Convex Multi-Choice Games: Characterizations and Solution Concepts

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1. Definition and basic characterizations

A game $v \in MC^{N,m}$ is called convex if

$$v(s \land t) + v(s \lor t) \geq v(s) + v(t) \quad (1.1)$$

for all $s, t \in \mathcal{M}^N$. For a convex game $v \in MC^{N,m}$ it holds that

$$v(s + t) - v(s) \geq v(\bar{s} + t) - v(\bar{s}) \quad (1.2)$$

for all $s, \bar{s}, t \in \mathcal{M}^N$ satisfying $\bar{s} \leq s$, $\bar{s}_i = s_i$ for all $i \in car(t)$ and $s + t \in \mathcal{M}^N$. This can be obtained by putting $s$ and $\bar{s} + t$ in the roles of $s$ and $t$, respectively, in expression (1.1). In fact, every game satisfying expression (1.2) is convex.
Relation (1.2) can be seen as the multi-choice extension of the property of increasing marginal contributions for coalitions in traditional convex games. As a particular case, specifically when $t$ is of the form $(0_{-i}, t_i)$ with $t_i \in M_i^+$, we obtain that the property of increasing marginal contributions for players holds true for convex multi-choice games as well, even at a more refined extent, because players can gradually increase their participation levels. In the following we denote the class of convex multi-choice games with player set $N$ and maximal participation profile $m$ by $CMC^{N,m}$. For these games we can say more about the relation between the core and the Weber set.
**Theorem 1.1.**

Let \( v \in CMC^{N,m}. \) Then \( W(v) \subset C(v). \)

**Proof.** Note that convexity of both \( C(v) \) and \( W(v) \) implies that it suffices to prove that \( w^\sigma \in C(v) \) for all \( \sigma \in \Xi(v). \) So, let \( \sigma \in \Xi(v). \) Efficiency of \( w^\sigma \) follows immediately from its definition. That \( w^\sigma \) is level increase rational follows directly when we use expression (1.2). Now, let \( s \in M^N. \) The ordering \( \sigma \) induces an admissible ordering \( \sigma' : \{(i,j) \mid i \in N, j \in \{1, \ldots, s_i\}\} \rightarrow \{1, \ldots, \sum_{i \in N} s_i\} \) in an obvious way. Since \( s^{\sigma',\sigma'}(i,j) \leq s^{\sigma,\sigma}(i,j) \) for all \( i \in N \) and \( j \in \{1, \ldots, s_i\}, \) the convexity of \( v \) implies \( w_{ij}^{\sigma'} \leq w_{ij}^\sigma \) for all \( i \in N \) and \( j \in \{1, \ldots, s_i\}. \) Hence,

\[
\sum_{i \in N} \sum_{j=0}^{s_i} w_{ij}^\sigma \geq \sum_{i \in N} \sum_{j=0}^{s_i} w_{ij}^{\sigma'} = v(s).
\]

We conclude that \( w^\sigma \in C(v). \)
1. Definition and basic characterizations

In contrast with convex crisp games for which \( C(v) = W(v) \) holds, the converse of Theorem 1.1 is not true for convex multi-choice games. We provide an example of a game \( v \in CMC^N,^m \) with \( W(v) \subset C(v), \ W(v) \neq C(v) \).

**Example 1.1.**
Let \( v \in CMC^{\{1,2\},^m} \) with \( m = (2,1) \) and
\[
v((1,0)) = v((2,0)) = v((0,1)) = 0, \ v((1,1)) = 2 \text{ and } v((2,1)) = 3.
\]
There are three marginal vectors,
\[
w_1 = \begin{bmatrix} 0 & 0 \\ 3 & \ast \end{bmatrix}, \ w_2 = \begin{bmatrix} 0 & 1 \\ 2 & \ast \end{bmatrix}, \ w_3 = \begin{bmatrix} 2 & 1 \\ 0 & \ast \end{bmatrix}.
\]

Some calculation shows that \( C(v) = co \{ w_1, w_2, w_3, x \} \), where
\[
x = \begin{bmatrix} 3 & 0 \\ 0 & \ast \end{bmatrix}. \text{ We see that } x \notin co \{ w_1, w_2, w_3 \} = W(v).\]
The core element \( x \) in Example 1.1 seems to be too large: note that \( w_3 \) is weakly smaller than \( x \) and \( w_3 \) is still in the core \( C(\nu) \). This inspires the following

**Definition 1.2.**
For a game \( \nu \in MC^{N,m} \) the set \( C_{\text{min}}(\nu) \) of minimal core elements is defined as follows

\[
\{ x \in C(\nu) \mid \nexists y \in C(\nu) \text{ s.t. } y \neq x \text{ and } y \text{ is weakly smaller than } x \}.
\]

Now, we can formulate

**Theorem 1.2.**
Let \( \nu \in CMC^{N,m} \). Then \( W(\nu) = \text{co} \left( C_{\text{min}}(\nu) \right) \).
1. Definition and basic characterizations

**Proof.** We start by proving that all marginal vectors are minimal core elements. Let $\sigma \in \Xi(v)$. Then $w^\sigma \in C(v)$ (cf. Theorem 1.1). Suppose $y \in C(v)$ is such that $y \neq w^\sigma$ and $y$ is weakly smaller than $w^\sigma$. Let $i \in N$ and $j \in M^+_i$ be such that $Y(je^i) < \sum_{l=1}^j w_{il}^\sigma$ and consider $t := s^{\sigma,\sigma((i,j))}$. Then,

$$Y(t) = \sum_{k \in N} Y(t_k e^k) < \sum_{k \in N} \sum_{l=0}^{t_k} w_{kl}^\sigma = v(t), \quad (1.3)$$

where the inequality follows from the fact that $t_i = j$ and the last equality follows from the definitions of $t$ and $w^\sigma$. Now, (1.3) implies that $y \notin C(v)$. Hence, we see that $w^\sigma \in C_{\min}(v)$. This immediately implies that

$$W(v) \subset \text{co} \left( C_{\min}(v) \right). \quad (1.4)$$
Now, let $x$ be a minimal core element. We prove that $x \in W(v)$. Then we can find a payoff vector $y \in W(v)$ that is weakly smaller than $x$. Using (1.4) we see that $y \in co (C_{\text{min}}(v)) \subset C(v)$. Since $x$ is minimal we may conclude that $x = y \in W(v)$. Hence, $W(v) = co (C_{\text{min}}(v))$.

Note that Theorem 1.2 implies that for a convex crisp game the core coincides with the Weber set. The converse of Theorem 1.2 also holds, as shown in

**Theorem 1.3.**

Let $v \in MC^{N,m}$ with $W(v) = co (C_{\text{min}}(v))$. Then $v \in CMC^{N,m}$.

**Proof.** Let $s, t \in M^N$. Clearly, there is an order $\sigma$ that is admissible for $v$ and that has the property that there exist $k, l$ with $0 \leq k \leq l \leq \sum_{i \in N} m_i$ such that $s \land t = s^{\sigma,k}$ and $s \lor t = s^{\sigma,l}$. Note that for the corresponding marginal vector $w^\sigma$ we have that $w^\sigma \in co (C_{\text{min}}(v)) \subset C(v)$. Using this we see
Proof continues

\[ v(s) + v(t) \leq \sum_{i \in N} \sum_{j=1}^{s_i} w_{ij}^\sigma + \sum_{i \in N} \sum_{j=1}^{t_i} w_{ij}^\sigma \]

\[ = \sum_{i \in N} \sum_{j=1}^{(s \land t)_i} w_{ij}^\sigma + \sum_{i \in N} \sum_{j=1}^{(s \lor t)_i} w_{ij}^\sigma \]

\[ = v(s \land t) + v(s \lor t), \]

where the last equality follows from the definition of \( w^\sigma \). Hence, \( v \)

is convex.

From Theorems 1.2 and 1.3 we immediately obtain

**Corollary 1.1.**

Let \( v \in MC^{N,m} \). Then \( v \in CMC^{N,m} \) if and only if \( W(v) = co(C_{min}(v)) \).
2. New characterizations

Now, we extend two characterizations for classical convex games to the multi-choice setting.

**Lemma 2.1.** Let $v \in CMC^{N,m}_N$ and let $u \in M^N_+$. Then, $v^{-u} \in CMC^{N,m-u}_N$.

**Proof.** Note that for $s, t \in M^N_{m-u}$ we have

$$v^{-u}(s \vee t) + v^{-u}(s \wedge t) = v((s \vee t) + u) + v((s \wedge t) + u) - 2v(u)$$

$$= v((s + u) \vee (t + u)) + v((s + u) \wedge (t + u))$$

$$\geq v(s + u) + v(t + u) - 2v(u)$$

$$= v^{-u}(s) + v^{-u}(t),$$

where the inequality follows from the convexity of $\langle N, m, v \rangle$. 
Since each convex game is also superadditive, we conclude from Lemma 2.1 that if $v \in \text{CMC}^N,m$ then all its marginal games are superadditive. Next, we prove that the converse also holds true. This result has been independently obtained for traditional cooperative games $\langle N, v \rangle$ by Branzei, Dimitrov and Tijs (2004a) and Martinez-Legaz (1997, 2006).

**Theorem 2.1.**

Let $v \in \text{MC}^N,m$. Then the following assertions are equivalent:

(i) for each $u \in \mathcal{M}_+^N$, the $u$-marginal game $v^{-u}$ of $v$ is superadditive;

(ii) $v$ is a convex game.

**Proof.** We still need to prove that (i)$\Rightarrow$(ii). Suppose that $v^{-u}$ is superadditive. Then

$$v(s \land t) + v(s \lor t) \geq v(s) + v(t)$$

holds true for all $s, t \in \mathcal{M}^N$ with $s \land t = \emptyset$ because $v = v^{-0}$ is superadditive.
For $s \land t = f \neq 0$, take $p = s - f$ and $q = t - f$. Since $\langle N, m - f, v^{-f} \rangle$ is superadditive, we obtain

$$0 \leq v^{-f}(p \lor q) - v^{-f}(p) - v^{-f}(q)$$
$$= v(p \lor q + f) - v(p + f) - v(q + f) + v(f)$$
$$= v(s \lor t) - v(s) - v(t) + v(s \land t),$$

i.e. $v$ is convex.

For a traditional cooperative game $\langle N, v \rangle$, Biswas et al. (1999) (see also Azrieli and Lehrer (2007)) proved that the game is convex if and only if each subgame $\langle S, v \rangle$, with $S \subset N$, is an exact game. In the sequel, we prove that a similar characterization holds true in the multi-choice setting.
Proposition 2.1.
Each convex multi-choice game \( v \) is an exact game.

Proof. According to Theorem 11.12 in Branzei, Dimitrov and Tijs (2005), for \( v \in CMC^{N,m} \), \( W(v) = \text{co}(C_{\text{min}}(v)) \), implying that all marginal vectors \( w^{\sigma,v} \) are core elements. Take \( \sigma \) such that \( s \) is one of the intermediate coalitions between \( 0 \) and \( m \). Then, \( X(s) = w^{\sigma,v}(s) = v(s) \).

Theorem 2.2.
Let \( v \in MC^{N,m} \). Then the following assertions are equivalent:

(i) \( \langle N, m, v \rangle \) is convex;

(ii) \( \langle N, u, v_u \rangle \) is exact for each \( u \in \mathcal{M}^N_+ \).
Proof. (i)⇒(ii) follows from Proposition 2.1 because each subgame of a convex game is convex, and hence exact. (ii)⇒(i) Take $s, t \in \mathcal{M}^N$. Since the subgame $v_{s \lor t}$ is exact, there is $x \in C(v_{s \lor t})$ such that $X(s \land t) = v_{s \lor t}(s \land t) = v(s \land t)$. Now, using $X(s \lor t) = v_{s \lor t}(s \lor t) = v(s \lor t)$, we obtain
\[ v(s \lor t) + v(s \land t) = X(s \lor t) + X(s \land t) = X(s) + X(t) \geq v(s) + v(t). \]

We note that the two characterizations of convex multi-choice games provided by Theorems 3.1 and 3.2 do not represent a benefit from the computational point of view from the original definition of a convex multi-choice game, but they are interesting from the theoretical point of view.
3. The constrained egalitarian solution

Now, we introduce the constrained egalitarian solution of a game $v \in CMC^N,m$ by using the (per-level) average worth of a multi-choice coalition with respect to $v$ and an adjusted version of the Dutta-Ray algorithm (Dutta and Ray, 1989). Let $v \in MC^N,m$ and let $s \in \mathcal{M}^N$. Let $\|s\| = \sum_{i=1}^{n} s_i$ be the aggregate number of levels of players according to the participation profile $s$. Given $v \in MC^N,m$ and $s \in \mathcal{M}^N_+$, we denote by $\alpha(s, v)$ the (per level) average worth of $s$ with respect to $v$, i.e.

$$\alpha(s, v) := \frac{v(s)}{\|s\|}.$$

Note that $\alpha(s, v)$ can be interpreted as a per (one-unit) level value of coalition $s$ in the game $v$. 
To formulate the Dutta-Ray algorithm in a multi-choice setting we need to prove that for each convex multi-choice game there exists a unique multi-choice coalition with the largest aggregate number of levels of players among all coalitions with the highest (per level) average worth.

**Lemma 3.1.**
Let $v \in \text{CMC}^{N,m}$. Then, the set

$$A(v) := \left\{ t \in \mathcal{M}^N_+ \mid \alpha(t, v) = \max_{s \in \mathcal{M}^N_+} \alpha(s, v) \right\}$$

is closed with respect to the join operator $\vee$. 

Proof. Let \( \bar{\alpha} = \max_{s \in \mathcal{M}^N_+} \alpha(s, \nu) \) and take \( t^1, t^2 \in A(\nu) \). We have to prove that \( t^1 \lor t^2 \in A(\nu) \), that is \( \alpha(t^1 \lor t^2, \nu) = \bar{\alpha} \). Since \( \nu(t^1) = \bar{\alpha}\|t^1\| \) and \( \nu(t^2) = \bar{\alpha}\|t^2\| \), we obtain

\[
\bar{\alpha}\|t^1\| + \bar{\alpha}\|t^2\| = \nu(t^1) + \nu(t^2) \leq \nu(t^1 \lor t^2) + \nu(t^1 \land t^2) \\
\leq \bar{\alpha}\|t^1 \lor t^2\| + \bar{\alpha}\|t^1 \land t^2\| = \bar{\alpha}\|t^1\| + \bar{\alpha}\|t^2\|,
\]

where the first inequality follows from the convexity of \( \nu \), and the second inequality follows from the definition of \( \bar{\alpha} \) and the fact that \( \nu(0) = 0 \) (in case \( t^1 \land t^2 = 0 \)). This implies that \( \nu(t^1 \lor t^2) = \bar{\alpha}\|t^1 \lor t^2\| \) in case \( t^1 \land t^2 \in A(\nu) \) as well as in case \( \|t^1 \land t^2\| = 0 \).
We can conclude from the proof of Lemma 3.1 that for any \( t^1, t^2 \in A(v) \) not only \( t^1 \lor t^2 \in A(v) \) holds true, but also \( t^1 \land t^2 \in A(v) \) if \( t^1 \land t^2 \neq 0 \). Further, \( A(v) \) is closed with respect to finite "unions", where \( t^1 \lor t^2 \) is seen as the "union" of \( t^1 \) and \( t^2 \). Thus, Proposition 3.1 holds true.

**Proposition 3.1.**

Let \( v \in CMC^{N,m} \). Then, there exists a unique element in \( \arg \max_{s \in \mathcal{M}_+^N} \alpha(s, v) \) with the maximal aggregate number of levels.

**Proof.** The set \( A(v) \cup \{0\} \) has a lattice structure and \( \lor_{t \in A(v)} t \) is the largest element in \( A(v) \).
3. The constrained egalitarian solution

Now, we introduce, in a similar way to that of Dutta and Ray (1989), an egalitarian rule on the class of convex multi-choice games. In view of Lemma 2.1 and Proposition 3.1, it is easy to adjust the Dutta-Ray algorithm for convex multi-choice games. In Step 1, one puts $m^1 := m$, $v_1 := v$, and considers the unique element in $\arg\max_{s \in \mathcal{M}_{m^1}} \{0\} \alpha(s, v_1)$ with the maximal aggregate number of levels, say $s^1$. Define $d_{ij} := \alpha(s^1, v_1)$ for each $i \in \text{car}(s^1)$ and $j \in M^1_i$. If $s^1 = m$, then we stop. Otherwise, in Step 2, we consider the convex multi-choice game $\langle N, m^2, v_2 \rangle$, where $m^2 := m^1 - s^1$ and for each $s \in \mathcal{M}_{m^2}$, $v_2(s) := v_1(s + s^1) - v_1(s^1)$. Once again, by using Proposition 3.1, we can take the largest element $s^2$ in $\arg\max_{s \in \mathcal{M}_{m^2}} \{0\} \alpha(s, v_2)$ and define $d_{ij} := \alpha(s^2, v_2)$ for all $i \in \text{car}(s^2)$ and $j \in \{s_i^1 + 1, ..., s_i^1 + s_i^2\}$. If $s^1 + s^2 = m$ we stop; otherwise we go to Step 3.
Step $p$: Suppose that $s^1, s^2, \ldots, s^{p-1}$ have been defined recursively and $s^1 + s^2 + \ldots + s^{p-1} \neq m$. We define a new multi-choice game with player set $N$ and maximal participation profile $m^p := m - \sum_{i=1}^{p-1} m^i$. For each multi-choice coalition $s \in \mathcal{M}_m^N$, we define $v_p(s) := v_{p-1}(s + s^{p-1}) - v_{p-1}(s^{p-1})$. The game $\langle N, m^p, v_p \rangle$ is convex. We denote by $s^p$ the (unique) largest element in $\arg\max_{s \in \mathcal{M}_m^N \setminus \{0\}} \alpha(s, v_p)$ and define $d_{ij} := \alpha(s^p, v_p)$ for all $i \in \text{car}(s^p)$ and $j \in \left\{\left(\sum_{k=1}^{p-1} s^k_i\right) + 1, \ldots, \sum_{k=1}^{p} s^k_i\right\}$.

In $P \leq |M^+|$ steps the algorithm will end, and the constructed allocation $(d_{ij})_{(i,j) \in M^+}$ is called the constrained egalitarian solution $d(\nu)$ of the convex multi-choice game $\nu$.

The next example illustrates the Dutta-Ray algorithm for convex multichoice games.
Example 3.1.
Consider the game $\langle N, m, v \rangle$ with $N = \{1, 2\}$, $m = (2, 1)$, $v(0, 0) = 0$, $v(1, 0) = 3$, $v(2, 0) = 4$, $v(0, 1) = 2$, $v(1, 1) = 8$, $v(2, 1) = 10$. The game is convex and we apply the Dutta-Ray algorithm. In Step 1, $\alpha_1 = 4$, $t^1 = s^1 = (1, 1)$, and we have $d_{11} = d_{21} = 4$. In Step 2, $\alpha_2 = 2$, $t^2 = (1, 1) + (1, 0)$ and we have $d_{12} = 2$. Thus, we obtain $d(v) = (4, 2, 4)$. Note that $\alpha_1 > \alpha_2$. This is true in general, as we show in Proposition 3.2.
Note that the above described Dutta-Ray algorithm determines in \( P \) steps for each \( v \in CMC^N,m \) a sequence of (per level) average values \( \alpha_1, \alpha_2, ..., \alpha_P \) with \( \alpha_p := \alpha(s^p, v_p) \) for each \( p \in \{1, ..., P\} \), and a sequence of multi-choice coalitions in \( M^N_{\pm} \), which we denote by \( t^1 := s^1, t^2 := s^1 + s^2, ..., t^p := s^1 + ... + s^p, ..., t^P := s^1 + ... + s^P = m \). Thus, a unique path \( \langle t^0, t^1, ..., t^P \rangle \), with \( t^0 = 0 \), from 0 to \( m \) is obtained, with which we can associate a suitable ordered partition \( D^1, D^2, ..., D^P \) of \( M \), such that for all \( p \in \{1, ..., P\} \), \( D^p := \{(i, j) \mid i \in \text{car}(t^p - t^{p-1}), j \in \{t^{p-1}_i + 1, ..., t^p_i\}\} \), where for each \((i, j) \in D^p\), \( d_{ij} = \alpha_p \), and the coalition \( t^p - t^{p-1} \) is the maximal participation profile in the ”box” \( D^p \) with (per level) average worth \( \alpha_p \). In order to avoid repetitions, we keep this notation throughout this section.
Proposition 3.2.
Let \( v \in CMC_{N,m} \) and let \( \alpha_p = \max_{s \in M_{mp}\setminus\{0\}} \frac{v_p(s)}{\|s\|} \) be the egalitarian distribution share determined in Step \( p \) of the Dutta-Ray algorithm. Then \( \alpha_p > \alpha_{p+1} \) for all \( p \in \{1, \ldots, P - 1\} \).

Proof. By definition of \( v_p \) and \( \alpha \), we have

\[
\frac{v(t^p) - v(t^{p-1})}{\|t^p - t^{p-1}\|} > \frac{v(t^{p+1}) - v(t^{p-1})}{\|t^p - t^{p-1}\| + \|t^{p+1} - t^p\|}.
\]

By adding and subtracting \( v(t^p) \) in the numerator of the right-hand term, we obtain

\[
\frac{v(t^p) - v(t^{p-1})}{\|t^p - t^{p-1}\|} > \frac{v(t^{p+1}) - v(t^p) + v(t^p) - v(t^{p-1})}{\|t^p - t^{p-1}\| + \|t^{p+1} - t^p\|}.
\]
Proof continues
This inequality is equivalent to

\[(v(t^p) - v(t^{p-1}))\|t^p - t^{p-1}\| + (v(t^p) - v(t^{p-1}))\|t^{p+1} - t^p\| > (v(t^{p+1}) - v(t^p))\|t^p - t^{p-1}\| + (v(t^p) - v(t^{p-1}))\|t^p - t^{p-1}\|,\]

which is, in turn, equivalent to

\[(v(t^p) - v(t^{p-1}))\|t^{p+1} - t^p\| > (v(t^{p+1}) - v(t^p))\|t^p - t^{p-1}\|.\]

Next, we focus on properties of the constrained egalitarian solution for convex multi-choice games.
Lemma 3.2.
Let $v \in CMC^{N;m}$. Then, for each $s \in \mathcal{M}^N_+$ and each $p \in \{1, \ldots, P\}$,
$v(s \wedge t^p - s \wedge t^{p-1} + t^{p-1}) - v(t^{p-1}) \geq v(s \wedge t^p) - v(s \wedge t^{p-1})$.

Proof. First, notice that, for each $i \in N$,
$$\min\{s_i, t_i^{p-1}\} = \min\{\min\{s_i, t_i^p\}, t_i^{p-1}\}$$
because $t_i^p \geq t_i^{p-1}$, implying that $s \wedge t^{p-1} = (s \wedge t^p) \wedge t^{p-1}$.
Second, notice that, for $i \in N$, either $\min\{s_i, t_i^{p-1}\} = t_i^{p-1}$ or $\min\{s_i, t_i^{p-1}\} = s_i$, and in both situations we have
$$\min\{s_i, t_i^p\} - \min\{s_i, t_i^{p-1}\} + t_i^{p-1} = \max\{\min\{s_i, t_i^p\}, t_i^{p-1}\},$$
implying that
$$(s \wedge t^p) - (s \wedge t^{p-1}) + t^{p-1} = (s \wedge t^p) \vee t^{p-1}.$$
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Proof continues
Now, by convexity of \( v \) (with \( s \land t^p \) in the role of \( s \) and \( t^{p-1} \) in the role of \( t \)), we obtain

\[
v((s \land t^p) \lor t^{p-1}) + v(s \land t^{p-1}) \geq v(s \land t^p) + v(t^{p-1}).\]

**Theorem 3.1.**
Let \( v \in CMC^{N,m} \). Then the constrained egalitarian allocation

\[
(d_{ij})_{i \in N, j \in M_i^+}
\]

belongs to the precore \( PC(v) \) of \( v \).
Proof. Note that each \( s \in \mathcal{M}_+^N \) can be expressed as

\[
s = (s \wedge t^1) + (s \wedge t^2 - s \wedge t^1) + \ldots + (s \wedge t^P - s \wedge t^{P-1}),
\]

where some of the terms could be zero. Then, by the definition of \( \alpha_p, \ p \in \{1, \ldots, P\} \), \( D(s) \) can be written as follows:

\[
D(s) = \sum_{i \in N} \sum_{j=1}^{s_i} d_{ij} \\
= \|s \wedge t^1\| \alpha_1 + \|s \wedge t^2 - s \wedge t^1\| \alpha_2 + \ldots + \|s \wedge t^P - s \wedge t^{P-1}\| \alpha_p \\
= \|s \wedge t^1\| \frac{v(t^1)}{||t^1||} + \|s \wedge t^2 - s \wedge t^1\| \frac{v(t^2) - v(t^1)}{||t^2 - t^1||} \\
+ \ldots + \|s \wedge t^P - s \wedge t^{P-1}\| \frac{v(t^P) - v(t^{P-1})}{||t^P - t^{P-1}||}.
\]
Proof continues. Now, we obtain

\[ D(s) \geq \|s \land t^1\| \frac{v(t^1)}{\|s \land t^1\|} + \|s \land t^2 - s \land t^1\| \frac{v((s \land t^2) - (s \land t^1) + t^1) - v(t^1)}{\|s \land t^2 - s \land t^1\|} \]
\[ + \ldots + \|s \land t^P - s \land t^{P-1}\| \frac{v((s \land t^P) - (s \land t^{P-1}) - t^{P-1}) - v(t^{P-1})}{\|s \land t^P - s \land t^{P-1}\|} \]
\[ \geq \|s \land t^1\| \frac{v(s \land t^1)}{\|s \land t^1\|} + \|s \land t^2 - s \land t^1\| \frac{v(s \land t^2) - v(s \land t^1)}{\|s \land t^2 - s \land t^1\|} \]
\[ + \ldots + \|s \land t^P - s \land t^{P-1}\| \frac{v(s \land t^P) - v(s \land t^{P-1})}{\|s \land t^P - s \land t^{P-1}\|} \]
\[ = v(s \land t^1) + (v(s \land t^2) - v(s \land t^1)) + \ldots + (v(s \land t^P) - v(s \land t^{P-1})) \]
\[ = v(s \land t^P) = v(s), \]

where the last inequality follows from Lemma 26. Hence, \( D(s) \geq v(s) \) for each \( s \in \mathcal{M}_N^+ \). Finally, \( D(m) = v(m) \) follows from the constructive definition of \( d \), too.
The constrained egalitarian solution of a multi-choice game does not necessarily belong to the imputation set of the game, as we illustrate in the next example.

**Example 3.2.**
Consider the convex multi-choice game \(<N, m, v>\), with \(N = \{1, 2\}\), \(m = (3, 2)\), and \(v(0, 0) = 0\), \(v(1, 0) = v(0, 1) = 1\), \(v(2, 0) = v(1, 1) = v(0, 2) = 2\), \(v(2, 1) = v(1, 2) = 3\), \(v(3, 0) = v(2, 2) = 5\), \(v(3, 1) = 6\), \(v(3, 2) = 12\). The constrained egalitarian allocation is \(d(v) = (2.4, 2.4, 2.4, 2.4, 2.4)\). Note that \(d_{13} = 2.4 < v(3e^1) - v(2e^1) = 5 - 2 = 3\). Hence, \(d(v) \notin I(v)\).
Our next goal is to prove that the constrained egalitarian solution Lorenz dominates each other element of the precore of $v$ and, hence, is a Lorenz undominated element. To achieve this we need some lemmas, which are gathered in the Appendix.

**Theorem 3.2.**
Let $v \in CMC^{N,m}$. The constrained egalitarian solution $d(v)$ of $v$ Lorenz dominates each other element of the precore of $v$. 
Proof. (Following the scheme of Proof of Theorem 3 in Dutta and Ray, 1989). Let \( x \in \mathcal{P}C(v) \) such that \( x \neq d(v) \). Two cases are possible:

- \( X(t^s) = \nu(t^s) \) for all \( s = 1, 2, \ldots, P \). Then, it is easy to check that we are in the conditions of Lemma A.2; hence, \( d(v) \) Lorenz dominates \( x \).

- \( X(t^s) > \nu(t^s) \) for some \( s = 1, 2, \ldots, P \). Then, it is easy to check that we are in the conditions of Lemma A.3; hence, \( d(v) \) Lorenz dominates \( x \).

Therefore, the proof is finished.

Remark 3.1.
Since \( C(\nu) \subseteq \mathcal{P}C(\nu) \) for each multi-choice game \( \nu \), we straightforwardly obtain from Theorem 3.2 that in case the constrained egalitarian solution of a convex multi-choice game is a core element, then it Lorenz dominates each other core allocation.
In the sequel, we introduce the equal division core for arbitrary multi-choice games and investigate the membership of the constrained egalitarian allocation of a convex multi-choice game to the equal division core of the game.

Given a cooperative multi-choice game \( v \in MC^{N,m} \), we define the equal division core \( EDC(v) \) of \( v \) as the set
\[
\{ x : M \rightarrow \mathbb{R} \mid X(m) = v(m); \ \forall s \in M^N_+ \text{ s.t. } \alpha(s, v) > x_{ij} \text{ for all } i \in car(s), j \in M_i^+ \}. \]

So, \( x \in EDC(v) \) is a distribution of the value of the grand coalition \( m \), where for each multi-choice coalition \( s \), there exists a player \( i \) with a positive participation level in \( s \) and an activity level \( j \in M_i^+ \) such that the payoff \( x_{ij} \) is at least as good as the equal division share \( \alpha(s, v) \) of \( v(s) \).

The relation between the equal division core of a game \( v \in G^N \) and the corresponding core can be extended for multi-choice games with the precore of such games in the role of the core for games \( v \in G^N \).
Proposition 3.3.
Let $v \in MC^{N,m}$. Then $PC(v) \subset EDC(v)$.

Proof. Take $x \notin EDC(v)$. Then there exists $s \in M^N_+$ such that $\alpha(s, v) > x_{ij}$ for all $i \in car(s)$ and $j \in \{1, \ldots, s_i\}$. Then

$$X_{is_i} = \sum_{j=1}^{s_i} x_{ij} < s_i \alpha(s, v),$$

implying that

$$X(s) = \sum_{i \in N} X_{is_i} < \sum_{i \in N} s_i \alpha(s, v) = \alpha(s, v) \cdot \sum_{i \in N} s_i = v(s).$$

So, $x \notin PC(v)$.

We notice that the inclusion relation in Proposition 3.3 may be strict, as in the case of traditional cooperative games (which are special multi-choice games).

Corollary 3.1.
For each $v \in MC^{N,m}$, $C(v) \subset PC(v) \subset EDC(v)$. 
From Theorem 3.1 and Proposition 3.3, we obtain \( d(v) \in PC(v) \subseteq EDC(v) \).

**Remark 3.2.**
In Peters and Zank (2005), a Shapley-type value, called the egalitarian solution for multi-choice games, was introduced and axiomatically characterized by the properties of efficiency, zero contribution, additivity, anonymity, and level-symmetry. However, this solution concept has no connection with the average worth of a multi-choice coalition and with the Lorenz criterion, and makes incomplete use of information regarding the characteristic function. It is therefore not surprising that on the class of convex multi-choice games the egalitarian solution neither coincides with the constrained egalitarian solution nor belongs to the equal division core.
To end our analysis on the constrained egalitarian solution for convex multi-choice games, we recall some properties of the solution concepts in this context and, later on, based on these properties we give an axiomatic characterization of the constrained egalitarian solution.

A solution \( \varphi \) on \( CMC^N,m \) satisfies\(^1\):

**Equal Division Stability (EDS)** if for all \( \langle N, v, m \rangle \in CMC^N,m \) and for all \( s \in \mathcal{M}_+^N \) with \( s \leq m \) there exists \( ij, i \in \text{car}(s) \) and \( j \leq s_i \) such that \( \varphi_{ij}(N, v, m) \geq \alpha(s, v) = \frac{v(s)}{\|s\|} \).

\(^1\)With an abuse of notation we write \( \varphi_{ij}(N, v, m) \) instead of \( \varphi_{ij}(\langle N, v, m \rangle) \).
3. The constrained egalitarian solution

**Bounded Maximum Payoff (BMP)** if for all $\langle N, v, m \rangle \in CMC^{N,m}$ it holds

$$\sum_{i \in \text{car}(s^{\text{max}})} \sum_{j \leq s_i^{\text{max}}} \varphi_{ij}(N, v, m) \leq v(s^{\text{max}})$$

where $s_i^{\text{max}} = \max\{j \leq m_i \mid ij \in \text{arg max}\{\varphi_{ij}(N, v, m)\}\}$. For further use, we introduce here the notation

$S^{\text{max}} = \text{car}(m - s^{\text{max}})$.

**Lower Level First to Receive (LLFR)** if

$$\varphi_{ij}(N, v, m) \geq \varphi_{i(j+1)}(N, v, m)$$

for all $i \in N$ and $1 \leq j \leq m_i - 1$. 
Davis and Maschler Max Consistency (DMMC) if for all games \( \langle N, \nu, m \rangle \in CMC^{N,m} \) and all \( s \geq s^{\max}, i \in S^{\max}, j \in \{1, ..., m_i - s_i^{\max}\} \),

\[
\varphi_i(j+s_i^{\max})(N, \nu, m) = \varphi_{ij}(S^{\max}, \nu_{-s^{\max}}, m - s^{\max}),
\]

where \( \nu_{-s^m} \) is the reduced subgame defined by

\[
\nu_{-s^{\max}}(t) := \begin{cases} 
0 \text{ if } t = 0 \\
\nu(t+s^{\max}) - \sum_{i \in \text{car}(s^{\max})} \sum_{j \leq s_i^{\max}} \varphi_{ij}(N, \nu, m) \end{cases} \text{ if } 0 < t.
\]

The following result shows that the constrained egalitarian solution on the class of convex multi-choice games satisfies the above-mentioned properties.

**Proposition 3.4.**

The constrained egalitarian solution \( d \) satisfies (EDS), (BMP), (LLFR) and (DMMC) on \( CMC^{N,m} \).
3. The constrained egalitarian solution

**Proof.** (EDS) Let us assume that an $s \in M^N_+$ such that $d_{ij}(N, \nu, m) < \alpha(s, \nu)$ exists, for all $i \in \text{car}(s)$ and for all $j \leq s_i$. Then, we have that

$$\sum_{i \in \text{car}(s)} \sum_{j \leq s_i} d_{ij}(N, \nu, m) < \|s\| \alpha(s, \nu) = \nu(s).$$

This implies $D(s) < \nu(s)$ and this contradicts $d(N, \nu, m) \in \mathcal{PC}(\nu)$. 

(BMP) From Proposition 3.2 we have that $s^1 = s^{\max}$ and

$$\sum_{i \in \text{car}(s^1)} \sum_{j \leq s^1_i} d_{ij}(N, \nu, m) = \nu(s^1) = \nu(s^{\max}).$$

(LLFR) This property follows from the definition of the constrained egalitarian solution and Proposition 3.2. 

(DMMC) From Proposition 3.2 we have that $s^1 = s^{\max}$ and

$$\sum_{i \in \text{car}(s^1)} \sum_{j \leq s^1_i} d_{ij}(N, \nu, m) = \nu(s^1) = \nu(s^{\max}),$$

hence we can write
Proof continues

\[ v_{-s^{\text{max}}}(t) := \begin{cases} 
0 & \text{if } t = 0 \\
v(m) - v(s^{\text{max}}) & \text{if } t = m - s^{\text{max}} \\
\max_{s \leq s^{\text{max}}} \{ v(t + s) - \sum_{i \in \text{car}(s)} \sum_{j \leq s_i} d_{ij}(N, v, m) \} & \text{if } 0 < t < m - s^{\text{max}}. 
\end{cases} \]

On the other hand, since \( d(N, v, m) \in \mathcal{PC}(v) \) and the game is convex, we can derive

\[ v(t+s) - \sum_{i \in \text{car}(s)} \sum_{j \leq s_i} d_{ij}(N, v, m) \leq v(t+s) - v(s) \leq v(t+s^{\text{max}}) - v(s^{\text{max}}). \]

Therefore,

\[ v_{-s^{\text{max}}}(t) = v(t + s^{\text{max}}) - v(s^{\text{max}}) = v_{-s^{\text{max}}}(t) \]

and by the definition of the constrained egalitarian solution we obtain the desired result.
Theorem 3.3.
If a solution $\varphi$ satisfies (EDS), (BMP), (LLFR) and (DMMC), then $\varphi$ is the constrained egalitarian solution.

Proof. We are going to prove that if $\varphi$ satisfies the properties, then $\varphi = d$. The proof will be done by induction on the total number of levels.
For $\|m\| = 1$ there is a single player with only one level of activity. Then, we have that $\varphi(N, v, m) = d(N, v, m) = v(1)$. Suppose that for some $p \geq 2$ we have that $\varphi(N, v, m) = d(N, v, m)$ for all multi-choice convex games $(N, v, m)$ with $\|m\| \leq p - 1$. Now we have to prove that $\varphi(N, v, m) = d(N, v, m)$ for all multi-choice convex games with $\|m\| \leq p$. 
Proof continues We know that $\alpha(s^1, v) = \alpha(s^{\text{max}}, v)$; therefore by the definition of $s^1$ and we have $s^{\text{max}} \leq s^1$. On the other hand, we have $v_{-s^{\text{max}}}(t) = v(s^{\text{max}} + t) - v(s^{\text{max}})$ and, consequently, it is convex.

Let us assume that $s^{\text{max}} < s^1$, hence

$$v_{-s^{\text{max}}}(s^1 - s^{\text{max}}) = v(s^1) - v(s^{\text{max}}).$$

Furthermore, since $\alpha(s^1, v) = \alpha(s^{\text{max}}, v)$ we have

$$\alpha(s^1, v) = \frac{v(s^1)}{\|s^1\|} = \frac{v(s^1) - v(s^{\text{max}})}{\|s^1 - s^{\text{max}}\|}.$$ 

Indeed, suppose that \( \frac{v(s^1)}{\|s^1\|} \neq \frac{v(s^1) - v(s^{\text{max}})}{\|s^1 - s^{\text{max}}\|} \), then

$$\|s^1 - s^{\text{max}}\| v(s^1) \neq \|s^1\| (v(s^1) - v(s^{\text{max}})),$$

and therefore we obtain $\|s^{\text{max}}\| v(s^1) \neq \|s^1\| v(s^{\text{max}})$ which is a contradiction.
Proof continues
Since the reduced game is convex, $\varphi$ satisfies (DMMC), the definition of $s^{\max}$ and $\alpha(s^{1}, v) = \alpha(s^{\max}, v)$ we can deduce

\[
\varphi_{ij}(S^{\max}, v_{-s^{\max}}, m - s^{\max}) = \varphi_{i(j+s_{i}^{\max})}(N, v, m) < \alpha(s^{1}, v)
\]

\[
= \frac{v_{-s^{\max}}(s^{1} - s^{\max})}{\|s^{1} - s^{\max}\|},
\]

for all $i \in car(s^{1} - s^{\max})$ and $1 \leq j \leq s_{i}^{1} - s_{i}^{\max}$, which contradicts (EDS) regarding $s^{1} - s^{\max}$ in the reduced game $(S^{\max}, v_{-s^{\max}}, m - s^{\max})$. Therefore, we conclude that $s^{1} = s^{\max}$. 
Proof continues

Further, the definition of $s^1$ and $s^1 = s^{\text{max}}$ imply that

$$\varphi_{ij}(N, v, m) = \frac{v(s^1)}{\|s^1\|} = d_{ij}(N, v, m),$$

for all $i \in \text{car}(s^1)$ and $j \leq s^1_i$. Then, if $s^1 = m$ we have finished. If $s^1 \neq m$, then since $\varphi$ and $d$ satisfy (DMMP) and taking into account the induction hypothesis we obtain

$$\varphi_{i(j+s^1_i)}(N, v, m) = \varphi_{ij}(S^1, v_{s^1}, m - s^1) = d_{ij}(S^1, v_{s^1}, m - s^1)$$

$$= d_{i(j+s^1_i)}(N, v, m),$$

for all $i \in S^1 = \text{car}(m - s^1)$ and $j \leq m - s^1_i$. Therefore, the desired result holds.
Inspired by Sprumont (1990) (see also Hokari (2000), Thomson (1983, 1995)) who introduced and studied the interesting notion of population monotonic allocation scheme (pmas) for traditional cooperative games, we introduce here for multi-choice games the notion of level-increase monotonic allocation scheme (limas).

Recall that a pmas for a (traditional) cooperative game \( \langle N, v \rangle \) is an allocation scheme \([a_S, i]_{S \in 2^N \setminus \{\emptyset\}, i \in S}\) such that:

1. \((a_S, i)_{i \in S} \in C(v_S)\) for each \(S \in 2^N \setminus \{\emptyset\}\), where \(v_S\) is the subgame corresponding to \(S\);
2. \(a_S, i \leq a_T, i\) for all \(S, T \in 2^N \setminus \{\emptyset\}\) with \(S \subset T\) and \(i \in S\).
Let $v \in MC^{N,m}$ and let $t \in \mathcal{M}_{+}^{N}$. For $i \in N$, denote the set 
$\{1, 2, \ldots, t_{i}\}$ by $M_{i}^{t}$. A scheme $a = [a_{ij}^{t}]_{i \in N, j \in M_{i}^{t}}$ is called a
level-increase monotonic allocation scheme (limas) if:

(i) $a^{t} \in C(v_{t})$ for all $t \in \mathcal{M}_{+}^{N}$ (stability condition);
(ii) $a_{ij}^{s} \leq a_{ij}^{t}$ for all $s, t \in \mathcal{M}_{+}^{N}$ with $s \leq t$, for all $i \in \text{car}(s)$ and
for all $j \in M_{i}^{s}$ (level-increase monotonicity condition).

The level-increase monotonicity condition implies that, if the
scheme is used as regulator for the (level) payoff distributions, in
the multi-choice subgames players are paid for each one-unit level
increase (weakly) more in larger coalitions than in smaller
coalitions.
4. Level increase monotonic schemes

We notice that a necessary condition for the existence of a limas for a multi-choice game \( v \) is the existence of core elements for \( v_t \) for each \( t \in \mathcal{M}^N \). But this is not sufficient, as in the case of traditional cooperative games which can be seen as multi-choice games where each player has exactly two participation levels. A sufficient condition for the existence of a limas is the convexity of the game as we see in Theorem 4.1.

Let \( v \in MC_{N,m} \) and \( x \in W(v) \). Then we call \( x \) limas extendable if there exists a limas \( [a_{ij}^t]_{t \in \mathcal{M}^N_+}^{i \in N, j \in M_i^+} \) such that \( a_{ij}^m = x_{ij} \) for each \( i \in N \) and \( j \in M_i^+ \).
In the next theorem we show that convex multi-choice games have a limas. Specifically, we prove that each Weber set element of a convex multi-choice game is limas extendable. In the proof, restrictions of $\sigma \in \Xi(v)$ to subgames $v_t$ of $v$ will play a role. It will be useful to look at such $\sigma$ as being a sequence of flags $f^i$, $i \in N$, signaling the players’ turns to one-unit level increase according to their sets of participation levels. Then, for each $t \in M^N_+$, the restriction of $\sigma$ to $t$, denoted here by $\sigma_t$, can be obtained from the sequence of flags of $\sigma$ by ”removing” (notation ”*”) for each player $i \in N$ exactly $m_i - t_i$ flags $f^i$ starting from the back of that sequence. We illustrate this procedure in Example 4.1.
Example 4.1.
Consider a convex multi-choice game $\langle N, m, \nu \rangle$ with
$N = \{1, 2, 3\}$, $m = (2, 1, 2)$ and $\sigma^1 \in \Xi(\nu)$ expressed in terms of
flags as $\sigma^1 = (f^3, f^1, f^3, f^2, f^1)$. Note that this ordering
generates the following maximal chain of multi-choice
collecions in $M^N$:

$$
(0, 0, 0) \xrightarrow{f^3} (0, 0, 1) \xrightarrow{f^1} (1, 0, 1) \xrightarrow{f^3} (1, 0, 2) \xrightarrow{f^2} (1, 1, 2) \xrightarrow{f^1} (2, 1, 2).
$$

Now, consider the multi-choice coalition $t = (1, 1, 1)$ and the
 corresponding subgame $\langle N, t, \nu_t \rangle$. Then, the restriction of $\sigma^1$ to
$t$ is the ordering $\sigma^1_t$ which can be expressed in terms of flags as
$(f^3, f^1, *, f^2, *)$; it generates the following maximal chain of
multi-choice coalitions in $M^N_t$:

$$
(0, 0, 0) \xrightarrow{f^3} (0, 0, 1) \xrightarrow{f^1} (1, 0, 1) \xrightarrow{f^2} (1, 1, 1).
$$
**Theorem 4.1.**
Let $v \in CMC^N,m$ and let $x \in W(v)$. Then $x$ is limas extendable.

**Proof.** Since $x$ is in the convex hull of the marginal vectors $w^\sigma,v$ of $v$, it suffices to prove that each marginal vector $w^\sigma,v$ is limas extendable, because then the right convex combination of these limas extensions gives a limas extension of $x$.

Take $\sigma \in \Xi(v)$ and define $[a^t_{ij}]_{i \in N, j \in M^t_i}$ by $a^t_{ij} := w^\sigma_{ij}^t$ for each $t \in M^N_+, i \in N$ and $j \in M^t_i$, where $\sigma_t$ is the restriction of $\sigma$ to $t$ (obtained via the procedure described above and illustrated in Example 4.1). We claim that this scheme is a limas extension of $w^\sigma,v$.

Clearly, $a^m_{ij} = w^\sigma_{ij}^m$ for each $i \in N$ and $j \in M^+_i$ since $v_m = v$.

Further, each multi-choice subgame $v_t$, $t \in M^N_+$, is a convex game, and since $w^\sigma_{t},v_t \in W(v_t)$ and $W(v_t) \subset C(v_t)$ (cf. Theorem 11.9 in Branzei, Dimitrov and Tijs (2005)), it follows that $(a^t_{ij})_{i \in N, j \in M^t_i} \in C(v_t)$. Hence, the scheme satisfies the stability condition.
Proof continues To prove the level-increase monotonicity condition, take \( s, t \in \mathcal{M}^N_+ \) with \( s \leq t \), \( i \in \text{car}(s) \), and \( j \in M^s_i \subset M^t_i \). We have to show that \( a^s_{ij} \leq a^t_{ij} \). Now,

\[
a^s_{ij} = w_{ij}^{\sigma^s, v_s} = v(u_{-i}, j) - v(u_{-i}, j - 1),
\]

where \((u_{-i}, j)\) is the intermediary multi-choice coalition in the maximal chain generated by the restriction of \( \sigma \) to \( s \), when player \( i \) increased his/her participation level from \( j - 1 \) to \( j \). Similarly,

\[
a^t_{ij} = w_{ij}^{\sigma^t, v_t} = v(\bar{u}_{-i}, j) - v(\bar{u}_{-i}, j - 1).
\]

Note that, since \( s \leq t \), in the maximal chain generated by \( \sigma_s \) the turn of \( i \) to increase his/her participation level from \( j - 1 \) to \( j \) will come not later than the same turn in the maximal chain generated by \( \sigma_t \), implying that \((u_{-i}, j) \leq (\bar{u}_{-i}, j)\). Furthermore, \((\bar{u}_{-i}, j) \leq m\).
Proof continues
Then,

\[ a_{ij}^s = v(u_{-i}, j) - v(u_{-i}, j - 1) \leq v(\bar{u}_{-i}, j) - v(\bar{u}_{-i}, j - 1) = a_{ij}^t, \]

where the inequality follows from the convexity of \( v \). Specifically, we used relation

\[ v(s + t) - v(s) \geq v(\bar{s} + t) - v(\bar{s}) \]

of \( \bar{s} \), \( (\bar{u}_{-i}, j - 1) \) in the role of \( s \), and \( (0_{-i}, 1) \) in the role of \( t \). Hence, \( [a_{ij}^t]_{i \in \mathcal{N}, j \in \mathcal{M}_i^+} \) is a limas extension of \( w^{\sigma, v} \).
Further, the total Shapley value (cf. Nouweland et al. (1995)) of a convex multi-choice game, which is the scheme \( [\Phi_{ij}(v_t)]_{i \in N, j \in M^t_i} \) with the Shapley value of each multi-choice subgame \( v_t \), is a limas. One can represent a limas as a defective \( |M^N_+| \times |M^+| \)-matrix, whose rows correspond to multi-choice coalitions and whose columns correspond to elements of \( M^+ \) arranged according to the increasing ordering for players and for each player with respect to his/her participation levels. In each row \( t \) there is a core element of the multi-choice subgame \( v_t \), with "*" for all components \( x_{ij} \), with \( i \in N \) and \( j \in \{ t_i + 1, \ldots, m_i \} \).

**Example 4.2.**

Consider the convex multi-choice game \( \langle N, m, v \rangle \) with \( N = \{1, 2\}, m = (2, 1), v((0, 0)) = 0, v((1, 0)) = 5, v((2, 0)) = 6, v((0, 1)) = 3, v((1, 1)) = 9, v((2, 1)) = 13. \)
Example continues There are three orderings on $M^+ = \{(1, 1), (1, 2), (2, 1)\}$: $\sigma^1 = (f^1, f^1, f^2)$, $\sigma^2 = (f^1, f^2, f^1)$ and $\sigma^3 = (f^2, f^1, f^1)$. The corresponding marginal vectors $w^{\sigma^1, \nu}$, $w^{\sigma^2, \nu}$, $w^{\sigma^3, \nu}$ are extendable to the following level-increase monotonic schemes:

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Then, the total Shapley value $\Phi(\nu)$ generates the limas

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5. Properties of solution concepts

As in the case of convex crisp games, solution concepts on convex multi-choice games have nice properties. First, note that the Shapley value \( \Phi(v) \) of \( v \in CMC^N,m \) belongs to the core \( C(v) \) of \( v \). Moreover, we have already seen that the corresponding extended Shapley value is a limas of the game and hence, the convexity of a multi-choice game is a sufficient condition for the existence of such monotonic allocation schemes. Second, the core of a convex multi-choice game is the unique stable set of the game.

**Theorem 5.1.**
Let \( v \in CMC^N,m \). Then \( C(v) \) is the unique stable set.
In the next result we only consider multi-choice clan games where for each player $i$ the level $m_i$ is effective, i.e. there is $s_i$ such that $v(s_i, m_i) - v(s_{i-1}, m_{i-1}) > 0$. Note that this condition is not a very restrictive one, because we are excluding those highest levels which do not contribute.

As it happens to traditional clan games (see Potters et al., 1989) convexity of multi-choice clan games is characterized either by the equality in the union property constraints (plus certain non-negativity of marginal contributions), or by monotonicity plus stability of the core.
Theorem 5.2.
Let $v \in MC_C^{N,m}$ such that the level $m_i$ is effective, for each player $i$. The following assertions are equivalent:

1. $v$ is convex;
2. $w_{ij} \geq 0$, for each $i \in N$ and $j \in \{1, \ldots, m_i\}$, and
   \[ v(s) + \sum_{i \in N \setminus C} w_{is_i^+} (m, v) = v(m), \text{ for all } s \text{ with } s_C = 1_C; \]
3. $v$ is monotonic, $v(m) - v(m_{-i}, m_i - 1) > 0$ for each $i \in N \setminus C$, and $C(v)$ is a stable set.
Appendix

Let \((x_1, x_2, \ldots, x_n)\) be the vector obtained from \(x\) by ordering its components in decreasing order: \(x_1 \geq x_2 \geq \ldots \geq x_n\). In the sequel, we use the following equivalent definition of Lorenz dominance: \(x\) Lorenz dominates \(y\) if \(\sum_{i=1}^{k} x(i) \leq \sum_{i=1}^{k} y(i)\) for each \(k = 1, 2, \ldots, n\), with at least one strict inequality.

**Lemma A.1**

*The vector \(a = 1_n a\) Lorenz dominates each other element of the set \(\{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = na > 0\}\), where \(1_n\) represents the vector whose \(n\) coordinates are equal to 1.*
**Proof.** Let us consider \( x \neq a \). It is obvious that \( a = a(1) < x(1) \).

Let us assume that \( ka \leq \sum_{i=1}^{k} x(i) \). Now, we are going to prove that \( (k + 1)a \leq \sum_{i=1}^{k+1} x(i) \). Let us suppose on the contrary that 
\[ (k + 1)a > \sum_{i=1}^{k+1} x(i). \]
This implies that 
\[ \sum_{i=k+2}^{n} x(i) > (n - k - 1)a. \]
Therefore, there exists an \( i^* > k + 1 \) such that \( x(i^*) > a \). On the other hand, we have

\[
(n - 1)a = ka + (n - k - 1)a < \sum_{i=1}^{k} x(i) + \sum_{i=k+2}^{n} x(i) = na - x(k+1).
\]

Hence, we obtain \( x(k+1) < a \). Further, since \( i^* > k + 1 \) implies 
\( x(k+1) \geq x(i^*) \), we have \( a > x(k+1) \geq x(i^*) > a \). Therefore, the result holds.
Lemma A.1 bis The vector \( a = 1_n a \) Lorenz dominates each other element of the set \( \{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \geq na > 0 \} \).

Proof. Let us consider \( x^{(1)} \geq x^{(2)} \geq \ldots \geq x^{(n)} \) and define \( x_{(n)}' = x^{(n)} - \left( \sum_{i=1}^n x_i - na \right) \). Then, \( x^{(1)} \geq x^{(2)} \geq \ldots \geq x_{(n)}' \) and \( \sum_{i=1}^{n-1} x_i + x_{(n)}' = na \). Thus, applying Lemma A.1 we obtain the desired result.

Lemma A.2

The vector \( a = (1_{n_1} a_1, 1_{n_2} a_2, \ldots, 1_{n_t} a_t) \), such that \( a_1 \geq a_2 \geq \ldots \geq a_t > 0 \) and \( \sum_{i=1}^t n_i = n \), Lorenz dominates each other element of the set

\[
\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n_1} x_i = n_1 a_1, \sum_{i=n_1+1}^{n_2} x_i = n_2 a_2, \ldots, \sum_{i=\sum_{j=1}^{t-1} n_j+1}^{n} x_i = n_t a_t \}.
\] (A1)
Proof. Let us consider $x \neq a$ and the vector obtained from vector $x$ by rearranging in decreasing order the elements inside each block displayed in (A1):

$$(x(1,1), \ldots, x(1,n_1); x(2,1), \ldots, x(2,n_2); \ldots; x(t,1), \ldots, x(t,n_t))$$ with

\[
\begin{align*}
x(1,1) & \geq \cdots \geq x(1,n_1) \\
x(2,1) & \geq \cdots \geq x(2,n_2) \\
& \cdots \\
x(t,1) & \geq \cdots \geq x(t,n_t) .
\end{align*}
\]

By Lemma A.1 we know that for each block $i = 1, 2, \ldots, t$ we have

$$\sum_{j=1}^{k} a_{i,j} \leq \sum_{j=1}^{k} x_{(i,j)}$$ for each $k = 1, 2, \ldots, n_i$, \hfill (A2)

and at least one inequality is strict, where $x_{(i,j)}$ is the $j$-th element in the decreasing order within the $i$-th block of vector $x$. 
Now, we are going to prove that $x$ is Lorenz dominated by $a$. For each $k$, $1 \leq k \leq n$, we have

$$k = \sum_{h=0}^{r} n_h + r(k)$$

for some $r \leq t$, $n_0 = 0$ and $0 \leq r(k) \leq n_{r+1}$. Then,

$$\sum_{i=1}^{k} x(i) \geq \sum_{h=1}^{r} \sum_{j=1}^{n_h} x(h,j) + \sum_{j=1}^{r(k)} x(r+1,j) \geq \sum_{h=1}^{r} \sum_{j=1}^{n_h} a(h,j)$$

$$+ \sum_{j=1}^{r(k)} a(r+1,j) = \sum_{i=1}^{k} a(i),$$

where the first inequality follows from the decreasing order inside each block, the second inequality follows from (A2) and the equality from the definition of $a$. Note that we use the alternative notations for ordered vectors $a(\cdot)$ and $a(\cdot, \cdot)$, where the former refers to blocks and the latter to the elements inside each block.
Now, since $x \neq a$ there exists a block $q$ and an $l$-th coordinate inside of it such that $\sum_{j=1}^l a_q < \sum_{j=1}^l x(q,j)$. Take 
$\hat{k} = \sum_{h=0}^{q-1} n_h + l$. Then,

$$\sum_{i=1}^{\hat{k}} x(i) \geq \sum_{h=1}^{q-1} \sum_{j=1}^{n_h} x(h,j) + \sum_{j=1}^{l} x(q,j) > \sum_{h=1}^{q-1} \sum_{j=1}^{n_h} a(h,j) + \sum_{j=1}^{l} a(q,j) = \sum_{i=1}^{\hat{k}} a(i),$$

where the strict inequality follows from (A1) and the above comment. Therefore, the result holds.
**Lemma A.2 bis**

The vector $a = (1_{n_1} a_1, 1_{n_2} a_2, ..., 1_{n_t} a_t)$, such that $a_1 \geq a_2 \geq ... \geq a_t > 0$ and $\sum_{i=1}^{t} n_i = n$, Lorenz dominates each other element of the set

$$\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n_1} x_i \geq n_1 a_1, \sum_{i=n_1+1}^{n_2} x_i \geq n_2 a_2, ..., \sum_{i=\sum_{j=1}^{t-1} n_j+1}^{n} x_i \geq n_t a_t \}.$$  

**Proof.** The proof can be derived straightforwardly taking into account Lemma A.2 and Lemma A.1 bis.
Lemma A.3  The vector $a = (1_{n_1}a_1, 1_{n_2}a_2, ..., 1_{n_t}a_t)$, where $a_1 \geq a_2 \geq ... \geq a_t > 0$ and $\sum_{i=1}^{t} n_i = n$, Lorenz dominates each other element $x \in \mathbb{R}^n$ such that

\[
\sum_{i=1}^{n_1} x_i \geq n_1a_1, \quad \sum_{i=1}^{n_1+n_2} x_i \geq \sum_{i=1}^{2} n_i a_i, \quad ... \quad \sum_{i=1}^{n-n_t} x_i \geq \sum_{i=1}^{t-1} n_i a_i, \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{t} n_i a_i.
\]

(A3)

Proof. Let us consider $x \neq a$. The goal is to show that $a$ Lorenz dominates $x$, i.e $\sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} a(i)$ for each $k = 1, ..., n$ with at least one strict inequality. If for this vector $x$ every inequality in (A3) is an equality, then applying Lemma A.2 we obtain the result. Therefore, let us suppose that at least one inequality in (A3) is strict.
Proof continues.
In this case, there exist $r$ such that

$$\sum_{i=1}^{n_1+n_2+\ldots+n_{r-1}} x_i = \sum_{i=1}^{n_1+n_2+\ldots+n_{r-1}} a_i \quad \text{and} \quad \sum_{i=1}^{n_1+n_2+\ldots+n_r} x_i > \sum_{i=1}^{n_1+n_2+\ldots+n_r} a_i.$$  

For the $r$-th block, by applying an analogous argument as in Lemma A.1, we obtain that $\sum_{j=1}^{k} a_r \leq \sum_{j=1}^{k} x_{(r,j)}$ for each $k = 1, \ldots, n_r$ and at least one inequality is strict (for example the last one). On the other hand, since $\sum_{i=1}^{n} x_i = \sum_{i=1}^{t} n_i a_i$, we also know that there is an $s$ such that $r < s \leq t$ and

$$\sum_{j=1}^{n_s} x_{(s,j)} < n_s a_s.$$  
Let $s^*$ be the first block for which

$$\sum_{j=1}^{n_{s^*}} x_{(s^*,j)} < n_{s^*} a_{s^*}.$$  
Now, we consider the vector $(x'_1, x'_2, \ldots, x'_{(n_1+n_2+\ldots+n_{s^*}-1)})$ obtained by arranging in decreasing order the first $n_1 + n_2 + \ldots + n_{s^*} - 1$ coordinates of vector $x$, that is

$$x'_1 \geq x'_2 \geq \ldots \geq x'_{(n_1+n_2+\ldots+n_{s^*}-1)}.$$
Proof continues. Applying Lemma A.2 bis, we obtain

\[ \sum_{i=1}^{k} x'_i \geq \sum_{i=1}^{k} a(i), \]

for each \( k \leq n_1 + n_2 + \ldots + n_{s^*} - 1 \), with at least one strict inequality. In particular, we have

\[ \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s^*} - 1} x(i) \geq \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s^*} - 1} x'(i) > \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s^*} - 1} a(i). \]

Let \( A = \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s^*} - 1} x'_i - \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s^*} - 1} a(i) \); then, from the definition of the set \((A3)\), we obtain

\[ A \geq n_{s^*} a_{s^*} - \sum_{j=1}^{n_{s^*}} x(s^*, j) = B > 0. \]

Now, we take \( x(s^*, 1) \), which is a largest element of block \( s^* \), and distinguish between two cases:
Proof continues

- \( B \geq a_{s^*} - x_{(s^*,1)} > 0 \). We construct a new decreasing ordered vector \( x'' \) using the first \( n_1 + n_2 + \ldots + n_{s^*-1} \) coordinates of vector \( x \) and \( x_{(s^*,1)} \), that is

\[
x''(1) \geq x''(2) \geq \ldots \geq x''(n_1+n_2+\ldots+n_{s^*-1}+1).
\]

We have

\[
\sum_{i=1}^{n_1+n_2+\ldots+n_{s^*-1}+1} x(i) \geq \sum_{i=1}^{n_1+n_2+\ldots+n_{s^*-1}+1} x''(i)
= \sum_{i=1}^{n_1+n_2+\ldots+n_{s^*-1}} x'(i) + x(s^*,1).
\]
Proof continues On the other hand, we have

\[ n_1 + n_2 + \ldots + n_{s* - 1} + 1 \]
\[ \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s* - 1}} x''(i) - \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s* - 1}} a(i) \]
\[ = \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s* - 1}} x'(i) + x(s*,1) - \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s* - 1}} a(i) - a_{s*} \]
\[ = A + x(s*,1) - a_{s*} \geq A - B \geq 0. \]

Therefore, we obtain

\[ \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s* - 1} + 1} x(i) \geq \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s* - 1} + 1} x'(i) \geq \sum_{i=1}^{n_1 + n_2 + \ldots + n_{s* - 1} + 1} a(i) \]

▶ \[ x(s*,1) \geq a_{s*} \]. The proof of this case is straightforward.
Proof continues At this point, we can do the same for each other element \( l \) in the \( s^* \)-th block, by considering the two cases: 
\[ B \geq l_{s} s^* - \sum_{j=1}^{l} x(s^*,j) > 0 \text{ and } \sum_{j=1}^{l} x(s^*,j) \geq l_{s} s^*. \]
So, at the end, we will obtain a new decreasing ordered vector \( x'' \) with the first \( n_1 + n_2 + \ldots + n_{s^* - 1} + n_{s^*} \) coordinates of \( x \) and

\[
\sum_{i=1}^{n_1+n_2+\ldots+n_{s^*}} x(i) \geq \sum_{i=1}^{n_1+n_2+\ldots+n_{s^*}} x''(i) \geq \sum_{i=1}^{n_1+n_2+\ldots+n_{s^*}} a(i).
\]

If another block \( p > s^* \) such that \( \sum_{j=1}^{n_p} x(p,j) < n_p a_p \) exists, we can repeat exactly the same reasoning so that we obtain
\[
\sum_{i=1}^{n_1+n_2+\ldots+n_p} x(i) \geq \sum_{i=1}^{n_1+n_2+\ldots+n_p} a(i)
\]
and, finally, we can conclude that \( a \) Lorenz dominates each other element \( x \) of the set (A3).
References

