II. Contributions to cooperative interval games
CHAPTER 1

The model of cooperative interval games

1.1 Motivation and location within game theory

Uncertainty affects our decision making activities on a daily basis and may have influence on cooperation. To incorporate uncertainty in cooperative game theory is motivated by the need to handle uncertain outcomes in collaborative situations. There are many sources of uncertainty in the real world. We refer here to technological and market uncertainty, noise in observation and experimental design, incomplete information and vagueness in decision making. On many occasions uncertainty is so severe that we can only predict some upper and lower bounds for the outcome of our actions, i.e., payoffs lie in some intervals. A suitable game theoretic model to support decision making in collaborative situations with interval data is that of cooperative interval games. In this model, interval uncertainty affects coalition values, i.e., for each nonempty coalition \( S \) the realized value belongs to an interval of real numbers instead of being sharply defined. The theory of cooperative interval games has been recently born (Branzei, Dimitrov and Tijs (2003)). In Alparslan Gök, Branzei, Branzei and Tijs (2011) we supply both empirical and theoretical background and motivation for the study of cooperative interval games.

Interval uncertainty is the simplest and the most natural type of uncertainty which may influence cooperation because lower and upper bounds for future outcomes or costs of cooperation can always be estimated based on available economic data. Differently, stochastic uncertainty and fuzzy uncertainty, which have already been considered within cooperative game theory, make use of more sophisticated information which can be difficult to obtain or to argue. Basic models of cooperative games which consider stochastic uncertainty are cooperative games in stochastic characteristic function (Granot (1977), Suijs et al. (1999)) and cooperative games with random payoffs (Timmer (2001), Timmer, Borm and Tijs (2005)).
Recently a general game-theoretic model – the model of partially ordered cooperative games (Puerto, Fernández and Hinojosa (2008)) - has been introduced which allows the payoff of any coalition to be an element of any partially ordered linear space. This model catches as particular instances cooperative stochastic games, games with random payoffs, cooperative interval games, and also cooperative vector-valued games\(^1\) (Fernández, Hinojosa and Puerto (2002)).

In the sequel we consider the model of cooperative interval games as a distinct one within cooperative game theory. To construct such a game one observes a lower and an upper bound of the considered coalitions or, more generally, the characteristic function interval-values may result from solving general optimization problems. This is very important, for example from a computational and algorithmic viewpoint.

The model of cooperative interval games is an extension of the classical model of cooperative games with transferable utility (TU games) from the point of view of the nature of payoffs for coalitions.

Let \(I(\mathbb{R})\) be the set of all closed and bounded intervals in \(\mathbb{R}\). A cooperative interval game (in coalitional form) is an ordered pair \(<N, w>\) where \(N = \{1, \ldots, n\}\) is the set of players, and \(w : 2^N \to I(\mathbb{R})\) is the characteristic function with \(w(\emptyset) = [0, 0]\), which assigns to each coalition \(S \in 2^N\) a closed and bounded interval \([\underline{w}(S), \overline{w}(S)]\). A classical cooperative game \(<N, v>\) can be identified with \(<N, w>\), where \(w(S) = [v(S), v(S)]\) for each \(S \in 2^N\).

We end this section with two examples of cooperative interval games arising from interactive situations with interval data.

**Example 1.1.1** Consider the following holding situation with interval data. Players 1 and 2 have each one container which they want to store, and player 3 is the owner of a holding house which has capacity for one container. If player 1 is allowed to store his/her container then the benefit belongs to \([10, 30]\) and if player 2 is allowed to store his/her container then the benefit belongs to \([50, 70]\). The situation is described by the interval game \(<N, w>\) with \(N = \{1, 2, 3\}\), \(w(S) = [0, 0]\) if \(3 \notin S\), \(w(\emptyset) = w(3) = [0, 0]\), \(w(1, 3) = [10, 30]\) and \(w(N) = w(2, 3) = [50, 70]\).

\(^1\) Cooperative vector-valued games arise naturally from collaborative situations where players face a reward/cost sharing problem according to a finite set of criteria, and for each criterion sharp values for each coalition can be evaluated. Mathematically, cooperative interval games can be looked at as special cooperative vector-valued games in the case only two criteria - one pessimistic and the other one optimistic - are used to predict the outcomes of coalitions.
Example 1.1.2 Consider a sequential production situation with 3 departments involved in the working process of a raw material. Each department assures one stage of processing and there is a hierarchy between them: the material is processed at stage $i$ only after processed in stages $1, \ldots, i-1$. At any stage $i$, $i=1, 2, 3$, there is a fixed cost necessary to process the material. However, the cost at stage 2 may increase with an additional amount, for example due to a machinery accident and related maintenance. Suppose that the cost at stages 1 and 3 are 7 and 12, respectively, whereas the cost of stage 2 is in between 5 and 10. The uncertainty due to department 2 affects the departments that are not its superiors. This situation is modeled as the cooperative interval game $< N, w >$ with $N = \{1, 2, 3\}$ and $w(\{1\}) = w(\{1, 3\}) = [7, 7]$, $w(\{1, 2\}) = [7 + 5, 7 + 10]$, $w(\{1, 2, 3\}) = [7 + 5 + 12, 7 + 10 + 12]$, and $w(S) = [0, 0]$ in any other case.

1.2 On interval calculus and cooperative interval games

Definition 1.2.1 A cooperative interval game in coalitional form is an ordered pair $< N, w >$, where $N = \{1, 2, \ldots, n\}$ is the set of players, and $w : 2^N \rightarrow \mathbb{I}(\mathbb{R})$ is the characteristic function which assigns to each coalition $S \subseteq 2^N$ a closed interval $w(S) \in \mathbb{I}(\mathbb{R})$, such that $w(\emptyset) = [0, 0]$.

We denote by $IG^N$ the family of all interval games with player set $N$. For developing a theory of cooperative interval games we need interval calculus.

Let $I(\mathbb{R})$ be the set of all closed and bounded intervals in $\mathbb{R}$, and let $I, J \in I(\mathbb{R})$ with $I = [I, \bar{I}], J = [J, \bar{J}], |I| = \bar{I} - I$ and $\alpha \in \mathbb{R}_+$. Then,

(i) $I + J = [I + J, \bar{I} + \bar{J}]$;

(ii) $\alpha I = [\alpha I, \alpha \bar{I}]$.

By (i) and (ii) we see that $I(\mathbb{R})$ has a cone structure.

The standard subtraction operator on $I(\mathbb{R})$, denoted by $\ominus$ was defined by Moore (1979) as follows: $I \ominus J = [I, \bar{I}] - [J, \bar{J}] = [I - J, \bar{I} - \bar{J}]$. We notice that this operator may cause that $(I \ominus J) + (J \ominus K) \neq I \ominus K$ for some $I, J, K \in I(\mathbb{R})$. To prevent such a situation, a partial subtraction operator was introduced by Alparslan Gök, Branzei and Tijs (2009) as follows: $I - J$, only if $|I| \geq |J|$, by $I - J = [I - J, \bar{I} - \bar{J}]$. Note that $\bar{I} - J \leq \bar{I} - \bar{J}$. This partial subtraction operator plays a central role in what follows.

We denote by $I(\mathbb{R}_+)$ the set of all closed intervals in $\mathbb{R}_+$. Let $a, b \in I(\mathbb{R}_+)$, then $a \cdot b = [\underline{a} \cdot b, \bar{a} \cdot \bar{b}]$. 3
The division operator is defined in Moore (1979) by \( a/b = (a/b, \overline{a}/b) \), with \((a, b) \in I(\mathbb{R}^+) \times I(\mathbb{R}^+ \setminus \{0\})\). Another division operation is more convenient in cooperative interval game theory and its applications. This division operator is defined, only if \( a\overline{b} \leq b\overline{a} \) and \( \overline{b}, \overline{a} \neq 0 \), by \( \frac{a}{b} = \frac{a}{\overline{b}}, \frac{a}{\overline{a}} \).

Now we introduce a partial order on \( I(\mathbb{R}) \). Let \( I = [L, \overline{L}] \) and \( J = [J, \overline{J}] \) be two intervals. We say that \( I \) is weakly better than \( J \), which we denote by \( I \succ J \), if and only if \( L \geq J \) and \( \overline{L} \geq \overline{J} \). Note that in the case \( I \succ J \), then for each \( x \in J \) there exists \( y \in I \) such that \( x \leq y \) and for each \( y \in I \) there exists \( x \in J \) such that \( x \leq y \). We say that \( I \) is better than \( J \), which we denote by \( I \succ J \), if and only if \( I \succ J \) and \( I \neq J \). We also use the reverse notation \( I \prec J \), if and only if \( I \leq J \) and \( \overline{I} \leq \overline{J} \) and the notation \( I < J \), if and only if \( I \prec J \) and \( I \neq J \). We can define an indifference relation \( \sim \) on \( I(\mathbb{R}) \) as follows: \( I \sim J \) if \( I \succ J \) and \( J \succ I \), which is equivalent with \( I \sim J \) if and only if \( I = J \).

In the theory of cooperative interval games \( n \)-tuples of intervals \( I = (I_1, \ldots, I_n) \) where \( I_i \in I(\mathbb{R}) \) for each \( i \in N \), will play a key role. We denote by \( I(\mathbb{R})^N \) the set of all \( n \)-dimensional vectors with elements in \( I(\mathbb{R}) \), and notice that the payoff vectors \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N \) from the classical cooperative TU game theory are replaced by \( n \)-dimensional vectors \( (I_1, I_2, \ldots, I_n) \in I(\mathbb{R})^N \). We notice by \( I(\mathbb{R}^+)^N \) the set of all \( n \)-dimensional vectors with components in \( I(\mathbb{R}^+) \).

Let \( I_i = [L_i, \overline{L}_i] \) be the interval payoff of player \( i \), and let \( I = (I_1, \ldots, I_n) \) be an interval payoff vector. Then, according to Moore (1995), we have \( \sum_{i \in S} I_i = \left[ \sum_{i \in S} L_i, \sum_{i \in S} \overline{L}_i \right] \in I(\mathbb{R}) \) for each \( S \in 2^N \setminus \{\emptyset\} \).

Basic arithmetic of cooperative interval games has been a key tool for developing a theory of such games. For \( w_1, w_2 \in I^N \) and \( \lambda \in \mathbb{R}^+ \) we define \( \langle N, w_1 + w_2 \rangle \) and \( \langle N, \lambda w \rangle \) by \( (w_1 + w_2)(S) = w_1(S) + w_2(S) \) and \( \lambda w(S) = \lambda \cdot w(S) \) for each \( S \in 2^N \). For \( w_1, w_2 \in I^N \) we say that \( w_1 \preceq w_2 \) if \( w_1(S) \preceq w_2(S) \), for each \( S \in 2^N \). So, \( I^N \) endowed with \( \preceq \) is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For \( w_1, w_2 \in I^N \) with \( |w_1(S)| \geq |w_2(S)| \) for each \( S \in 2^N \), \( \langle N, w_1 - w_2 \rangle \) is defined by \( (w_1 - w_2)(S) = w_1(S) - w_2(S) \). Some classical TU-games associated with an interval game \( w \in I^N \) have played a key role, namely the border games \( \langle N, w \rangle, \langle N, \overline{w} \rangle \) (referred to as the lower game and the upper game, respectively), and the length game \( \langle N, |w| \rangle \), where \( |w|(S) = \overline{w}(S) - \underline{w}(S) \) for each \( S \in 2^N \).
CHAPTER 2

Solution concepts for cooperative interval games

A cooperative interval game can be used to solve reward/cost sharing situations with interval data either through its selections or using interval calculus. The first approach was used in Alparslan Gök, Miquel and Tijs (2009) to introduce the core $C(w)$ of a cooperative interval game $< N, w >$. Let $< N, w >$ be an interval game; then $v : 2^N \to \mathbb{R}$ is called a selection of $w$ if $v(S) \in w(S)$ for each $S \in 2^N$ (Alparslan Gök, Miquel and Tijs (2009)). The core $C(w)$ is the union of the cores of all selections of the interval game. The study in this chapter follows the second approach to introduce the interval core $C(w)$, the interval dominance core $DC(w)$, and stable sets according to Alparslan Gök et al. (2011), and the interval Weber set, the interval Shapley value according to Alparslan Gök, Branzei and Tijs (2009). Interval solutions are a timely extension to classical game theoretic solutions. Such solutions inform the players about the ranges of individual payoffs they could generate by cooperation within the grand coalition. Post cooperation, interval solutions explain how the final cut of the worth of the grand coalition, i.e., the sharp gain/cost achieved by the grand coalition, should be distributed among the players.

2.1 The interval core

An interval solution concept $\mathcal{F}$ on a class of $IG^N$ is a map assigning to each interval game $w$ in this class a set of $n$-dimensional vectors whose components belong to $I(\mathbb{R})$.

The interval imputation set $I(w)$ of the interval game $w$, is defined by

$$I(w) = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R})^N \mid \sum_{i \in N} I_i = w(N), w(i) \preceq I_i, \text{ for all } i \in N \right\}.$$  

We note that $\sum_{i \in N} I_i = w(N)$ (the efficiency condition) is equivalent with $\sum_{i \in N} \bar{I}_i = \bar{w}(N)$ and $\sum_{i \in N} \underline{I}_i = \underline{w}(N)$, and $w(i) \preceq I_i$ (the individual rationality condition) is equivalent with
\[ w(i) \leq I_i \text{ and } \bar{w}(i) \leq \bar{I}_i, \text{ for each } i \in N. \] Furthermore, \( \sum_{i \in N} I_i = w(N) \) implies for all \( i \in N \) and for all \( t \in w(N) \) there exists \( x_i \in I_i \) such that \( \sum_{i \in N} x_i = t. \) Notice that the interval uncertainty of coalition values propagates into the interval uncertainty of individual payoffs and we obtain interval payoff vectors as building blocks of interval solutions. The interval imputation set consists of those interval payoff vectors which assure the distribution of the uncertain worth of the grand coalition such that each player can expect a weakly better interval payoff than what he/she can expect on his/her own.

In the definition of the interval core \( \sum_{i \in S} I_i \succ w(S), S \in 2^N \setminus \{\emptyset\}, \) are the coalitional rationality (or stability) conditions of the interval payoff vectors.

The interval core \( C(w) \) of the interval game \( w, \) is defined by

\[
C(w) = \left\{(I_1, \ldots, I_n) \in \left( I(\mathbb{R}) \right)^N | \sum_{i \in N} I_i = w(N), \sum_{i \in S} I_i \succ w(S), \forall S \in 2^N \setminus \{\emptyset\}\right\}.
\]

**Example 2.1.1** Consider the auction interval game\(^1\) given by \( w(\{1\}) = [14, 28], w(\{1, 2\}) = [34, 68], w(\{1, 2, 3\}) = [50, 100], \) and \( w(S) = [0, 0], \) for any other coalition \( S. \) One can easily check that the interval allocation \(([27, 54], [\frac{53}{7}, \frac{20}{3}], [\frac{16}{5}, \frac{68}{3}])\) belongs to the interval core of the game.

The interval core consists of those interval payoff vectors which assure the distribution of the uncertain worth of the grand coalition such that each coalition of players can expect a weakly better interval payoff than what that group can expect on its own, implying that no coalition has any incentives to split off. Clearly, \( C(w) \subset I(w) \) for each \( w \in IG^N. \) Notice that for two-person cooperative interval games the interval imputation set coincides with the interval core.

**Remark 2.1.1** If the worth of the grand coalition is given by a degenerate interval then the elements of the interval core are tuples of degenerate intervals. Under this assumption, the necessary and sufficient condition for the nonemptiness of the interval core is the balancedness of the upper game.

In the view of Remark 2.1.1 we only consider further cooperative interval games for which the value of the grand coalition is affected by interval uncertainty, that is \( |w(N)| \neq 0. \)

**Remark 2.1.2** The interval core of a cooperative interval game can be obtained as a particular instance of the (extended) core of a partially ordered cooperative game (Definition 3.1, Interval games arising from second price sealed bid auctions with one object where the bidders are facing interval uncertainty are defined from interval valuations in Branzei, Mallozzi and Tijs (2010).
p.146 in Puerto, Fernández and Hinojosa (2008)) in the case the characteristic function takes values in the cone (not a linear space) \( I(\mathbb{R}) \) endowed with the partial order \( \succcurlyeq \).

Some basic properties of the interval core are straightforward extensions of the corresponding properties of the core of traditional cooperative games (Gillies (1959)). Specifically, the interval core correspondence \( C : IG^N \rightarrow I(\mathbb{R})^N \) is a superadditive map; for each \( w \in IG^N \) the set \( C(w) \) is a convex set, and the interval core is relative invariant with respect to strategic equivalence, i.e. for all \( w, a \in IG^N \), and for each \( k > 0 \) we have \( C(kw + a) = kC(w) + C(a) \), where \( < N, a > \) is defined by \( a(S) = \sum_{i \in S} a(\{i\}) \).

Since from the mathematical point of view a cooperative interval game can be expressed as a vector-valued game with two components, and balanced vector-valued games are special instances of balanced partially ordered cooperative games, we can use Example 3.3 in Puerto, Fernández and Hinojosa (2008) to obtain a formal definition of an \( I \)-balanced cooperative interval game.

**Definition 2.1.1** A cooperative interval game \( < N, w > \) with \( |w(N)| \neq 0 \) and \( ((w(S))_{S \subseteq N}) \neq ((0, 0))_{S \subseteq N} \) is \( I \)-balanced \(^2\) if there exist \( t^{S,j} \geq 0 \) with \( S \in 2^N \setminus \{\emptyset, N\} \) and \( j = 1, \ldots, 2n \) such that

1. \( \sum_{S \subseteq N, k \in S} t^{S,2k-1} = 1 \), \( \sum_{S \subseteq N, k \in S} t^{S,2k} = 1 \) for any \( k = 1, \ldots, n \);
2. \( w(N) \geq \sum_{S \subseteq N} \sum_{k=1}^n t^{S,2k-1} w(S), \bar{w}(N) \geq \sum_{S \subseteq N} \sum_{k=1}^n t^{S,2k} \bar{w}(S) \).

This means that a cooperative interval game \( w \in IG^N \) is called \( I \)-balanced if there is a balanced map \( T : I(\mathbb{R})^{2^n-2} \rightarrow I(\mathbb{R}) \) which is maximal regarding the partial order \( \succcurlyeq \), that is \( w(N) \succcurlyeq T((w(S))_{S \subseteq N}) \), and does not exist \( T' \) such that \( T'((w(S))_{S \subseteq N}) \succcurlyeq T((w(S))_{S \subseteq N}) \).

In the classical theory of cooperative games it is proved by Bondareva (1963) and Shapley (1967) that a game \( v \in G^N \) is balanced if and only if \( C(v) \) is nonempty. In Theorem 2.1.1 \(^3\) we extend this result to cooperative interval games.

**Theorem 2.1.1** Let \( w \in IG^N \). Then the following two assertions are equivalent:

(i) \( C(w) \neq \emptyset \);

(ii) The game \( w \) is \( I \)-balanced.

\(^2\) The same terminology “\( I \)-balancedness” had previously been used by Iehlé (2007) in the context of hedonic games, and in Alparslan Göck et al. (2011) in the context of cooperative interval games, but these two concepts are different from each other and also from the one in Definition 2.1.1.

\(^3\) This theorem is a corollary of Theorem 3.3 in Puerto, Fernández and Hinojosa (2008).
2.2 The interval dominance core and stable sets

Let \( w \in IG^N \), \( I = (I_1, \ldots, I_n), J = (J_1, \ldots, J_n) \in \mathcal{I}(w) \) and \( S \in 2^N \setminus \{ \emptyset \} \). We say that \( I \) dominates \( J \) via coalition \( S \), and denote it by \( I \text{ dom}_S J \), if

(i) \( I_i > J_i \) for all \( i \in S \),

(ii) \( \sum_{i \in S} I_i \preceq w(S) \).

For \( S \in 2^N \setminus \{ \emptyset \} \) we denote by \( D(S) \) the set of those elements of \( \mathcal{I}(w) \) which are dominated via \( S \). For \( I, J \in \mathcal{I}(w) \), we say that \( I \) dominates \( J \) and denote it by \( I \text{ dom} J \) if there is an \( S \in 2^N \setminus \{ \emptyset \} \) such that \( I \text{ dom}_S J \). \( I \) is called undominated if there exist no \( J \) and no coalition \( S \) such that \( J \text{ dom}_S I \).

The interval dominance core \( DC(w) \) of an interval game \( w \in IG^N \) consists of all undominated elements in \( \mathcal{I}(w) \).

For \( w \in IG^N \) a subset \( A \) of \( \mathcal{I}(w) \) is an interval stable set \(^4\) if the following conditions hold:

(i) \((\text{Internal stability})\) There do not exist \( I, J \in A \) such that \( I \text{ dom} J \).

(ii) \((\text{External stability})\) For each \( I \notin A \) there exists \( J \in A \) such that \( J \text{ dom} I \).

The next theorem establishes relations between the interval core, the interval dominance core and the interval stable sets for cooperative interval games.

**Theorem 2.2.1** Let \( w \in IG^N \) and let \( A \) be a stable set of \( w \). Then, \( C(w) \subset DC(w) \subset A \).

Next, we introduce unanimity interval games, give an explicit description of the interval core for such games and establish that the interval core and the interval dominance core coincide on the class of unanimity interval games. Let \( J \in I(\mathbb{R}_+) \) and let \( T \in 2^N \setminus \{ \emptyset \} \). The unanimity interval game based on \( J \) and \( T \) is defined by

\[
 u_{T,J}(S) = \begin{cases} 
 J, & T \subset S \\
 [0,0], & \text{otherwise}, 
\end{cases}
\]

for each \( S \in 2^N \). We notice that the interval core of the unanimity interval game based on the degenerate interval \( J = [1,1] \) corresponds to the core of the unanimity game in the traditional case.

We define \( \mathcal{K} \) as follows:

\[
 \mathcal{K} = \left\{ (I_1, \ldots, I_n) \in I(\mathbb{R}_+)^N \mid \sum_{i \in N} I_i = J, L_i \geq 0 \text{ for all } i \in N, I_i = [0,0] \text{ for } i \in N \setminus T \right\}.
\]

\(^4\) The classical notion of stable sets was introduced by von Neumann and Morgenstern (1944).
Proposition 2.2.2 Let $< N, u_{T,J} >$ be the unanimity interval game based on coalition $T$ and the interval payoff $J \ni [0,0]$. Then, $\mathcal{D}(u_{T,J}) = C(u_{T,J}) = \mathcal{K}$.

2.3 The interval Weber set

In this section we introduce interval marginal operators on the class of size monotonic interval games and define the Weber set on this class of games.

We call a game $< N, w >$ size monotonic if $< N, |w| >$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SMIG^N$ the class of size monotonic interval games with player set $N$.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \rightarrow N$. Let $w \in SMIG^N$. We introduce the notions of interval marginal operator corresponding to $\sigma$, denoted by $m^\sigma$, and of interval marginal vector of $w$ with respect to $\sigma$, denoted by $m^\sigma(w)$. The marginal vector $m^\sigma(w)$ corresponds to a situation, where the players enter a room one by one in the order $\sigma(1), \sigma(2), \ldots, \sigma(n)$ and each player is given the marginal contribution he/she creates by entering. If we denote the set of predecessors of $i$ in $\sigma$ by $P_\sigma(i) = \{ r \in N | \sigma^{-1}(r) < \sigma^{-1}(i) \}$, where $\sigma^{-1}(i)$ denotes the entrance number of player $i$, then $m^\sigma_{\sigma(k)}(w) = w(P_\sigma(\sigma(k))) - w(P_\sigma(\sigma(k))) = w(P_\sigma(i) \cup \{i\}) - w(P_\sigma(i))$. We notice that $m^\sigma(w)$ is an efficient interval payoff vector for each $\sigma \in \Pi(N)$. For size monotonic games $< N, w >$, $w(T) - w(S)$ is well defined for all $S, T \in 2^N$ with $S \subset T$ since $|w(T)| = |w|(T) = |w|(S) = |w(S)|$. Now, we notice that for each $w \in SMIG^N$ the interval marginal vectors $m^\sigma(w)$ are defined for each $\sigma \in \Pi(N)$, because the monotonicity of $|w|$ implies $\overline{w}(S \cup \{i\}) - \underline{w}(S \cup \{i\}) \geq \overline{w}(S) - \underline{w}(S)$, which can be rewritten as $\overline{w}(S \cup \{i\}) - \underline{w}(S) \geq \overline{w}(S) - \underline{w}(S)$. So, $w(S \cup \{i\}) - w(S)$ is defined for each $S \subset N$ and $i \not\in S$.

The interval Weber set $\mathcal{W}$ on the class of size monotonic interval games is defined by $\mathcal{W}(w) = \text{conv}(m^\sigma(w)|\sigma \in \Pi(N))$ for each $w \in SMIG^N$. We notice that for traditional TU-games we have $W(v) \neq \emptyset$ for all $v \in G^N$, while for interval games it might happen that $\mathcal{W}(w) = \emptyset$ (in case none of the interval marginal vectors $m^\sigma(w)$ is defined). Clearly, $\mathcal{W}(w) \neq \emptyset$ for all $w \in SMIG^N$. Further, $\mathcal{W}(w) \subset C(w)$ for all $w \in SMIG^N$.

Proposition 2.3.1 Let $w \in SMIG^N$ and let $\sigma \in \Pi(N)$. Then, $m^\sigma_i(w) = \left[ m^\sigma_i(w), m^\sigma_i(\overline{w}) \right]$ for all $i \in N$. 

9
2.4 Interval solutions obtained with the square operator

Let \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) with \(a \leq b\). Then, we denote by \(a \Box b\) the vector \([a_1, b_1], \ldots, [a_n, b_n]\) \(\in I(\mathbb{R})^N\) generated by the pair \((a, b)\) \(\in \mathbb{R}^N\). Let \(A, B \subset \mathbb{R}^N\). Then, we denote by \(A \Box B\) the subset of \(I(\mathbb{R})^N\) defined by \(A \Box B = \{a \Box b | a \in A, b \in B, a \leq b\}\).

With the use of the \(\Box\) operator, we give a procedure to extend classical multi-solutions on \(G^N\) to interval multi-solutions on \(IG^N\).

For a multi-solution \(F : G^N \rightarrow \mathbb{R}^N\) we define \(\Phi_i : SMIG^N \rightarrow I(\mathbb{R})^N\) by \(\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^{\sigma}(w)\), for each \(w \in SMIG^N\).

We can write (1) as follows

\[
\Phi_i(w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} (w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))).
\]
The terms after the summation sign in (2) are of the form \( w(S \cup \{i\}) - w(S) \), where \( S \) is a subset of \( N \) not containing \( i \).

Note that there are exactly \(|S|!(n - 1 - |S|)!\) orderings for which one has \( P^S((i)) = S \). The first factor, \(|S|!\), corresponds to the number of orderings of \( S \) and the second factor, \((n - 1 - |S|)!\), is just the number of orderings of \( N \setminus (S \cup \{i\}) \). Using this, we can rewrite (2) as

\[
\Phi_i(w) = \sum_{S \vdash N} \frac{|S|!(n - 1 - |S|)!}{n!} (w(S \cup \{i\}) - w(S)).
\]  

(3)

**Example 2.5.1** The interval Shapley value of the game in Example 1.1.2 is \( \frac{1}{6}([81, 96], [39, 54], [24, 24]) \) and belongs to the interval dominance core of the game.

In the sequel we give some properties of the interval Shapley value on the class of size monotonic interval games.

**Proposition 2.5.1** The interval Shapley value \( \Phi : SMIG^N \to I(\mathbb{R})^N \) is additive.

Let \( w \in SMIG^N \) and \( i, j \in N \). Then, \( i \) and \( j \) are called symmetric players, if \( w(S \cup \{j\}) - w(S) = w(S \cup \{i\}) - w(S) \), for each \( S \) with \( i, j \notin S \).

**Proposition 2.5.2** Let \( i, j \in N \) be symmetric players in \( w \in SMIG^N \). Then, \( \Phi_i(w) = \Phi_j(w) \).

Let \( w \in SMIG^N \) and \( i \in N \). Then, \( i \) is called a dummy player if \( w(S \cup \{i\}) = w(S) + w(\{i\}) \), for each \( S \in 2^{N \setminus \{i\}} \).

**Proposition 2.5.3** The interval Shapley value \( \Phi : SMIG^N \to I(\mathbb{R})^N \) has the dummy player property, i.e. \( \Phi_i(w) = w(\{i\}) \) for all \( w \in SMIG^N \) and for all dummy players \( i \) in \( w \).

**Proposition 2.5.4** The interval Shapley value \( \Phi : SMIG^N \to I(\mathbb{R})^N \) is efficient, i.e., \( \sum_{i \in N} \Phi_i(w) = w(N) \).

**Proposition 2.5.5** Let \( w \in SMIG^N \) and let \( \sigma \in \Pi(N) \). Then, \( \Phi_i(w) = \left[ \phi_i(w), \phi_i(\overline{w}) \right] \) for all \( i \in N \).

Let \( S \in 2^N \setminus \{\emptyset\} \), \( I \in I(\mathbb{R}) \) and let \( u_S \) be the unanimity game based on \( S \). The cooperative interval game \( \langle N, Iu_S \rangle \) is defined by \( (Iu_S)(T) = u_S(T)I \) for each \( T \in 2^N \setminus \{\emptyset\} \), and its Shapley value is given by

\[
\Phi_i(Iu_S) = \begin{cases} 
I/|S|, & i \in S \\
[0, 0], & i \notin S.
\end{cases}
\]
We denote by $KIG^N$ the additive cone generated by the set

$$K = \{ I_S u_S | S \in 2^N \setminus \{\emptyset\}, I_S \in I(\mathbb{R}) \}.$$ 

So, each element of the cone is a finite sum of elements of $K$. We notice that $KIG^N \subset SMIG^N$, and axiomatically characterize the restriction of the interval Shapley value to the cone $KIG^N$.

**Theorem 2.5.6** There is a unique solution $\Psi : KIG^N \to I(\mathbb{R})^N$ satisfying the properties of additivity, efficiency, dummy-player and symmetry. This solution is the interval Shapley value.
CHAPTER 3

Procedures for using interval solutions after the interval uncertainty is resolved

In collaborative situations with interval data, to settle cooperation within the grand coalition $N$ using the cooperative game theory as a tool, that is the cooperative interval game $<N, w>$ arising from the interactive situation with interval data, the players should jointly choose:

(i) An interval solution concept that captures the interval uncertainty with regard to the coalition values under the form of an interval allocation, say $J = (J_1, \ldots, J_n)$. Notice that in the case a value-type interval solution $\Psi$ is chosen, then we have $J_i = \Psi_i(w)$ for all $i \in N$;

(ii) A procedure, specifying the allocation process and the allocation rule(s) to be used during the allocation process, in order to transform the interval allocation $(J_1, \ldots, J_n)$ into a payoff vector $(x_1, \ldots, x_n) \in \mathbb{R}^N$ such that $J_i \leq x_i \leq J_i$ for each $i \in N$ and $\sum_{i \in N} x_i = R$, where $R$ is the revenue for the grand coalition at the end of cooperation.

Two procedures are presented in Sections 3.2 and 3.3 that transform an interval allocation into a payoff vector, under the assumption that only the uncertainty with regard to the value of the grand coalition has been resolved. In both procedures the vector of computed payoff shares belongs to the core $C(\nu)$ of a selection $<N, \nu>$ of the interval game $<N, w>$. Recall that a selection of a cooperative interval game $w$ is a classical game $\nu : 2^N \rightarrow \mathbb{R}$ where $\nu(S) \in w(S)$ for each $S \in 2^N$. Good candidates for allocation rules used in the procedures in Sections 3.2 and 3.3 are presented in Section 3.1. This chapter is based on Branzei, Tijs and Alparslan Gök (2010).
3.1 Allocation rules

Let $N$ be a set of players that consider cooperation under interval uncertainty of coalition values, i.e. knowing what each group $S$ of players (coalition) can obtain between two bounds, $\underline{w}(S)$ and $\bar{w}(S)$, via cooperation. If the players use cooperative game theory as a tool, they can choose an interval solution concept, say the value-type solution $\Psi$, that associates with the related cooperative interval game $<N, w>$ the interval allocation $\Psi(w) = (J_1, \ldots, J_n)$ which guarantees for each player $i \in N$ a final payoff within the interval $J_i = [\underline{J}_i, \bar{J}_i]$ when the value of the grand coalition is known with certainty. Clearly, $\underline{w}(N) = \sum_{i \in N} \underline{J}_i$ and $\bar{w}(N) = \sum_{i \in N} \bar{J}_i$.

For each $i \in N$ the interval $[\underline{J}_i, \bar{J}_i]$ can be seen as the interval claim of $i$ on the realization $R$ of the payoff for the grand coalition $N$, where $\underline{w}(N) \leq R \leq \bar{w}(N)$. One should determine payoffs $x_i \in [\underline{J}_i, \bar{J}_i], i \in N$ (the feasibility condition) such that $\sum_{i \in N} x_i = R$ (the efficiency condition). Notice that in the case $R = \underline{w}(N)$ the payoff vector $x$ equals $(\underline{J}_1, \ldots, \underline{J}_n)$, in the case $R = \bar{w}(N)$ we have $x = (\bar{J}_1, \ldots, \bar{J}_n)$, but in the case $\underline{w}(N) < R < \bar{w}(N)$ there are infinitely many ways to determine allocations $(x_1, \ldots, x_n)$ satisfying both the efficiency and the feasibility conditions.

In the last case, we need suitable allocation rules to determine fair allocations $(x_1, \ldots, x_n)$ of $R$ satisfying the above conditions. As players prefer as large payoffs as possible and the amount $R$ to be divided between them is smaller than $\sum_{i \in N} \bar{J}_i$, the players are facing a bankruptcy-like situation, implying that bankruptcy rules are good candidates for transforming an interval allocation $(J_1, \ldots, J_n)$ into a payoff vector $(x_1, \ldots, x_n)$.

A bankruptcy situation with set of claimants $N$ is a pair $(E, d)$, where $E \geq 0$ is the estate to be divided and $d \in \mathbb{R}_+^N$ is the vector of claims such that $\sum_{i \in N} d_i \geq E$. We denote by $BR^N$ the set of bankruptcy situations with player set $N$. There is a huge literature on bankruptcy situations and related bankruptcy rules and cooperative games. We refer here to O’Neill (1982), Aumann and Maschler (1985), Curiel, Maschler and Tijs (1987) and Thomson (2003).

A bankruptcy rule is a function $f : BR^N \rightarrow \mathbb{R}^N$ which assigns to each bankruptcy situation $(E, d) \in BR^N$ a payoff vector $f(E, d) \in \mathbb{R}^N$ such that $0 \leq f(E, d) \leq d$ (reasonability) and $\sum_{i \in N} f_i(E, d) = E$ (efficiency). In this thesis we only use three bankruptcy rules: the proportional rule (PROP), the constrained equal awards (CEA) rule and the constrained equal losses (CEL) rule.

The rule $PROP_i(E, d) = \frac{d_i}{\sum_{i \in N} d_i} E$ for each bankruptcy problem $(E, d)$ and all $i \in N$. The rule $CEA_i(E, d) = \min\{d_i, \alpha\}$, where $\alpha$ is determined by $\sum_{i \in N} CEA_i(E, d) = E$, for each bankruptcy problem $(E, d)$ and all $i \in N$. The rule $CEL$
is defined by $CEL_i(E,d) = \max \{d_i - \beta, 0\}$, where $\beta$ is determined by $\sum_{i \in N} CEL_i(E,d) = E$, for each bankruptcy problem $(E,d)$ and all $i \in N$. All these rules provide allocations in the core of the (pessimistic) bankruptcy game associated to a bankruptcy situation and are widely used in applications. For further use, we introduce the notation $F = \{CEA, CEL, PROP\}$ and let $f \in F$. The choice of one specific $f \in F$ in a certain bankruptcy situation is based on the preference of the players involved in that situation; other bankruptcy rules could be also considered as elements of a larger $F$.

When the value of the grand coalition becomes known in multiple stages, i.e., updated estimates of the outcome of cooperation within the grand coalition are considered during an allocation process, more general division problems than bankruptcy problems may arise. In the sequel we briefly present the rights-egalitarian ($f_{RE}$) rule (Herrero, Maschler and Villar (1999)) which has a larger domain than bankruptcy rules. This rule is defined by $f_{RE}^i(E,d) = d_i + \frac{1}{2}(E - \sum_{i \in N} d_i)$, for each division problem $(E,d)$ and all $i \in N$. The rights-egalitarian rule divides equally among the agents the difference between the total claim $D = \sum_{i \in N} d_i$ and the available amount $E$, being suitable for all circumstances of division problems; in particular, the amount to be divided can be either positive or negative, the vector of claims $d = (d_1, \ldots, d_n)$ may have negative components, and the amount to be divided may exceed or fall short of the total claim $D$.

### 3.2 The one-stage procedure

Let $(J_1, \ldots, J_n)$ be an interval allocation, with $J_i = [J_i^-, J_i^+]$, $i \in N$, satisfying $\sum_{i \in N} J_i = w(N)$ and $\sum_{i \in N} J_i = \bar{w}(N)$, and let $R$ be the realization of $w(N)$. One can write $R$ and $\bar{J}_i$, $i \in N$, as:

$$R = w(N) + (R - w(N)), \quad (1)$$

$$\bar{J}_i = J_i + (\bar{J}_i - J_i), \quad (2)$$

implying that the problem $(R - w(N), (\bar{J}_i - J_i)_{i \in N})$ is a bankruptcy problem. Since $R$ is the realization of $w(N)$, one can expect that

$$w(N) \leq R \leq \bar{w}(N). \quad (3)$$

Next we describe and illustrate a simple (one-stage) procedure to transform an interval allocation $(J_1, \ldots, J_n) \in I(\mathbb{R})^N$ into a payoff vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$ which satisfies

$$J_i^- \leq x_i \leq J_i^+ \text{ for each } i \in N; \quad (4)$$
\[
\sum_{i \in N} x_i = R. \quad (5)
\]

Procedure **One-Stage**;

**Input data:** \( n, (J_i)_{i=1,n}, R; \)

**function** \( f; \)

**begin**

compute \( w(N) \left\{ w(N) = \sum_{i \in N} J_i \right\}; \)

for \( i = 1 \) to \( n \) do

\( d_i = J_i - J_i \)

[endfor]

for \( i = 1 \) to \( n \) do

\( p_i = f_i(R - w(N), (d_i)_{i=1,n}) \)

[endfor]

for \( i = 1 \) to \( n \) do

\( x_i := J_i + p_i \)

[endfor]

**Output data:** \( x = (x_1, \ldots, x_n); \)

[end procedure].

**Example 3.2.1** Let \( < N, w > \) be the three-person interval game with \( w(S) = [0, 0] \) if \( 3 \notin S, \)

\( w(\emptyset) = w(3) = [0, 0], w(1, 3) = [20, 30] \) and \( w(N) = w(2, 3) = [50, 90]. \) We assume that the realization of \( w(N) \) is \( R = 60 \) and consider that cooperation within the grand coalition was set-tled based on the use of the interval Shapley value. Then, \( \Phi(w) = ([3\frac{1}{3}, 5], [18\frac{1}{3}, 35], [28\frac{1}{3}, 50]). \)

We determine individual uncertainty-free shares distributing the amount \( R - w(N) = 10 \) among the three agents. Note that we deal here with a classical bankruptcy problem \( (E, d) \) with \( E = 10, d = (1\frac{2}{3}, 16\frac{2}{3}, 21\frac{2}{3}). \) Using the one-stage procedure three times with PROP, CEA and CEL in the role of \( f, \) respectively, we have

\[
\begin{array}{c|c|c|c}
 f & PROPE(d) & CEA(d) & CEL(d) \\
p & \left(\frac{5}{12}, 4\frac{1}{6}, 5\frac{5}{12}\right) & \left(1\frac{2}{3}, 4\frac{1}{6}, 4\frac{4}{6}\right) & \left(0, 2\frac{1}{2}, 7\frac{1}{2}\right) \\
\end{array}
\]

Then, we obtain \( x \) as \( (3\frac{1}{3}, 18\frac{1}{3}, 28\frac{1}{3}) + f(10, (1\frac{2}{3}, 16\frac{2}{3}, 21\frac{2}{3})), f \in \mathcal{F}, \) shown in the next table.

\[
\begin{array}{c|c|c|c}
 f & PROPE(d) & CEA(d) & CEL(d) \\
x & (3\frac{1}{3}, 22\frac{1}{3}, 33\frac{1}{3}) & (5, 22\frac{1}{3}, 32\frac{1}{3}) & (3\frac{1}{3}, 20\frac{5}{6}, 35\frac{5}{6}) \\
\end{array}
\]
A comparison of the payoff vectors obtained using PROP, CEA and CEL can be useful in practice to support the choice of the preferred bankruptcy rule \( f \) to be implemented.

We notice that players are given their payoffs \( x_i, i \in N \), either in one blow when \( R \) is known, or by handing in the partial payoff \( J_i \) in advance (i.e. before the uncertainty on the payoff for the grand coalition is resolved) and the portion \( f_i(E, d) \) at the end of cooperation.

Finally, an alternative approach for designing one-stage procedures is to use taxation rules instead of bankruptcy rules by handing out first \( J_i \) and then taking away with the aid of a taxation rule the deficit \( T = \sum_{i \in N} J_i - R \) based on \( d_i = J_i - J_i^f \) for each \( i \in N \).

3.3 The multi-stage procedure

In this section we introduce some dynamics in allocation processes for procedures to transform an interval allocation \((J_1, \ldots, J_n) \in I(\mathbb{R})^N\) into a payoff vector \( x \in \mathbb{R}^N \) satisfying conditions (4) and (5). We assume that a finite sequence of updated estimates of the outcome of the grand coalition, \( R^{(t)} \) with \( t \in \{1, 2, \ldots, T\} \), is available because the value of the grand coalition is known in multiple stages, where

\[
\underline{w}(N) \leq R^{(1)} \leq R^{(2)} \leq \ldots \leq R^{(T)} \leq \bar{w}(N). \tag{8}
\]

At any stage \( t \in \{1, 2, \ldots, T\} \) a budget of fixed size, \( R^{(t)} - R^{(t-1)} \), where \( R^{(0)} = \underline{w}(N) \), is distributed among the players. The decision as which portion of the budget each player will receive at that stage depends on the historical allocation and is specified by a predetermined allocation rule. As allocation rules at each stage we consider either a bankruptcy rule \( f \) (in the case when a bankruptcy problem arises) or a general division rule (for example \( f^{RE} \)) otherwise.

Procedure Multi-Stage;

\textbf{Input data:} \( n, (J_i)_{i=1,n}, T, (R^{(j)})_{j=1,T} \);

\textbf{function} \( f, g \);

\textbf{compute} \( \underline{w}(N) \) \{ \( \underline{w}(N) = \sum_{i \in N} J_i \) \};

\textbf{begin}

\( R^{(0)} := \underline{w}(N) \);

\textbf{for} \( i = 1 \) to \( n \) \textbf{do}

\( d_i = J_i - J_i^f; \ sp_i := 0 \)

\textbf{[endfor]}

17
for $t = 1$ to $T$ do
begin
$D := 0$;
for $i = 1$ to $n$ do
$D := D + d_i$
{endfor}
if $D > R(j) - R(j-1)$ then
for $i = 1$ to $n$ do $p_i = f_i(R(j) - R(j-1), (d_i)_{i=1,n})$ {endfor}
else for $i = 1$ to $n$ do $p_i = g_i(R(j) - R(j-1), (d_i)_{i=1,n})$ {endfor}
{endif}
for $i = 1$ to $n$ do
$d_i := d_i - p_i$;
$sp_i := sp_i + p_i$
{endfor}
{end}
{endfor}
for $i = 1$ to $n$ do
$x_i := J_i + sp_i$
{endfor}
Output data: $x = (x_1, \ldots, x_n)$;
{end procedure}.

We notice that the One-Stage procedure appears as a special case of the Multi-Stage procedure where $T = 1$. At each stage $t \in \{1, \ldots, T\}$ of the allocation process the fixed amount $R^{(t)} - R^{(t-1)}$, where $R^{(t)}$ is the estimate of the payoff for the grand coalition at stage $t$, with $R^{(0)} = w(N)$ is distributed among the players by taking into account the players’ updated claims at the previous stage, $d_i$, $i \in N$, to determine the payoff portions, $p_i$, $i \in N$. The calculation of the individual payoff portions is done using the specified bankruptcy rule $f$ when we deal with a bankruptcy problem, i.e. when the total claim $D$ is greater than $R(j) - R(j-1)$ (and all the individual claims are nonnegative). These payoff portions are used further to update both the aggregate portions $sp_i$ and the individual claims $d_i$, $i \in N$. Notice that under the assumption (8) our procedure assures that all the individual claims are nonnegative as far as we apply a bankruptcy rule $f$ (Example 3.3.1 illustrates such a situation where we use PROP
in the role of $f$). However, the condition $D > R^{(j)} - R^{(j-1)}$ may be not satisfied requiring the use of a general division rule $g$ like $f^{RE}$.

**Example 3.3.1** Consider the interval game and the interval Shapley value as in Example 3.2.1. But, suppose there are the following 3 updated estimates of the realization of the payoff for the grand coalition: $R^{(1)} = 60; R^{(2)} = 65$ and $R^{(3)} = 80$.

Thus we start with $R^{(0)} = 50; d = (1\frac{5}{7}, 16\frac{2}{3}, 21\frac{2}{3}); sp = (0, 0, 0);

Stage 1. The amount $R^{(1)} - R^{(0)} = 10$ is distributed over agents in $N$ according to the claims $d = (1\frac{2}{3}, 16\frac{2}{3}, 21\frac{2}{3})$. Note that $D = 40 > 10$, so the bankruptcy rule PROP can be applied at this stage yielding $p = (\frac{5}{12}, 4\frac{1}{6}, 5\frac{5}{12})$. Clearly, $sp = (\frac{5}{12}, 4\frac{1}{6}, 5\frac{5}{12})$. The vector of claims becomes $d = (1\frac{1}{3}, 12\frac{1}{2}, 16\frac{1}{4})$.

Stage 2. The amount $R^{(2)} - R^{(1)} = 5$ is distributed over agents in $N$ according to $d = (1\frac{1}{4}, 12\frac{1}{2}, 16\frac{1}{4})$.

Note that $D = 30 > 5$, so the bankruptcy rule PROP can be applied yielding $p = (\frac{5}{24}, 2\frac{1}{12}, 2\frac{11}{24})$. Then the adjusted vector of claims is $d = (1\frac{1}{4}, 10\frac{5}{12}, 13\frac{13}{24})$ and $sp$ equals now $(\frac{5}{8}, 6\frac{1}{4}, 8\frac{1}{8})$.

Stage 3. The amount $R^{(3)} - R^{(2)} = 15$ is distributed over agents in $N$ according to $d = (1\frac{1}{4}, 10\frac{5}{12}, 13\frac{13}{24})$.

Since $D = 24\frac{3}{8} > 15$, we can apply the bankruptcy rule PROP obtaining $p = (\frac{5}{8}, 6\frac{1}{4}, 8\frac{1}{8})$.

Then we obtain $sp = (1\frac{1}{4}, 12\frac{1}{2}, 16\frac{1}{4})$ (No claims are further needed because $T = 3$).

Finally, $x = (3\frac{1}{2} + 1\frac{1}{4}, 18\frac{1}{3} + 12\frac{1}{2}, 28\frac{1}{3} + 16\frac{1}{4}) = (4\frac{7}{12}, 30\frac{5}{6}, 44\frac{7}{12})$. 

19
CHAPTER 4

Convex interval games

In this chapter, we introduce the class of convex interval games and extend classical results regarding characterizations of convex games and properties of solution concepts to the interval setting. This chapter is based on Branzei, Tijs and Alparslan Gök (2008), Alparslan Gök, Branzei and Tijs (2009) and Yanovskaya, Branzei and Tijs (2010).

4.1 Some characterizations

We call a game \( < N, w > \) supermodular if
\[
w(S) + w(T) \preceq w(S \cup T) + w(S \cap T) \quad \text{for all } S, T \in 2^N.
\] (1)

From formula (1) it follows that a game \( < N, w > \) is supermodular if and only if its border games \( < N, w > \) and \( < N, \overline{w} > \) are supermodular (convex).

We introduce the notion of convex interval game and denote by \( CIG_N \) the class of convex interval games with player set \( N \). We call a game \( w \in IG_N \) convex if \( < N, w > \) is supermodular and its length game \( < N, |w| > \) is also supermodular.

Next we give as a motivating example a situation with an economic flavour leading to a convex interval game.

**Example 4.1.1** Let \( N = \{1, 2, \ldots, n\} \) and let \( f : [0, n] \to I(\mathbb{R}) \) be such that \( f(x) = [f_1(x), f_2(x)] \) for each \( x \in [0, n] \) and \( f(0) = [0, 0] \). Suppose that \( f_1 : [0, n] \to \mathbb{R} \), \( f_2 : [0, n] \to \mathbb{R} \) and \((f_2 - f_1) : [0, n] \to \mathbb{R} \) are convex monotonic increasing functions. Then, we can construct a corresponding interval game \( w : 2^N \to I(\mathbb{R}) \) such that \( w(S) = f(|S|) = [f_1(|S|), f_2(|S|)] \) for each \( S \in 2^N \). It is easy to show that \( w \) is a convex interval game with the symmetry property \( w(S) = w(T) \) for each \( S, T \in 2^N \) with \(|S| = |T| \). We can see \( < N, w > \) as a production game if we interpret \( f(s) \) for \( s \in N \) as the interval reward which \( s \) players in \( N \) can produce by working together.
Proposition 4.1.1 Let \( w \in IG^N \). If \( < N, w > \) is convex, then it is size monotonic.

We notice that the nonempty set \( CIG^N \) is a subcone of \( IG^N \) and traditional convex games can be embedded in a natural way in the class of convex interval games because if \( v \in G^N \) is convex then the corresponding game \( w \in IG^N \) which is defined by \( w(S) = [v(S), v(S)] \) for each \( S \in 2^N \) is also convex. The next example shows that a supermodular interval game is not necessarily convex.

Example 4.1.2 Let \( < N, w > \) be the two-person interval game with \( w(\emptyset) = [0, 0] \), \( w(1) = w(2) = [0, 1] \) and \( w(1, 2) = [3, 4] \). Here, \( < N, w > \) is supermodular, but \( |w| (1) + |w| (2) = 2 > 1 = |w| (1, 2) + |w| (\emptyset) \). Hence, \( < N, w > \) is not convex.

Interesting examples of convex interval games are unanimity interval games.

For convex TU-games various characterizations are known. In the next theorem we give some characterizations of convex interval games inspired by Shapley (1971).

Theorem 4.1.2 Let \( w \in IG^N \) be such that \( |w| \in G^N \) is supermodular. Then, the following three assertions are equivalent:

(i) \( w \in IG^N \) is convex;

(ii) For all \( S_1, S_2, U \in 2^N \) with \( S_1 \subset S_2 \subset N \setminus U \) we have

\[
  w(S_1 \cup U) - w(S_1) \preceq w(S_2 \cup U) - w(S_2);
\]

(iii) For all \( S_1, S_2 \in 2^N \) and \( i \in N \) such that \( S_1 \subset S_2 \subset N \setminus \{i\} \) we have

\[
  w(S_1 \cup \{i\}) - w(S_1) \preceq w(S_2 \cup \{i\}) - w(S_2).
\]

A characterization of convex interval games with the aid of interval marginal vectors is given in the following theorem.

Theorem 4.1.3 Let \( w \in IG^N \). Then, the following assertions are equivalent:

(i) \( w \) is convex;

(ii) \( |w| \) is supermodular and \( m^\sigma (w) \in C(w) \) for all \( \sigma \in \Pi(N) \).

Proposition 4.1.4 Let \( w \in IG^N \). Then the following assertions hold:
(i) A game \( < N, w > \) is supermodular if and only if its border games \( < N, w > \) and \( < N, \overline{w} > \) are convex;

(ii) A game \( < N, w > \) is convex if and only if its length game \( < N, |w| > \) and its border games \( < N, w >, < N, \overline{w} > \) are convex;

(iii) A game \( < N, w > \) is convex if and only if its border game \( < N, w > \) and the game \( < N, \overline{w} - w > \) are convex.

We call a game \( w \in IG^N \) superadditive if for all \( S, T \subset N \) with \( S \cap T = \emptyset \),

\[
\begin{align*}
w(S \cup T) & \geq w(S) + w(T); \\
|w|(S \cup T) & \geq |w|(S) + |w|(T).
\end{align*}
\]

In the following we give two characterizations of convex interval games using the notions of superadditivity and exactness, respectively.

Given a game \( < N, w > \) and a coalition \( T \subset N \), the \( T \)-marginal interval game \( w^T : 2^{N \setminus T} \to I(\mathbb{R}) \) is defined by \( w^T(S) = w(S \cup T) - w(T) \) for each \( S \subset N \setminus T \). Marginal interval games are used in our first new characterization of convex interval games. The next proposition provides an affirmative answer to the question: If the original interval game is convex, are all its marginal interval games also convex?

**Proposition 4.1.5** Let \( < N, w > \) be a convex game and \( T \subset N \). Then, \( < N \setminus T, w^T > \) is a convex game.

**Theorem 4.1.6** Let \( w \in IG^N \). Then, the following assertions are equivalent:

(i) \( w \in CIG^N \);

(ii) \( < N \setminus T, w^T > \) is superadditive for each \( T \subset N \).

For a traditional cooperative game \( (N, v) \), Biswas et al. (1999) proved that the game is convex if and only if each subgame \( (S, v) \), with \( S \subset N \), is an exact game. In the sequel, we prove that a similar characterization holds true in the interval data setting.

We call a game \( w \in IG^N \) an exact interval game if for each \( S \in 2^N \):

(i) there exists \( I = (I_1, \ldots, I_n) \in C(w) \) such that \( \sum_{i \in S} I_i = w(S) \);

(ii) there exists \( x \in C(|w|) \) such that \( \sum_{i \in S} x_i = |w|(S) \).
Note that (ii) expresses the exactness of the length game $< N, |w| >$.

**Proposition 4.1.7** Each convex interval game $w \in IG^N$ is an exact interval game.

For a given $S \in 2^N$ and $I = (I_1, \ldots, I_n) \in C(w)$, $\sum_{i \in S} I_i = w(S)$ also delivers $(\bar{I}_1, \ldots, \bar{I}_n) \in C(w)$, $(\bar{I}_1 - I_1, \ldots, \bar{I}_n - I_n) \in C(|w|)$, with $\sum_{i \in S} \bar{I}_i = \overline{w}(S)$ and $\sum_{i \in S} (\bar{I}_i - I_i) = |w|(S)$. This can be used for extending the characterization of Biswas et al. (1999) to interval games.

**Theorem 4.1.8** Let $w \in IG^N$. Then the following assertions are equivalent:

(i) $w \in CIG^N$;

(ii) $< T, w_T >$ is exact for each $T \subset N$.

**Theorem 4.1.9** Let $w \in IBIG^N$. Then, the following assertions are equivalent:

(i) $w$ is convex;

(ii) $|w|$ is supermodular and $C(w) = W^C(w)$.

### 4.2 Properties of interval solution concepts

It is well known that $C(v) = W(v)$ if and only if $v \in G^N$ is convex. However, this result can not be extended to convex interval games.

**Proposition 4.2.1** Let $w \in CIG^N$. Then, $W(w) \subset C(w)$.

Since $\Phi(w) \in W(w)$ for each $w \in SMIG^N$, by Proposition 4.2.1 we have $\Phi(w) \in C(w)$ for each $w \in CIG^N$. Since $CIG^N \subset SMIG^N$ we obtain from Proposition 2.3.1 that $m^\sigma_i(w) = \left[ m^\sigma_i(w), m^\sigma_i(\overline{w}) \right]$ for each $w \in CIG^N$, $\sigma \in \Pi(N)$ and for all $i \in N$, implying that $\Phi_i(w) = \left[ \phi_i(w), \phi_i(\overline{w}) \right]$ for all $i \in N$. From Theorem 2.4.2 and Proposition 4.2.1 we obtain that $W(w) \subset W^C(w)$ for each $w \in CIG^N$.

**Proposition 4.2.2** The interval core $C : CIG^N \to I(\mathbb{R})^N$ is an additive map.

Since $CIG^N \subset IBIG^N$ we obtain that $C(w) = C(w) \square C(\overline{w})$ for each $w \in CIG^N$. For convex interval games we have $DC^C(w) = DC(w) \square DC(\overline{w}) = C(w) \square C(\overline{w}) = C^C(w) = C(w)$, where the second equality follows from the well known result in the theory of TU-games that for convex games the core and the dominance core coincide, and the last equality follows from
Proposition 2.4.1. From $DC^C(w) = C(w)$ for each $w \in CIG^N$ and $C(w) \subset DC(w)$ for each $w \in IG^N$ we obtain $DC(w) \supset DC^C(w)$ for each $w \in CIG^N$. We notice that this inclusion might be strict (see Example 4.1 in Alparslan Gök et al. (2011)).

**Definition 4.2.1** An interval value $\phi$ on a class of interval games $IG^N$ is generated by a TU game value $\varphi$ if

$$\phi(N, w) = [\varphi(N, w), \varphi(N, \overline{w})]. \quad (3)$$

Equality (3) implies that the inequality

$$\varphi(N, w) \leq \varphi(N, \overline{w}) \quad (4)$$

should hold, and, hence, not all TU game values can be extended to the generated interval values, and even if a value can be extended, then only for some special classes of TU and interval games.

Consider the class $CG^N$ of convex TU games with a finite set of players $N$. Define the class $CIG^N$ of convex interval games with the universal set of players $N$ by the following way:

$$< N, w > \in CIG^N \iff < N, \overline{w}> < N, w >, < N, \overline{w} - w > \in CG^N \text{ and } w(S) \leq \overline{w}(S) \text{ for all } S \subset N.$$ 

Given a TU value $\varphi$ for the class $CG^N$, the existence of the generated by it interval value $\phi$ on $CIG^N$, i.e. the fulfilment of inequality (4) is equivalent to the following monotonicity property of $\varphi$ :

**Convex monotonicity (CvM).** If $< N, v >, < N, v' >, < N, v' - v > \in CG^N$, and $v'(S) \geq v(S)$ for all $S \subset N$, then $\varphi(N, v') \geq \varphi(N, v)$.

### 4.3 The constrained egalitarian solution for convex interval games

Hokari and van Gellekom (2002) proved that the $DR$ solution over the class of convex games satisfies weak contribution monotonicity.

**Weak contribution monotonicity (WCM)** (Hokari, van Gellekom 2002) If for all $i \in N$ and all coalitions $S \neq i$ the inequalities $v'(S \cup \{i\}) - v'(S) \geq v(S \cup \{i\}) - v(S)$ hold, then $\varphi(N, v') \geq \varphi(N, v)$.

Proposition 1 in Yanowskaya, Branzei and Tijs (2010) establishes that weak contribution monotonicity implies convex monotonicity, providing the existence of the generated Dutta-Ray solution on the class of convex interval games.
Therefore, we define the interval Dutta–Ray solution for interval convex games as a mapping assigning to each convex interval game \(< N, w >\) the pair of vectors \((\text{DR}(N, \underline{w}, \text{DR}(N, \bar{w})))\). This definition can be done in the form of the Lorenz domination property as that for convex TU games. For this, first we should extend the Lorenz domination to sets of ordered pairs of vectors \((x, y) \in \mathbb{R}^N \times \mathbb{R}^N\) such that \(x \leq y\).

Let \(A = \{(x, y) \mid x \in \mathbb{R}^N, y \in \mathbb{R}^N, x \leq y\}\) be a set of pairs of vectors, \((x, y), (x’, y’) \in A\). We say that \((x, y)\) Lorenz dominates \((x’, y’),\) if the Lorenz curve \(L(x, y)\) Pareto dominates the Lorenz curve \(L(x’, y’).\) Note that in a weakly increasing ordering of the vector \((x, y)\) defining the Lorenz curve \(L(x, y),\) it may happen that \(x_i > y_j\) for some components \(i > j.\)

We denote here by \(C(N, w)\) the interval core (of the interval game \(< N, w >\)):

\[
C(N, w) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N \mid x \in C(N, w), y \in C(N, \bar{w}), x \leq y\}.
\]

We notice that this definition is different than the usual one, which regards the interval core as a set of \(|N|\)-dimensional vectors in \(I(\mathbb{R})^N\), but it is equivalent in its consequences.

**Proposition 4.3.1** For any convex interval game \(< N, w >\) the interval Dutta–Ray solution belongs to the interval core \(C(N, w)\) and Lorenz dominates all other vectors \((x, y) \in C(N, w)\).

**Proposition 4.3.2** For any finite \(N\) the interval DR solution on the class \(\text{CIG}^N\) is covariant with respect to identical affine transformations of players utilities, which may be different for lower and upper games: for arbitrary \(< N, w >\in \text{CIG}^N,\) numbers \(\alpha, \alpha’ \in \mathbb{R}_+, \alpha \leq \alpha’\), and vectors \(\underline{a} = (a, \ldots, a), \bar{b} = (b, \ldots, b) \in \mathbb{R}^N, a \leq b,\) it holds

\[
\text{DR}(N, \alpha(\underline{w} + \underline{a}, \bar{w} + \bar{b})) = (\alpha \text{DR}(N, \underline{w} + \underline{a}), \alpha \text{DR}(N, \bar{w} + \bar{b})),
\]

and if the border games are positive, i.e. \(\underline{w}(S) \geq 0\) for all \(S \subset N,\) then

\[
\text{DR}(N, \alpha \underline{w} + \underline{a}, \alpha’ \bar{w} + \bar{b}) = \alpha \text{DR}(N, \underline{w} + \underline{a}, \alpha’ \bar{w} + \bar{b}).
\]

Similar to classical TU games, given an interval game \(< N, w >\) we call the game \(< N, \alpha(\underline{w} + \underline{a}, \bar{w} + \bar{b}) >,\) where \(\alpha > 0, \underline{a} \leq \bar{b},\) strategically equivalent to the game \(< N, w >.\) For the case of positive border games the game \(\langle \alpha \underline{w} + \underline{a}, \alpha \bar{w} + \bar{b} >\) is called strategically equivalent to the game \(< N, w >.\)

Now we are going to define and to show consistency properties of the interval Dutta–Ray solution. Consistency properties of a solution connect the solution vectors of TU games with different sets of players. More exactly, a TU game solution \(\sigma\) is consistent, if, given a
TU game \(\langle N, v \rangle\) and a solution vector \(x \in \sigma(N, v)\), for any coalition \(S \subseteq N\) the vector \(x_{N \setminus S}\) belongs to the solution \(\sigma(N \setminus S, v^\prime)\) (\(\sigma(N \setminus S, v^\prime)\)) of the reduced game, obtained from \(\langle N, v \rangle\) after leaving the coalition \(S\). The characteristic function of the reduced game is defined in different ways depending on methods of aggregating the values \(v(T \cup Q)\) for \(T \subseteq N \setminus S\), \(Q \subseteq S\) and \(x_S\) (or on the solution \(\sigma\) itself) into a unique characteristic function value \(v^x_{N \setminus S}(T)\) (\(v^x_{N \setminus S}(T)\)) of the reduced game.

Thus, to consider consistency properties of a solution, we should put into consideration the classes of games with different sets of players. Let \(N\) be an arbitrary universal set of players, \(G^N(IG^N)\) be an arbitrary class of TU (interval) games with the player set \(N\). Denote by \(G^N = \bigcup_{N \subseteq N} G^N\), \(IG^N = \bigcup_{N \subseteq N} IG^N\) the classes of all TU games and interval games whose finite sets of players are contained in the universal set \(N\), and characteristic functions are defined by the classes \(G^N, IG^N, N \subseteq N\), respectively.

Dutta (1990) showed that the DR solution on the class of convex TU games \(CG^N\) is consistent in the definition of Davis–Maschler (max consistency) (Davis and Maschler (1965)) and of Hart–Mas-Colell (self consistency) (Hart and Mas-Colell (1989)). Yanovskaya, Branzei and Tijs (2010) extended the definitions of consistency of TU game solutions to the generated by them interval solutions by demanding consistency of the corresponding TU game solutions for both border games.

A solution \(\phi\) on the class \(CIG^N\) of interval games, generated by a TU game solution \(\varphi\), is consistent or satisfies the reduced game property in the sense of Hart–Mas-Colell if for any game \(\langle N, w \rangle \in IG^N\), a coalition \(S \subseteq N\), it holds

\[
(\varphi(N, w), \varphi(N, \overline{w}))_S = (\varphi(S, w^\varphi), \varphi(S, \overline{w}^\varphi)), \tag{5}
\]

where the reduced games \(\langle S, w^\varphi \rangle\), \(\langle S, \overline{w}^\varphi \rangle \in CIG^N\) are defined as follows:

\[
\begin{align*}
  w^\varphi(T) &= \overline{w}(T \cup (N \setminus S)) - \sum_{j \in N \setminus S} \varphi_j (T \cup (N \setminus S), w), \\
  \overline{w}^\varphi(T) &= \overline{w}(T \cup (N \setminus S)) - \sum_{j \in N \setminus S} \overline{\varphi_j}(T \cup (N \setminus S), \overline{w}),
\end{align*}
\]

where \(\langle T \cup (N \setminus S), w \rangle, \langle T \cup (N \setminus S), \overline{w} \rangle\) are the subgames of the lower and upper games \(\langle N, w \rangle, \langle N, \overline{w} \rangle\), respectively.

An interval solution \(\varphi\) is bilateral consistent in the sense of Davis–Maschler (Hart–Mas-Colell) if equality (5) holds only for two-person coalitions \(S\), i.e. \(|S| = 2\).

Since the definitions of consistency given above are applied separately to lower and upper games, it may seem that the results about consistency of TU games solutions can be directly
extended to interval games. However, convex interval games demand convexity not only of lower and upper games, but also convexity of the length game. Just this property can be violated by the classical reduced games that does not permit to extend consistency of the DR solution to the interval setting.

However, it is possible to establish the fact of bilateral Hart–Mas-Colell consistency of the DR solution:

**Proposition 4.3.3** The DR solution is bilateral consistent on the class $CIG^N$ for all $N, |N| \geq 3$.

**Theorem 4.3.4** For arbitrary universal set $N$ the interval Dutta–Ray solution is the unique solution on the class $CIG^N$ satisfying constrained egalitarianism for two-person games and bilateral consistency à la Hart–Mas-Colell.

The solution $CE$ of constrained egalitarianism on the class of two-person superadditive games is defined for each game $\langle \{i, j\}, v \rangle$ as follows:

$$CE_i(\{i, j\}, v) = \begin{cases} \frac{v(\{i, j\})}{2} & \text{if } \frac{v(\{i, j\})}{2} \geq \max\{v(\{i\}), v(\{j\})\}, \\ v(\{i\}) & \text{if } \frac{v(\{i, j\})}{2} < v(\{i\}), v(\{i\}) > v(\{j\}) \end{cases} \quad (6)$$

**4.4 Participation monotonic interval allocation schemes**

In the sequel we introduce the notion of population monotonic interval allocation scheme (pmias) for totally $I$-balanced interval games, which is a direct extension of the notion of population monotonic allocation scheme (pmas) for classical cooperative games (Sprumont (1990)). A game $w \in IG^N$ is called totally $I$-balanced if the game itself and all its subgames are $I$-balanced.

We say that for a game $w \in TIBIG^N$ a scheme $A = (A_{IS})_{i \in S, S \in 2^N \setminus \{\emptyset\}}$ with $A_{IS} \in I(\mathbb{R})^N$ is a pmias of $w$ if:

(i) $\sum_{i \in S} A_{IS} = w(S)$ for all $S \in 2^N \setminus \{\emptyset\}$,

(ii) $A_{IS} \preceq A_{IT}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and for each $i \in S$.

Notice that the total $I$-balancedness of an interval game is a necessary condition for the existence of a pmias for that game. A sufficient condition is the convexity of the interval game.
We notice that all subgames of a convex interval game are also convex. In what follows we focus on pmias on the class of convex interval games.

We say that for a game \( w \in CIG^N \) an imputation \( I = (I_1, \ldots, I_n) \in I(w) \) is pmias extendable if there exist a pmias \( A = (A_S)_{S \subseteq 2^N \setminus \emptyset} \) such that \( A_{iN} = I_i \) for each \( i \in N \).

**Theorem 4.4.1** Let \( w \in CIG^N \). Then, each element \( I \) of \( W(w) \) is extendable to a pmias of \( w \).

From Theorem 4.4.1 we obtain that the total interval Shapley value, i.e. the interval Shapley value applied to the game itself and all its subgames, generates a pmias for each convex interval game.

**Example 4.4.1** Let \( w \in CIG^N \) with \( w(\emptyset) = [0,0] \), \( w(1) = w(2) = w(3) = [0,0] \), \( w(1, 2) = w(1, 3) = w(2, 3) = [2,4] \) and \( w(1, 2, 3) = [9,15] \). It is easy to check that the interval Shapley value generates for this game the pmias depicted as

\[
\begin{bmatrix}
1 & 2 & 3 \\
{1,2} & [1,2] & [1,2] & * \\
{1,3} & [1,2] & * & [1,2] \\
{2,3} & * & [1,2] & [1,2] \\
{1} & [0,0] & * & * \\
{2} & * & [0,0] & * \\
{3} & * & * & [0,0]
\end{bmatrix}
\]
CHAPTER 5

Big boss interval games

This chapter introduces and studies a class of cooperative interval games suitable to model market situations with two corners where players face interval uncertainty regarding the outcome of cooperation. In one corner there is a powerful player called the big boss; the other corner contains players that need the big boss to benefit from cooperation. Various characterizations of big boss interval games are given. The interval core of a big boss interval game is explicitly described, bi-monotonic allocation schemes using interval core elements are introduced, and it is shown that each element of the interval core of a big boss interval game is extendable to such a scheme. Two value-type interval solution concepts are defined on the class of big boss interval games which generate for each such game the same interval core allocation which is extendable to a bi-monotonic interval allocation scheme. This chapter is based on Alparslan Gök, Branzei and Tijs (2009) and Alparslan Gök et al. (2011).

5.1 Definition and some characterizations

In most market situations with two corners and in many other economic situations, people or businesses considering cooperation can rather forecast lower and upper bounds for the outcome of their cooperation. To deal with reward sharing problems under interval uncertainty in two-corner market situations, big boss interval games and related solution concepts can be helpful. For example, let us consider a supply chain planning problem where one producer (let’s say a car manufacturer) orders material from several suppliers in order to meet some demand. The car manufacturer would be the big boss and he is naturally facing interval uncertainty regarding the supply demanded from different suppliers. Then, the outcome of cooperation between the manufacturer and different groups of suppliers is affected by interval uncertainty as well, i.e. coalition values are compact intervals. Thus, we have an interval game with a big boss and to solve the related reward sharing problem we need suitable sets of
Definition 5.1.1 A game \(< N, w >\) is called a big boss interval game if its border game \(< N, \overline{w} >\) and the length game \(< N, |w| >\) are classical total big boss games.

The interval game in Example 1.1.1 is a big boss interval game with player 3 as a big boss.

We denote by \(BBIG^N\) the set of all big boss interval games with player set \(N\) (without loss of generality we denote the big boss by \(n\)).

Proposition 5.1.1 Let \(w \in IG^N\). Then, \(w \in BBIG^N\) if and only if its length game \(< N, |w| >\) and its border games \(< N, \overline{w} >, < N, w >\) are (total) big boss games.

Example 5.1.1 Let \(< N, w >\) be a three-person interval game with \(w(1) = w(2) = w(3) = w(1, 2) = [0, 0], w(2, 3) = [5, 6], w(1, 3) = [6, 6]\) and \(w(N) = [9, 11]\). Here, \(< N, w >\) is a total big boss game, but the length game \(< N, |w| >\) is not.

Next we give an example with an economic flavour leading to a big boss interval game.

Example 5.1.2 Let us consider a production economy with one landlord and many peasants. Let \(N = \{1, 2, \ldots, n\}\) be the player set, where \(n\) is the landlord that can not produce anything alone, and \(1, 2, \ldots, n - 1\) are landless peasants. Let \(f : [0, n - 1] \to I(\mathbb{R})\) be the production function with interval values, where \(f(s)\) is the interval reward \([f_1(s), f_2(s)] \supseteq [0, 0]\) if \(s\) peasants are hired by the landlord, where \(f(0) = [0, 0], f_1\) and \(f_2 - f_1\) are concave with \(f_2 - f_1 \geq 0\). This situation corresponds to the big boss interval game \(< N, w >\) whose characteristic function is given by

\[
  w(S) = \begin{cases} 
  [0, 0], & n \notin S \\
  f(|S| - 1), & n \in S.
\end{cases}
\]

Various characterizations of big boss interval games are given in Theorems 5.1.2 and 5.1.3.

Theorem 5.1.2 Let \(w \in S MIG^N\). Then, the following two conditions are equivalent:

(i) \(w \in BBIG^N;\)

(ii) \(< N, w >\) satisfies:

(a) Veto power property:

\[
  w(S) = [0, 0] \text{ for each } S \in 2^N \text{ with } n \notin S ;
\]
(b) Monotonicity property:
\[ w(S) \leq w(T) \text{ for each } S, T \in 2^N \text{ with } n \in S \subset T ; \]

(c) Union property:
\[ w(T) - w(S) \geq \sum_{i \in T \setminus S} (w(T) - w(T \setminus \{i\})) \text{ for all } S, T \text{ with } n \in S \subset T. \]

Now, we give a concavity property for big boss interval games with $n$ as a big boss which plays an important role in Theorem 5.1.3.

(d) $n$-concavity property:
\[ w(S \cup \{i\}) - w(S) \geq w(T \cup \{i\}) - w(T), \]

for all $S, T \in 2^N$ with $n \in S \subset T \subset N \setminus \{i\}$. 

**Theorem 5.1.3** Let $w \in IG^N$ satisfying properties (a) and (b) from Theorem 5.1.2. Then properties (c) and (d) are equivalent.

**Remark 5.1.1** In view of Theorem 4.1.6 we obtain that a game $w \in IG^N$ is concave if and only if for each $T \in 2^N$ the marginal interval game $< N \setminus T, w^T >$ is subadditive.

**Remark 5.1.2** In view of Theorem 4.1.8, a game $w \in IG^N$ is concave if and only if $< T, w^T >$ is exact for each $T \subset N$.

We denote by $MV^{N,[n]}$ is the set of all monotonic games on $N$ satisfying the big boss property with respect to the big boss $n$ and $MIG^{N,[n]}$ the set of all size monotonic interval games on $N$ that satisfy the big boss property with respect to $n$ (the big boss player).

**Proposition 5.1.4** Let $w \in MIG^{N,[n]}$. Then, $w \in BBIG^N$ if and only if the marginal interval game $< N \setminus \{n\}, w^{[n]} >$ is a concave interval game.

Now, using Remarks 5.1.1 and 5.1.2 we obtain

**Proposition 5.1.5** Let $w \in MIG^{N,[n]}$. Then, the following assertions are equivalent:

1. $w \in BBIG^N$.
2. Each marginal interval game of $< N \setminus \{n\}, w^{[n]} >$ is subadditive.
3. Each (interval) subgame of $< N \setminus \{n\}, w^{[n]} >$ is exact.
5.2 The core of big boss interval games

We define the set \( \mathcal{K}(T, w) \) for each subgame \( < T, w > \) of \( < N, w > \) where \( n \in T \) by

\[
\mathcal{K}(T, w) = \{(I_1, \ldots, I_n) \in I(T, w) \mid [0, 0] \preceq I_i \preceq M_i(T, w) \text{ for each } i \in T \setminus \{n\}\}.
\]

**Proposition 5.2.1** Let \( w \in BBIG^N \) and \( < T, w > \) with \( n \in T \subset N \). Then,

\[
C(T, w) = \mathcal{K}(T, w).
\] (5.1)

Two additive maps \( B : BBIG^N \to I(\mathbb{R})^N \) and \( U : BBIG^N \to I(\mathbb{R})^N \) associate with each big boss interval game two special interval “points”.

**Definition 5.2.1** Let \( < T, w > \) be a big boss subgame of \( < N, w > \) with \( n \) as a big boss. The big boss interval point \( B(T, w) \) is defined by

\[
B_j(T, w) = \begin{cases}
[0, 0], & j \in T \setminus \{n\} \\
w(T), & j = n,
\end{cases}
\]

and the union interval point \( U(T, w) \) is defined by

\[
U_j(T, w) = \begin{cases}
M_j(T, w), & j \in T \setminus \{n\} \\
w(T) - \sum_{i \in T \setminus \{n\}} M_i(T, w), & j = n.
\end{cases}
\]

**Theorem 5.2.2** Let \( w \in IG^N \) be such that property (a) in Theorem 5.1.2 holds. Then, \( w \in BBIG^N \) if and only if for each \( T \subset N \) with \( n \in T \subset N \) the big boss interval point \( B(T, w) \) and the union interval point \( U(T, w) \) belong to the interval core of \( < T, w > \).

From the above theorem we learn that big boss interval games are totally \( I \)-balanced games.

5.3 Bi-monotonic participation interval allocation schemes

We denote by \( P_n \) the set \( \{S \subset N \mid n \in S\} \) of all coalitions containing the big boss.

**Definition 5.3.1** Let \( w \in BBIG^N \). A scheme \( B = (B_{iS})_{i \in S, S \in P_n} \) is a bi-monotonic interval allocation scheme (in short bi-mias) if it has the following two properties:

(i) Stability: \( (B_{iS})_{i \in S} \) is an interval core element of the subgame \( < S, w > \) for each coalition \( S \in P_n \);

(ii) Bi-monotonicity: For all \( S, T \in P_n \) with \( S \subset T \), \( B_{iS} \succeq B_{iT} \) for all \( i \in S \setminus \{n\} \) and \( B_{nS} \preceq B_{nT} \).
Property (i) says that in a bi-mias the interval payoff vector for each subgame with \( n \) as a big boss belongs to the interval core of that subgame, whereas property (ii) says that the big boss is weakly better off in larger coalitions, while the other players are weakly worse off.

Special bi-mias for a big boss interval game are obtained by applying the two additive maps \( \mathcal{B} : BBIG^N \to I(\mathbb{R})^N \) and \( \mathcal{U} : BBIG^N \to I(\mathbb{R})^N \) to the game itself and all its proper interval subgames, as the next example illustrates.

**Example 5.3.1** Consider the interval game in Example 5.1.2. Let \( < T, w > \) be a subgame of it with \( n \) as a big boss and let \( \mathcal{B}(T, w) \) and \( \mathcal{U}(T, w) \) be the big boss interval point and union interval point, respectively. For each \( i \neq n \) and for each \( S \subset T \), \( \mathcal{B}_i(S, w) = \mathcal{B}_i(T, w) = [0, 0] \); for \( i = n \) and for each \( S \subset T \), \( \mathcal{B}_n(S, w) = f(|S| - 1) \preceq f(|T| - 1) = \mathcal{B}_n(T, w) \). For each \( i \neq n \) and for each \( S \subset T \),

\[
\mathcal{U}_i(S, w) = M_i(S, w) = w(S) - w(S \setminus \{i\}) \geq w(T) - w(T \setminus \{i\}) = M_i(T, w) = \mathcal{U}_i(T, w);
\]

for \( i = n \) and for each \( S \subset T \),

\[
\mathcal{U}_n(S, w) = w(S) - \sum_{i \in S \setminus \{n\}} M_i(S, w) \preceq w(T) - \sum_{i \in T \setminus \{n\}} M_i(T, w) = \mathcal{U}_n(T, w).
\]

**Definition 5.3.2** Let \( w \in BBIG^N \) with \( n \) as a big boss. An interval imputation \( I = (I_1, \ldots, I_n) \in I(w) \) is said to be bi-mias extendable if there exists a bi-mias \( B = (B_i)_i \), \( S \in P_n \) such that \( B_iN = I_i \) for each \( i \in N \).

The next theorem, inspired by Voorneveld, Tijs and Grahn (2003), shows that each element of the interval core of a big boss interval game is extendable to a bi-mias.

**Theorem 5.3.1** Let \( w \in BBIG^N \) with \( n \) as a big boss and let \( I \in \mathcal{C}(N, w) \). Then \( I \) is bi-mias extendable.

### 5.4 Two value-type interval solutions for big boss interval games

In the sequel, we introduce two value-type interval solutions for big boss interval games called the \( \mathcal{T} \)-value and the \( \mathcal{AL} \)-value.

**Definition 5.4.1** Let \( w \in BBIG^N \). The \( \mathcal{T} \)-value of \( w \) is defined by

\[
\mathcal{T}(N, w) = \frac{1}{2}(\mathcal{U}(N, w) + \mathcal{B}(N, w))
\]
Example 5.4.1 Consider the interval game in Example 1.1.1. This game is a big boss interval game with player 3 as big boss because properties (a), (b) and (c) in Theorem 5.1.2 are satisfied. The $T$-value, in case of full cooperation, generates the interval allocation $T(N,w) = (\{0, 0\}, [20, 20], [30, 50])$, which indicates sharp shares for players 1 and 2 equal to 0 and 20, respectively. The payoff for player 3 depends, in this case, only on the realization $R$ of $w(N)$. Assuming that $R = 60$ player 3 will receive a payoff equal to 40. However, in general the actual individual shares depend not only on $R$, but also on the interval payoff vector agreed upon before starting cooperation, in our case $T(N,w)$. Finally, the total $T$-value generates a bi-mias represented by the following matrix:

$$
\begin{bmatrix}
1 & 2 & 3 \\
N & [0, 0] & [20, 20] & [30, 50] \\
(1, 3) & [5, 15] & * & [5, 15] \\
(2, 3) & * & [25, 35] & [25, 35] \\
(3) & * & * & [0, 0]
\end{bmatrix}
$$

Such a bi-mias extension of the interval core element $T(N,w)$ might be helpful in the decision making process regarding which coalitions should form and how to distribute the collective gains among the participants.

Definition 5.4.2 The interval average lexicographic value $AL$-value is defined by

$$
AL(N,w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^\sigma(N,w),
$$

where $\Pi(N)$ is the set of permutations $\sigma : N \rightarrow N$ and the lexicographic interval vector $L^\sigma(N,w)$ is defined by

$$
L^\sigma_{\sigma(i)}(N,w) = \begin{cases} 
M_{\sigma(i)}(N,w), & i < k \\
[0, 0], & i = k \\
w(N) - \sum_{j=1}^{k-1} M_j(N,w), & i = k
\end{cases}
$$

if $\sigma(k) = n$, where $\sigma(k)$ stands for the position of the player $k$.

Theorem 5.4.1 Let $w \in BBIG^N$ with $n$ as a big boss. Then,

$$
T(N,w) = AL(N,w) \in C(N,w)
$$

and the (total) $AL$-value generates a bi-mias for $w \in BBIG^N$.

The bi-mias in Example 5.4.1 can be interpreted, according to Theorem 5.4.1, in terms of lexicographic maximization within the interval core of the game and the interval cores of the subgames where the big boss $n$ is a player.
CHAPTER 6

Peer group situations with interval data and related games

In this chapter, by using the introduced notions, we approach the study of peer group games in an interval-valued setting. Peer group games have been introduced in Branzei, Fragnelli and Tijs (2002) and correspond to situations where the players have a hierarchical relationship and a positive real value is given to each player only if all his superiors cooperate with him. Peer group games are a subclass of the class of trading games (Deng and Papadimitriou (1994)) and also of the class of games with permission structure (Gilles, Owen and van den Brink (1992)). We introduce interval peer group situations and for the related interval peer group games we give properties of monotonicity, convexity and specify the Shapley value and the Weber set. In the last section all the results are applied to sealed bid second price auctions with uncertainty. This chapter is based on Mallozzi, Branzei and Tijs (2010).

6.1 Interval peer group situations and games: theoretical results

Now we approach classical peer group situations in an interval-valued games setting. Namely we suppose that each agent \(i\) does not know exactly how much he gains from the cooperation with his superiors in \(P(i) = [1, i]\) in the tree hierarchy; he knows only a lower and an upper bound of his gain given by a positive real interval \(A_i = [A_{i}, \bar{A}_i]\).

In this case we call an interval peer group situation or shortly ipg-situation any triplet \(<N, P, A>\) where \(N = \{1, \ldots, n\}\), \(P : N \rightarrow 2^N\) and \(A \in I(\mathbb{R}_N^+))\) is the vector of agents’ gain intervals.

**Definition 6.1.1** An interval-valued peer group game (ipg-game) corresponding to a ipg-situation \(<N, P, A>\) is an interval-valued cooperative game \(<N, w_{P,A}>\) or shortly \(<N, w>\) where \(w\) is given

\[w(\emptyset) = [0, 0];\]
\[ w(S) = \sum_{i : P(i) \subseteq S} A_i, \quad \text{for each } S \subseteq N. \]

Whenever \( 1 \not\in S \), then \( w(S) = [0, 0] \).

**Example 6.1.1 (Sequential Production Situations).** We consider a production situation with \( n \) departments involved in the working process of a raw material. There is a hierarchy between them: the material is processed at stage \( k \) only after the processes in stages \( 1, \ldots, k - 1 \). At any stage there is a fixed cost necessary to process the material. Sometimes the cost may have an additional amount, for example due to a machinery accident. Let us suppose that the cost at stage \( k \) is in between \( A_k \) and \( \bar{A}_k \). If we consider \( N = \{1, \ldots, n\} \), \( P : N \to 2^N \) s.t. \( P(k) = [1, k] \) and \( w(S) = \sum_{i : P(i) \subseteq S} A_i, \quad \text{for each } S \subseteq N \), the game \( \langle N, w_P, A \rangle \) is an interval peer group game. Note that \( w(S) = 0 \) for any coalition \( S \) not containing the root 1.

For example, let us consider \( N = \{1, 2, 3\} \) and \( A_i = [a_i, a_i], \ a_i \in \mathbb{R}, \ i = 1, 3, A_2 = [\bar{A}_2, \tilde{A}_2] \). We have

\[
\begin{align*}
w(\{1\}) &= w(\{1, 3\}) = [a_1, a_1], \\
w(\{1, 2\}) &= [a_1 + A_2, a_1 + \bar{A}_2], \\
w(\{1, 2, 3\}) &= [a_1 + A_2 + a_3, a_1 + \bar{A}_2 + a_3].
\end{align*}
\]

and \( w(S) = [0, 0] \) in any other case. The uncertainty due to department 2 affects the departments that are not its superiors. In a non-cooperative setting, the production situations have been studied in Voorneveld, Tijs and Mallozzi (1999).

Any interval-valued peer group game can be expressed in terms of unanimity interval-valued games as specified in the following.

**Proposition 6.1.1** Interval-valued peer group games with fixed \( N, P \) form a cone

\[ C^{\text{ipg}} = \{ \langle N, w_{PA} \rangle, \ A \in I(\mathbb{R}^+)^N \}. \]

Moreover for any \( \langle N, w_{PA} \rangle \in C^{\text{ipg}} \) we have

\[ w_{PA} = \sum_{i=1}^{n} u_{\lfloor i \rfloor, \lfloor PA \rfloor}. \]

Some solution concepts, well-known in the theory of cooperative interval games, have special properties for interval peer group games as specified below.
Proposition 6.1.2 Let \( < N, w_{PA} > \) be an interval-valued peer group game. Then:

1. \( < N, w_{PA} > \) is size monotonic;
2. \( < N, w_{PA} > \) is convex;
3. the interval Shapley value \( \Phi_i(w) = \sum_{j \in P(j)} A_j |P(j)|, \ i \in N \), where \( |P(j)| \) is the number of elements in \( P(j) \);
4. the interval Weber set is contained in the interval core: \( \emptyset \neq \mathcal{W}(w) \subset \mathcal{C}(w) \).

6.2 Applications of interval peer group games in second-price sealed bid auctions with interval uncertainty

A seller of an object decides a reservation price \( r \) as the lowest price for which he wants to sell the object and it is known to the bidders (Tijs (2003)). There are \( n \) bidders \( \{1, 2, ..., n\} \), the players, that submit a secret bid: the bidder with the highest bid obtains the object at the price of the second highest bid. We suppose that \( w_i \geq r > 0 \) for \( i = 1, ..., n \) is the value for player \( i \) of the object, and that

\[ w_1 > w_2 > w_3 > ... > w_n \geq r. \]

The value \( w_i \) is known only by player \( i \).

Now, we suppose that the bid \( w_i \) of player \( i \) is not exactly determined: the bidder submits an interval of values \( W_i = [w_i - \delta_i, w_i + \delta_i] \) where \( \delta_i, i = 1, ..., n \) represents bidder \( i \)’s uncertainty. Here the bidder \( i \)’s value can be any element in \( W_i = [w_i - \delta_i, w_i + \delta_i] \). These value bounds are not known to the other players. In our model we suppose that the bidder with the highest bid obtains the object at the highest interval price of the second highest bid with respect to the operator \( \succeq \) in \( I(\mathbb{R}) \).

Suppose that for the given \( \delta_1, ..., \delta_n \), the following assumptions hold:

\[ W_1 \succeq W_2 \succeq W_3 \succeq ... \succeq W_n \succeq [r, r], \]  \( (1a) \)

\[ |W_1| > |W_2| > |W_3| > ... > |W_n| > 0. \]  \( (1b) \)

Note that \( W_n = [\underline{w}_n, \bar{w}_n] \succeq [r, r] \) means \( \underline{w}_n \geq r \). Such \( \delta_i, i = 1, ..., n \) exist. For example, if we consider

\[ \delta_i = \frac{w_i - \bar{w}_n}{n}, i = 1, ..., n \]

we have

\[ w_{i+1} - \delta_{i+1} \leq w_i - \delta_i \iff (w_i - w_{i+1})(1 - \frac{1}{n}) \geq 0 \]
and also
\[ w_{i+1} + \delta_{i+1} \leq w_i + \delta_i \iff (w_{i+1} - w_i)(1 + \frac{1}{n}) \leq 0 \]
so that \( W_i \succeq W_{i+1} \) for all \( i = 1, \ldots, n - 1 \), implying that assumption \((1a)\) holds. Moreover,
\[ 2 \frac{w_{i+1} - r}{n} \leq 2 \frac{w_i - r}{n} \iff w_{i+1} \leq w_i \]
so that \( |W_i| > |W_{i+1}| \) for all \( i = 1, \ldots, n - 1 \), implying that assumption \((1b)\) holds.

For any possible coalition \( S \subset \{1, 2, \ldots, n\} \), let us define the payoff as follows.
i) \( S = \{i\} \): player \( i \) bids \( W_i \) that gives to him a payoff \( w(\{i\}) = [0, 0] \) if \( i \neq 1 \), \( w(\{1\}) = W_1 - W_2 \) because player 1 obtains the object at price in \( W_2 \);
ii) \( S = N \), all the players cooperate: optimal bid for player 1 is to choose in \( W_1 \), for the others is to choose \([r, r]\); player 1 obtains the object at price \( r \) and the payoff to \( N \) is \( w(N) = W_1 - [r, r] \);
iii) \( S \subset N \), let \( \hat{i}(S) \) be the player with the highest (interval) value with respect to the operator \( \succeq \) in \( I(\mathbb{R}) \):
- if \( 1 \notin S \) then \( \hat{i}(S) \) bids \( W(\hat{i}(S)) \), the others bid \( r \). The object goes to player 1 and the value of coalition \( S \) is \( w(S) = [0, 0] \);
- if \( 1 \in S \) then the highest bid is in \( W_1 \) and the second highest is in \( W_{k+1} \) if \([1, k] \subset S \) and \( k+1 \notin S \), since \( W_i = [r, r] \) for \( i = 2, \ldots, k \). In this case the value of the coalition is \( w(S) = W_1 - W_{k+1} \).

Let us define \( \mathcal{T} \) the line graph with root 1 and arcs \((i, i+1), i = 1, \ldots, n - 1\), and the hierarchy function \( P : N \rightarrow 2^N \), \( P(i) = \{1, \ldots, i\} \). Let us define \( A_i = W_i - W_{i+1} \) for any \( i \in N \). Then the triplet \(<N, P, A>\) with \( A \in I(\mathbb{R})^N_+ \) is an interval peer group situation.

The auction peer group interval game is the cooperative interval game \(<N, w_{PA} >\) or \(<N, w >\) where
\[ w_{PA}(S) = \sum_{i=1}^{n} u_{\{1,i\},W_i-W_{i+1}}(S) \quad \text{for each } S \in 2^N, \]
where \( W_{n+1} = [r, r] \). In fact for any \( S \subset N \) we have: \( w(S) = [0, 0] \) if \( 1 \notin S \), \( w(S) = \sum_{i=1}^{k} (W_i - W_{i+1}) = W_1 - W_{k+1} \) if \([1, k] \subset S \) and \( k+1 \notin S \).

**Example 6.2.1** As in Example 4 in Branzei, Fragnelli and Tijs (2002), we consider three bidders in an auction have values for the object of 100, 80, 50, respectively, and the reservation price is \( r = 25 \). Let us suppose that the three bidders are facing interval uncertainty with
\( \delta_i = \frac{w_i - r}{3}, i = 1, 2, 3. \) We have

\[
W_1 = [100 - 25, 100 + 25], \ W_2 = [80 - \frac{55}{3}, 80 + \frac{55}{3}], \ W_3 = [50 - \frac{25}{3}, 50 + \frac{25}{3}].
\]

Let us define the auction peer group interval game \( < N, w > \) where \( N = \{1, 2, 3\} \) and the characteristic function is

\[
w(\{1\}) = W_1 - W_2 = [\frac{115}{3} - 25, \frac{115}{3} + 25],
\]

\[
w(\{1, 2\}) = W_1 - W_3 = [\frac{175}{3} - 25, \frac{175}{3} + 25],
\]

\[
w(\{1, 2, 3\}) = W_1 - r = [75 - 25, 75 + 25],
\]

and for any other coalition \( S \subset N \), we have \( w(S) = [0, 0] \).