a) = a$ (the inclusion). Restriction of a function $f: X \to Y$ to the subset $A$ becomes a composite $f|_A = f \circ i: A \to Y$. To topologize a subset $A$ of $X$, a topological space, we want that restriction to $A$ of a continuous function on $X$ be continuous. This is accomplished by giving $A$ a topology for which $i: A \to X$ is continuous.

**Definition 4.1.** Let $X$ be a topological space with topology $T$ and $A$, a subset of $X$. The **subspace topology** on $A$ is given by $T_A = \{ U \cap A \mid U \in T \}$, also called the **relative topology** on $A$.

**Proposition 4.2.** The collection $T_A$ is a topology on $A$ and with this topology the inclusion $i: A \to X$ is continuous.

**Proof.** If $U$ is open in $X$, then $i^{-1}(U) = U \cap A$, which is open in $A$. The fact that $T_A$ is a topology on $A$ is easy to prove and, in fact, it is the smallest topology on $A$ making $i: A \to X$ continuous. We leave it to the reader to prove these assertions. \qed

Example 1. Some interesting spaces are the **spheres** in $\mathbb{R}^n$, for $n \geq 1$. They are given by

$$S^{n-1} = \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}.$$
Thus \( S^0 = \{-1, 1\} \subset \mathbb{R}, \) and \( S^1 \subset \mathbb{R}^2 \) is the unit circle. Open sets in \( S^1 \) are easy to picture: the intersection of an open ball in \( \mathbb{R}^2 \) with \( S^1 \) gives a sort of 'interval' in \( S^1 \). To be precise, take any point \( z \in S^1 \) with \( z = (\cos \theta_0, \sin \theta_0) \), and let \( w: (-\varepsilon, \varepsilon) \rightarrow S^1 \) be the mapping \( r \mapsto (\cos(\theta_0 + r), \sin(\theta_0 + r)) \). Then let \( \rho = d(z, (\cos(\theta_0 + \varepsilon), \sin(\theta_0 + \varepsilon))) \). For small \( \varepsilon \), we get \( w^{-1}(B(z, \rho)) = (-\varepsilon, \varepsilon) \) and the mapping \( w \) is a homeomorphism. Thus each point of \( S^1 \) has a neighborhood around it homeomorphic to an open set in \( \mathbb{R} \). This condition is special and characterizes \( S^1 \) as a 1-dimensional manifold. More on this later.

**Example 2.** Some interesting subspaces of \( \mathbb{R}^3 \) are pictured here: they are the cylinder and the Möbius band. (Are they homeomorphic?)

![Diagram of cylinder and Möbius band]

If a space \( X \) has a topological property, does a subset \( A \) of \( X \) as a subspace share it? Such a property is called **hereditary**.

**Proposition 4.3.** Metrizability is a hereditary property. The Hausdorff condition is also hereditary.

**Proof.** That metrizability is hereditary is left to the reader to prove. To see how the Hausdorff condition is hereditary, suppose \( a, b \in A \). Then \( a, b \) are also in \( X \), which is Hausdorff. So there are open sets \( U, V \) in \( X \) with \( a \in U, b \in V, \) and \( U \cap V = \emptyset \). Consider \( U \cap A \) and \( V \cap A \). Since these are nonempty, disjoint, open sets in \( A \) with \( a \in U \cap A \) and \( b \in V \cap A \), we have that \( A \) is Hausdorff.

Reversing the notion of a hereditary property, we consider properties that, when they hold on a subspace, can be seen to hold on the whole space. For example, one can build continuous mappings this way:

**Theorem 4.4.** Suppose \( X = A \cup B \) is a space, \( A, B \) are open subsets of \( X \), and \( f: A \rightarrow Y, g: B \rightarrow Y \) are continuous functions (where \( A \) and \( B \) have the subspace topologies). If \( f(x) = g(x) \) for all \( x \in A \cap B \),
then \( F = f \cup g : X \to Y \) is a continuous function, where \( F \) is defined by

\[
F(x) = \begin{cases} 
  f(x), & \text{if } x \in A, \\
  g(x), & \text{if } x \in B.
\end{cases}
\]

**Proof.** The condition that \( f \) and \( g \) agree on \( A \cap B \) implies that \( F \) is well defined. Let \( U \) be open in \( Y \) and consider

\[
F^{-1}(U) = (f^{-1}(U) \cap A) \cup (g^{-1}(U) \cap B).
\]

The subset \( f^{-1}(U) \cap A \) is open in \( A \) so it equals \( V \cap A \), where \( V \) is open in \( X \). But since \( A \) is open, \( V \cap A \) is open in \( X \), so \( f^{-1}(U) \cap A \) is open in \( X \). Similarly \( g^{-1}(U) \cap B \) is open in \( X \) and their union is \( F^{-1}(U) \). Thus \( F \) is continuous. \( \square \)

If a space breaks up into disjoint open pieces, then continuity of a function defined on the whole space is determined by continuity on each piece.

There is a similar characterization for \( A, B \) closed in \( X \). A subset \( K \subset A \) is closed in \( A \) if there is an \( L \subset X \) closed in \( X \) with \( K = L \cap A \). To see this write \( A - K = A \cap (X - L) \).

More generally, when \( A \) is a subspace of \( X \) and \( f: A \to Y \) is a continuous function, is there an extension of \( f \) to all of \( X \), \( \hat{f}: X \to Y \), that is continuous, for which \( f = \hat{f} \circ i \)? This problem is called the **extension problem** and it is a common formulation of many problems in topology. An example where it is known to fail is the inclusion

\[
i: S^{n-1} \to e^n = \text{cls } B(0, 1) = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \subset \mathbb{R}^n,
\]

with respect to the mapping \( \text{id}: S^{n-1} \to S^{n-1} \) (Brouwer Fixed Point Theorem in Chapter 11). The corollaries of this failure are numerous.

An extension problem with a positive solution is the following result.

**Tietze Extension Theorem.** Any continuous function \( f: A \to \mathbb{R} \) from a closed subspace \( A \) of a metric space \( (X, d) \) has an extension \( g: X \to \mathbb{R} \) that is also continuous.

We first prove a couple of lemmas.
Chapter 5

Connectedness

We begin our introduction to topology with the study of connectedness—traditionally the only topic studied in both analytic and algebraic topology.

C. T. C. WALL, 1972

The property at the heart of certain key results in analysis is that of connectedness. The definition applies to any topological space.

Definition 5.1. A space $X$ is disconnected by a separation $\{U, V\}$ if $U$ and $V$ are open, nonempty, and disjoint ($U \cap V = \emptyset$) subsets of $X$ with $X = U \cup V$. If no separation of the space $X$ exists, then $X$ is connected.

Notice that $V = X - U$ is closed and likewise $U$ is closed. A subset that is both open and closed is sometimes called clopen. Closure leads to an equivalent condition.

Theorem 5.2. A space $X$ is connected if and only if whenever $X = A \cup B$ with $A, B$ nonempty, then $A \cap (\text{cls } B) \neq \emptyset$ or $(\text{cls } A) \cap B \neq \emptyset$.

Proof. If $A \cap (\text{cls } B) = \emptyset$ and $(\text{cls } A) \cap B = \emptyset$, then, since $A \cup B = X$, it will follow that $\{X - \text{cls } A, X - \text{cls } B\}$ is a separation of $X$. To see this, consider $x \in (X - \text{cls } A) \cap (X - \text{cls } B)$; then $x \notin \text{cls } A$ and $x \notin \text{cls } B$. But then $x \notin \text{cls } A \cup \text{cls } B = X$, a contradiction. Therefore $(X - \text{cls } A) \cap (X - \text{cls } B) = \emptyset$. Thus we have a separation.
Conversely, if \( \{U, V\} \) is a separation of \( X \), let \( A = X - V = U \) and \( B = X - U = V \). Since \( U \) and \( V \) are open, \( A \) and \( B \) are closed. Then \( X = U \cup V = A \cup B \). However, \( A \cap \text{cls } B = A \cap B = U \cup V = \emptyset \). \( \square \)

**Example.** The canonical connected space is the unit interval \([0, 1] \subset (\mathbb{R}, \text{usual})\). To see this, suppose \( \{U, V\} \) is a separation of \([0, 1]\). Suppose that \( 0 \in U \). Let \( c = \sup\{0 \leq t \leq 1 \mid [0, t] \subset U\} \). If \( c = 1 \), then \( V = \emptyset \), so suppose \( c < 1 \). Since \( c \in [0, 1] \), \( c \in U \) or \( c \in V \). If \( c \in U \), then there exists an \( \epsilon > 0 \) such that \((c - \epsilon, c + \epsilon) \subset U \) and there is a natural number \( N > 1 \) such that \( c < c + (\epsilon/N) < 1 \). But this contradicts \( c \) being a supremum since \( c + (\epsilon/N) \in [0, 1] \). If \( c \in V \), then there exists a \( \delta > 0 \) such that \((c - \delta, c + \delta) \subset V \). For some \( N' > 1 \), \( c + (\delta/N') < 1 \) and so \((c - (\delta/N'), c + (\delta/N')) \) does not meet \( U \), so \( c \) could not be a supremum. Since the set \( \{0 \leq t \leq 1 \mid [0, t] \subset U\} \) is nonempty and bounded, it has a supremum. It follows that \( c = 1 \) and so \([0, 1]\) is connected.

Is connectedness a topological property? In fact more is true:

**Theorem 5.3.** If \( f : X \to Y \) is continuous and \( X \) is connected, then \( f(X) \), the image of \( X \) in \( Y \), is connected.

**Proof.** Suppose \( f(X) \) has a separation. It would be of the form \( \{U \cap f(X), V \cap f(X)\} \) with \( U \) and \( V \) open in \( Y \). Consider the open sets \( \{f^{-1}(U), f^{-1}(V)\} \). Since \( U \cap f(X) \neq \emptyset \), we have \( f^{-1}(U) \neq \emptyset \) and similarly \( f^{-1}(V) \neq \emptyset \). Since \( U \cap f(X) \cup V \cap f(X) = f(X) \), we have \( f^{-1}(U) \cup f^{-1}(V) = X \). Finally, if \( x \in f^{-1}(U) \cap f^{-1}(V) \), then \( f(x) \in U \cap f(X) \) and \( f(x) \in V \cap f(X) \). But \( (U \cap f(X)) \cap (V \cap f(X)) = \emptyset \). Thus \( f^{-1}(U) \cap f^{-1}(V) = \emptyset \) and \( X \) is disconnected. \( \square \)

**Corollary 5.4.** Connectedness is a topological property.

**Example.** Suppose \( a < b \). Then there is a homeomorphism \( h : [0, 1] \to [a, b] \) given by \( h(t) = a + (b - a)t \). Thus, every \([a, b]\) is connected.

A subspace \( A \) of a space \( X \) is disconnected when there are open sets \( U \) and \( V \) in \( X \) for which \( A \cap U \neq \emptyset \neq A \cap V \), \( A \subset U \cup V \), and \( A \cap U \cap V = \emptyset \). Notice that \( U \cap V \) can be nonempty in \( X \), but \( A \cap U \cap V = \emptyset \).
Lemma 5.5. If $\{A_i \mid i \in J\}$ is a collection of connected subspaces of a space $X$ with $\bigcap_{i \in J} A_i \neq \emptyset$, then $\bigcup_{i \in J} A_i$ is connected.

Proof. Suppose $U$ and $V$ are open subsets of $X$ with $\bigcup_{i \in J} A_i \subset U \cup V$ and $\bigcup_{i \in J} A_i \cap U \cap V = \emptyset$. Let $p \in \bigcap_{j \in J} A_j$; then $p \in A_j$ for all $j \in J$. Suppose that $p \in U$. Since $U$ and $V$ are open, $\{U \cap A_j, V \cap A_j\}$ would separate $A_j$ if they were both nonempty. Since $A_j$ is a connected subspace, this cannot happen, and so $A_j \subset U$. Since $j \in J$ was arbitrary, we can argue in this way to show $\bigcup A_i \subset U$, and hence $\{U, V\}$ is not a separation. \ \qed

Example. Given an open interval $(a, b) \subset \mathbb{R}$, let $N > 2/(b-a)$. Then we can write $(a, b) = \bigcup_{n \geq N} [a + \frac{1}{n}, b - \frac{1}{n}]$, a union with nonempty intersection. It follows from the lemma that $(a, b)$ is connected. Also $\mathbb{R} = \bigcup_{n > 0} [-n, n]$ and so $\mathbb{R}$ is connected.

Let us review our constructions to see how they respect connectedness. A subset $A$ of a space $X$ is connected if it is connected in the subspace topology. Subspaces do not generally inherit connectedness; for example, $\mathbb{R}$ is connected but $[0, 1] \cup (2, 3) \subset \mathbb{R}$ is disconnected. A quotient of a connected space, however, is connected since it is the continuous image of the connected space. How about products?

Proposition 5.6. If $X$ and $Y$ are connected spaces, then $X \times Y$ is connected.

Proof. Let $x_0$ and $y_0$ be points in $X$ and $Y$, respectively. In the exercises of Chapter 4 you proved that the inclusions $j_{x_0} : Y \to X \times Y$, given by $j_{x_0}(y) = (x_0, y)$, and $i_{y_0} : X \to X \times Y$, given by $i_{y_0}(x) = (x, y_0)$, are continuous; hence $j_{x_0}(Y)$ and $i_{y_0}(X)$ are connected in $X \times Y$. Furthermore, $j_{x_0}(Y) \cap i_{y_0}(X) = (x_0, y_0)$ so $i_{y_0}(X) \cup j_{x_0}(Y)$ is connected. We express $X \times Y$ as a union of similar connected subsets:

$$X \times Y = \bigcup_{x \in X} i_{y_0}(X) \cup j_x(Y),$$

a union with intersection given by $\bigcap_{x \in X} i_{y_0}(X) \cup j_x(Y) = i_{y_0}(X)$, which is connected. By Lemma 5.5, $X \times Y$ is connected. \ \qed

Example. By induction, $\mathbb{R}^n$ is connected for all $n$. Wrapping $\mathbb{R}$ onto $S^1$ by $w : \mathbb{R} \to S^1$, given by $w(\gamma) = (\cos(2\pi \gamma), \sin(2\pi \gamma))$, shows that $S^1$ is connected and so is the torus $S^1 \times S^1$. We can also prove this
by arguing that $[0, 1] \times [0, 1]$ is connected and the torus is a quotient of $[0, 1] \times [0, 1]$. It also follows that $S^2$ is connected—$S^2 \cong \Sigma S^1$, a quotient of $S^1 \times [0, 1]$. By induction and Theorem 4.19, $S^n$ is connected for all $n \geq 1$.

A characterization of the connected subspaces of $\mathbb{R}$ has some interesting corollaries.

**Proposition 5.7.** If $W \subset (\mathbb{R}, \text{usual})$ is connected, then $W = (a, b)$, $[a, b)$, $(a, b]$, or $[a, b]$ for $-\infty \leq a \leq b \leq \infty$.

**Proof.** Suppose $c, d \in W$ with $c < d$. We show $[c, d] \subset W$, that is, that $W$ is convex. (In other words, if $c, d$ are both in $W$, then $(1 - t)c + td \in W$ for all $0 \leq t \leq 1$.) Otherwise there exists a value $r_0$, with $c < r_0 < d$ and $r_0 \notin W$. Then $U = (-\infty, r_0) \cap W$, $V = W \cap (r_0, \infty)$ is a separation of $W$. We leave it to the reader to show that a convex subset of $\mathbb{R}$ must be an open, closed, or half-open interval. \qed

**Intermediate Value Theorem.** If $f: [a, b] \to \mathbb{R}$ is a continuous function and $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there is a value $x_0 \in [a, b]$ with $f(x_0) = c$.

**Proof.** Since $f$ is continuous, $f([a, b])$ is a connected subset of $\mathbb{R}$. Furthermore, this subset contains $f(a)$ and $f(b)$. By Proposition 5.7, the interval between $f(a)$ and $f(b)$, which includes $c$, lies in the image of $[a, b]$, and so there is a value $x_0 \in [a, b]$ with $f(x_0) = c$. \qed

**Corollary 5.8.** Suppose $g: S^1 \to \mathbb{R}$ is continuous. Then there is a point $x_0 \in S^1$ with $g(x_0) = g(-x_0)$.

**Proof.** Define $\tilde{g}: S^1 \to \mathbb{R}$ by $\tilde{g}(x) = g(x) - g(-x)$. Wrap $[0, 1]$ onto $S^1$ by $w(t) = (\cos(2\pi t), \sin(2\pi t))$. Then $w(0) = -w(1/2)$.

Let $F = \tilde{g} \circ w$. It follows that

\[
F(0) = \tilde{g}(w(0)) = g(w(0)) - g(-w(0)) = -[g(-w(0)) - g(w(0))] = -[g(w(1/2)) - g(-w(1/2))] = -F(1/2).
\]
How many dimensions does our universe require for a comprehensive physical description? In 1905, Poincaré argued philosophically about the necessity of the three familiar dimensions, while recent research is based on 11 dimensions or even 23 dimensions. The notion of dimension itself presented a basic problem to the pioneers of topology. Cantor asked if dimension was a topological feature of Euclidean space. To answer this question, some important topological ideas were introduced by Brouwer, giving shape to a subject whose development dominated the twentieth century.

The basic notions in topology are varied and a comprehensive grounding in point-set topology, the definition and use of the fundamental group, and the beginnings of homology theory requires considerable time. The goal of this book is a focused introduction through these classical topics, aiming throughout at the classical result of the Invariance of Dimension.

This text is based on the author's course given at Vassar College and is intended for advanced undergraduate students. It is suitable for a semester-long course on topology for students who have studied real analysis and linear algebra. It is also a good choice for a capstone course, senior seminar, or independent study.