A simple algorithm for the nucleolus of airport profit games

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Abstract In this paper we present a procedure for calculating the nucleolus for airport profit games which are a generalization of the airport cost games.

Keywords Airport cost (profit) games · Nucleolus

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1 Introduction

When Littlechild and Thompson (1977) investigated the cost sharing problem arising from the construction of a landing strip for Birmingham airport they proposed a cooperative game-theoretic approach to solve this problem. In the characteristic function of the game only the airport runway cost function was taken into account. Taking advantage of the special structure of these “airport cost games”, Littlechild (1974) derived an extremely simple algorithm for the calculation of the nucleolus (Schmeidler 1969). Formerly, Littlechild and Owen (1973) had obtained a remarkably simplified formula for the Shapley value (Shapley 1953) (see also Dubey 1982). More recently, Potters and Sudhölter (1999) have studied some consistency properties for these games.

Later, Littlechild and Owen (1977) pointed out that: “airport cost games are only a partial representation of the actual situation, for they take no account of the revenues or other benefits generated by aircraft movements”. For instance, revenues are important in determining the optimal size of the runway, or the final payoffs to agents, since none of them would agree to pay a fee schedule higher than his revenue. Consequently, these authors introduced “airport profit games” defining the worth of a coalition as the maximum revenue minus cost attainable by the coalition, whenever that maximum is non-negative, otherwise setting it equal to 0. In their paper, these authors claim that “the fee schedule corresponding to the nucleolus of an airport profit game is independent of the revenues” and that the nucleolus for these games could be directly determined from the simple expression of the nucleolus for airport cost games derived by Littlechild (1974). This claim is not true\(^1\) and in this paper we determine a procedure to calculate the nucleolus for airport profit games. As we shall see, the proposed algorithm, based on the maximal excesses of coalitions, considers a sequence of airport problems, starting from the original one, where each problem is a “reduction” of the preceding one.

The outline of the paper is as follows. Section 2 contains the model and some preliminary results. In Sect. 3, we obtain the results concerning the coalitions with maximal excess. Section 4 describes the algorithm.

2 Airport problems

The tuple \((N, \preceq, C, b)\) is an airport problem if

(a) \(N\) is a finite nonempty set.
(b) \(\preceq\) is a total order relation on \(N\).
(c) \(C : N \to \mathbb{R}_+^+\) is non-decreasing (i.e., \(i \preceq j\) implies \(C(i) \leq C(j)\)).
(d) \(b \in \mathbb{R}_+^N\).

\(^1\)We acknowledge J. Arin by pointing out that simple examples of three-person games can be used to check that the two statements of Littlechild–Owen’s note are incorrect. As far as we know this mistake has not been corrected until now. This led us to amend a former version of this work.
The set $N$ represents the set of agents and every non-empty subset of $N$ is called a coalition. Every agent $i$ wishes to implement a project that generates a cost $C(i) \geq 0$. Moreover, $i \preceq j$ means that the project of agent $j$ is an extension of the project of agent $i$, and accordingly, we assume $C(i) \leq C(j)$. In addition, if the project of agent $i$ is carried out, then that player receives a revenue $b_i > 0$.

Remark 2.1 The ordering $\preceq$ has been included in the description of the model for the sake of clarity of the proofs. Needless to say, this ordering could have been any one induced by $C$. If there is no confusion we simply write $(N, C, b)$.

Let $x \in \mathbb{R}^N$, and $S \subseteq N, S \neq \emptyset$. We write $x_S$ for the restriction of $x$ to $\mathbb{R}^S$, and $x(S) = \sum_{i \in S} x_i$. Therefore, $b(S)$ is the total revenue of the members of $S$.

Since actually $C \in \mathbb{R}^N_{+}$, with some abuse of notation we denote

$$C(S) = \max \{C(i) : i \in S\}.$$ 

Thus, $C(S)$ represents the cost of attending to the needs of coalition $S$. By convention $C(\emptyset) = 0$.

With each problem $(N, C, b)$, we associate a TU game $(N, v^{(N,C,b)})$, called an airport profit game, where

$$v^{(N,C,b)}(S) = \max \{b(R) - C(R) : R \subseteq S\} \quad \text{for each } S \subseteq N,$$

represents the profits obtained by coalition $S$. Thus, the members of $S \subseteq N$ will carry out the most profitable project that is feasible, whenever that project generates a non-negative surplus.

The proof of the following proposition is standard and it is omitted.

Proposition 2.1 Let $(N, C, b)$ be an airport problem, then the associated profit game $(N, v)$ is convex.

Let $(N, \preceq, C, b)$ be an airport problem, $S$ a proper coalition of $N$, and $x \in \mathbb{R}^N$. The reduced airport profit of $(N, \preceq, C, b)$ with respect to $S$ and $x$, is the problem $(S, \preceq_S, C^{S,x}, b_S)$, where

$$C^{S,x}(i) = \left( \min \left\{ C(Q \cup \{i\}) - (b(Q) - x(Q)) : Q \subseteq N\backslash S \right\} \right)_+ \quad \text{for every } i \in S$$

($a_+$ denotes $\max\{a, 0\}$).

We simply write $C^x$, and $(S, C^x, b)$ if no confusion arises. Note that $C^{S,x}$ depends only on $x_{N\backslash S}$, so abusing notation, we also write $C^{S,x}$ when $x \in \mathbb{R}^N_+\backslash S$.

To illustrate the meaning of the reduced problem, suppose that agents in $N$ constructing a runway, agree that agent $i$ should receive a surplus of $x_i$, i.e. $i$'s payment to finance the runway is $b_i - x_i$. Then we imagine agent $i$ leaving the remaining agents negotiating, and assume agent $i$ specifies the exact part of the runway that he will pay for. It is reasonable to think that he will pay for a fragment between the origin and the last point of the runway that he may use.
If agent \( i \) defrays the part of the section of the runway closest to this last point, the remaining agents will face a ‘reduced’ airport problem, whose cost function is exactly \( C^x \), and their revenues are the same as in the original situation.

Let \( (N,v) \) be a TU game, \( S \subset N \) a coalition, and \( x \in \mathbb{R}^N_+ \). The reduced game of \( (N,v) \) with respect to \( S \) and \( x \) (Davis and Maschler 1965) is the game \((S,v^{S,x})\), given by

\[
v^{S,x}(T) = \begin{cases} 
v(N) - x(N \setminus S) & \text{if } T = S, \\
\max \{v(T \cup Q) - x(Q) : Q \subseteq N \setminus S\} & \text{if } T \neq \emptyset, S, \\
0 & \text{if } T = \emptyset.
\end{cases}
\]

Note that \( v^{S,x} \) depends only on \( x_{N \setminus S} \), so abusing notation, we also write \( v^{S,x} \) when \( x \in \mathbb{R}^{N \setminus S}_+ \).

**Proposition 2.2** Let \((N,C,b)\) be an airport problem, \( S \) a proper coalition of \( N \), and \( x \) a core element of \( v^{(N,C,b)} \). Then

\[
\left( v^{(N,C,b)} \right)^{S,x} = v^{(S,C^x,b)}.
\]

In words, the reduced airport surplus game is the game corresponding to the reduced airport problem.

**Proof** First note that if \( T \subseteq S \subset N \) are two coalitions, and \( x \) is a core element of \( v^{(N,C,b)} \), then \( \left( T, (C^{S,x})^{T,x}_S, (b_S)_T \right) = (T, C^{T,x}, b_T) \), and \( (v^{S,x})^{T,x}_S = v^{T,x} \). Hence it is enough to consider the case in which \( S = N \setminus \{i\} \) for some \( i \in N \).

First of all, notice that

\[
C^x(R) = \min \left\{ C(R), C(R \cup \{i\}) - (b_i - x_i) \right\} \quad \text{for every } R \subseteq N \setminus \{i\}.
\]

To simplify, write \( v = v^{(N,C,b)} \), and \( w = v^{(N\setminus i),C^x,b} \). Thus, we have to show that \( w(T) = v^x(T) \) for every \( T \subseteq N \setminus \{i\} \). We consider first the case \( T \neq N\setminus \{i\} \). Then

\[
v^x(T) = \max \left\{ v(T), v(T \cup \{i\}) - x_i \right\}
\]

\[
= \max_{R \subseteq T} \left\{ \max \left\{ b(R) - C(R), b(R \cup \{i\}) - C(R \cup \{i\}) - x_i \right\} \right\}
\]

\[
= \max_{R \subseteq T} \left\{ b(R) - \min \left\{ C(R), C(R \cup \{i\}) - (b_i - x_i) \right\} \right\}
\]

\[
= \max \left\{ b(R) - C^x(R) : R \subseteq T \right\} = w(T).
\]

Now, let \( T = N \setminus \{i\} \). Let \( \tilde{M} \subseteq N \), and \( M \subseteq N \setminus \{i\} \) be such that \( v(N) = b(\tilde{M}) - C(\tilde{M}) \), and \( v(N \setminus \{i\}) = b(M) - C(M) \) respectively. From the definition of \( w \) and \( C^x \) it follows
\[ w(N \setminus \{i\}) \geq b(\bar{M} \setminus \{i\}) - C^x(\bar{M} \setminus \{i\}) \geq b(\bar{M} \setminus \{i\}) - C(\bar{M}) + b_i - x_i \]

\[ = b(\bar{M}) - C(\bar{M}) - x_i = v(N) - x_i. \]

Thus, it is enough to show that \( b(R) - C^x(R) \leq v(N) - x_i \) for every \( R \subseteq N \setminus \{i\}. \)

We consider two cases:

(a) \( C^x(R) = C(R) \). Since \( x \) is a core element of \( v^{(N,C,b)} \) then \( x(\bar{M}) = b(\bar{M}) - C(\bar{M}) \) and \( x(M) \geq b(M) - C(M) \). Hence

\[ C(\bar{M}) - C(M) \leq (b(\bar{M}) - x(\bar{M})) - (b(M) - x(M)) \leq b(\bar{M} \setminus M) - x_i. \]

(Notice that if \( i \notin \bar{M}, \) then \( x_i = 0 \). Consequently

\[ b(R) - C^x(R) = b(R) - C(R) \leq b(M) - C(M) \]

\[ = b(\bar{M}) - x_i - C(\bar{M}) - b(\bar{M} \setminus M) + x_i - C(\bar{M}) = v(N) - x_i. \]

(b) \( C^x(R) = C(R \cup \{i\}) - (b_i - x_i) \). In this case

\[ b(R) - C^x(R) = b(R) - C(R \cup \{i\}) + (b_i - x_i) \]

\[ = b(R \cup \{i\}) - C(R \cup \{i\}) - x_i \leq v(N) - x_i. \]

\[ \square \]

3 The maximal excess at the nucleolus of an airport profit game

For the rest of the paper \((N, \leq, C, b)\) is a fixed airport problem with \(|N| \geq 2\). We also assume that \( N = \{1, 2, \ldots, n\} \), and \( \leq \) is the usual order. By \( v \) we denote the associated airport profit game, and \( v \) its nucleolus.

The following numbers are the basic tool for the algorithm that is presented in the next section.

\[ \alpha_i = \frac{b([i + 1, \ldots, n]) - (C(N) - C(i))}{n - i + 1}, \quad i = 1, \ldots, n - 1, \]

\[ \beta_i = \frac{C(i)}{i + 1}, \quad i = 1, \ldots, n - 1, \]

\[ \gamma_i = \frac{b_i}{2}, \quad i = 1, \ldots, n - 1, \]

\[ \delta = \frac{b(N) - C(N)}{|N|}. \]

Define also

\[ \lambda = \min \left( \{\alpha_i : i \neq n\} \cup \{\beta_i : i \neq n\} \cup \{\gamma_i : i \neq n\} \cup \{\delta\} \right). \]
Next we determine the nucleolus for some players. We consider separately two cases: $\lambda > 0$ and $\lambda \leq 0$.

3.1 Case $\lambda > 0$.

Remark 3.1 (a) Since $0 < \lambda \leq \alpha_i$ then $b((i, \ldots, n)) > C(n) - C(i-1)$ for every $i \in N$ (in particular $v(N) = b(N) - C(N)$).
(b) Since $0 < \lambda \leq \beta_i$ then $C(i) > 0$ for every $i \in N$.

In what follows we examine the structure of the collection of coalitions with maximal excess at the nucleolus for the case $\lambda > 0$.

Let $S \subseteq N$, and $x \in \mathbb{R}^N_+$. Then the excess of $S$ at $x$ is

$$e(S, x) = v(S) - x(S).$$

We also denote

$$D_1(x) = \{S \subseteq N : e(S, x) \geq e(T, x), \text{ for all } T \subseteq N, S \neq N, \emptyset\}.$$

i.e., the set of proper coalitions of $N$ with maximal excess at $x$.

If $\{S_1, \ldots, S_k\}$ is a partition of $N$, the family $\{N \setminus S_1, \ldots, N \setminus S_k\}$ formed by the complements is called an antipartition.

Some upcoming proofs use the following result.

Theorem 1 (Arin and Iñarra 1998) For every convex game on $N$, the family of proper coalitions with maximal excess at the nucleolus contains a partition or an antipartition of $N$.

Our goal is to determine some partitions or antipartitions in $D_1(v)$. We need some lemmas.

Let $(N, v)$ be a TU game, $x \in \mathbb{R}^N_+$, and $B$ a family of coalitions of $N$. Define

$$e(B, x) = \frac{\sum_{S \in B} e(S, x)}{|B|}$$

i.e., the average excess of coalitions in $B$ at $x$.

Remark 3.2 (Arin and Iñarra 1998) If $B$ is a partition or an antipartition of $N$ and $x(N) = v(N)$ then $e(B, x)$ is independent of $x$. Indeed, it is easy to check that

(a) if $\mathcal{P}$ is a partition, $e(\mathcal{P}, x) = \frac{\sum_{S \in \mathcal{P}} v(S) - v(N)}{|\mathcal{P}|},$

(b) if $\mathcal{A}$ is an antipartition, $e(\mathcal{A}, x) = \frac{\sum_{S \in \mathcal{A}} v(S) - (|\mathcal{A}|-1)v(N)}{|\mathcal{A}|}.$

$^2$ It can be shown that these cases correspond respectively to the situation when there are not dummy players in $v$ (case $\lambda > 0$) and to the opposite situation (case $\lambda \leq 0$).
**Lemma 3.1** If $\mathcal{P} = \{S_1, \ldots, S_k\}$ is a partition of $N$, then $v(N) > \sum_{S_j \in \mathcal{P}} v(S_j)$.

**Proof** If $v(S_j) = 0$ for every $j = 1, \ldots, k$, the result follows by Remark 3.1(a). So we can assume that $v(S_{j_0}) \neq 0$ for some $j_0$.

For every $j = 1, \ldots, k$, let $R_j \subseteq S_j$ be such that $v(S_j) = b(R_j) - C(R_j)$. Also let $T = \bigcup_{R_t : v(R_t) \neq 0} R_t$ and $\ell_t = \max\{k : k \in T\}$. We consider two cases.

(a) $\ell_t = n$. Let us assume w. l. o. g. that $n \in R_k$. Then

$$v(N) = b(N) - C(N) = b(S_1) + \cdots + b(S_{k-1}) + b(S_k) - C(N)$$

$$> v(R_1) + \cdots + v(R_{k-1}) + b(R_k) - C(N)$$

$$= v(S_1) + \cdots + v(S_{k-1}) + v(S_k),$$

where the inequality follows since $b(S_j) \geq v(R_j)$ for every $j = 1, \ldots, k$, and $b(S_{j_0}) > v(R_{j_0})$ by Remark 3.1(b).

(b) $\ell_t \neq n$. Then let $T_0 = \{1, \ldots, \ell_t\}$, and assume w. l. o. g. that $\ell_t \in R_k$. Then

$$v(N) = b(N) - C(N) = b(N \setminus T_0) - (C(N) - C(T_0)) + b(T_0) - C(T_0)$$

$$> b(T_0) - C(T_0) \geq b(R_1) + \cdots + b(R_{k-1}) + b(R_k) - C(R_k)$$

$$\geq v(R_1) + \cdots + v(R_k) = v(S_1) + \cdots + v(S_k),$$

where the first equality and the strict inequality follow from Remark 3.1(a).

\[\square\]

**Lemma 3.2** If $\mathcal{A} = \{S_1, \ldots, S_k\}$ is an antipartition of $N$, then $(k - 1)v(N) > \sum_{j=1}^{k} v(S_j)$.

**Proof** If $v(S_j) = 0$ for some $j$, then the proof is immediate by the monotonicity of $v$. So we can assume that $v(S_j) \neq 0$ for every $j = 1, \ldots, k$.

Let $R_j \subseteq S_j$ be such that $v(S_j) = b(R_j) - C(R_j)$ for every $j = 1, \ldots, k$. Assume w. l. o. g. that $C(R_1) = \min\{C(R_j) : j = 1, \ldots, k\}$. Since $v(S_1) > 0$ then $R_1 \neq \emptyset$; hence by Lemma 3.1 (b) we have $C(R_1) > 0$, and therefore

$$v(S_1) + v(S_2) + \cdots + v(S_k) < b(R_1) + b(R_2) - C(R_2) + \cdots + b(R_k) - C(R_k).$$

Further, since $\mathcal{A}$ is an antipartition $\{(N \setminus S_2) \cap R_1, \ldots, (N \setminus S_k) \cap R_1\}$ is a partition of $R_1$, and, moreover, since $R_j \subseteq S_j$ then $R_j \cap ((N \setminus S_j) \cap R_1) = \emptyset$ for $j = 2, \ldots, k$. So

$$b(R_1) + b(R_2) + \cdots + b(R_k)$$

$$= b\left( R_2 \cup ((N \setminus S_2) \cap R_1) \right) + \cdots + b\left( R_k \cup ((N \setminus S_k) \cap R_1) \right).$$
This, together with inequality (2), implies
\[
v(S_1) + v(S_2) + \cdots + v(S_k) < b\left(R_2 \cup ((N\setminus S_2) \cap R_1)\right) - C(R_2) \\
+ \cdots + b\left(R_k \cup ((N\setminus S_k) \cap R_1)\right) - C(R_k).
\]

Furthermore, since \(C(R_1) \leq C(R_j)\), then \(C\left(R_j \cup ((N\setminus S_j) \cap R_1)\right) = C(R_j)\) for every \(j = 2, \ldots, k\). Hence
\[
v(S_1) + v(S_2) + \cdots + v(S_k) < b\left(R_2 \cup ((N\setminus S_2) \cap R_1)\right) - C(R_2) \\
- b\left(R_2 \cup ((N\setminus S_2) \cap R_1)\right) + \cdots \\
+ b\left(R_k \cup ((N\setminus S_k) \cap R_1)\right) - C(R_k) \leq (k - 1)v(N),
\]
where the last inequality follows from the monotonicity of \(v\).

\[\square\]

**Lemma 3.3** For all \(S \in \mathcal{D}_1(v)\), we have \(e(S, v) < 0\).

**Proof** It follows from Theorem 1, Remark 3.2, and Lemmas 3.1 and 3.2. \(\square\)

**Lemma 3.4** For all \(i \in N\), \(0 < v_i < b_i\).

**Proof** Obviously \(v_i \geq 0\) for every \(i \in N\). If \(v_i = 0\) for some \(i \in N\), then \(e([i], v) = v([i]) \geq 0\). But this is in contradiction with Lemma 3.3.

Since \(v(S\cup\{i\}) - v(S) \leq b_i\) for every \(S \subseteq N\), it follows that \(v_i \leq b_i\) for every \(i \in N\). If \(v_i = b_i\) for some \(i \in N\), then by Lemma 3.3 we have \(0 > e(N\setminus\{i\}, v) = v(N\setminus\{i\}) - v(N) + b_i\). Hence \(v(N) - v(N\setminus\{i\}) > b_i\) and this is impossible. \(\square\)

**Lemma 3.5**

(a) If \(S \in \mathcal{D}_1(v)\) and \(|S| > 1\), then \(v(S) > 0\).

(b) If \(S \in \mathcal{D}_1(v)\) and \(|S| > 1\), then \(v(S) = b(S) - C(S)\).

(c) If \([j] \in \mathcal{D}_1(v)\), and \(j \neq 1\), then \(v([j]) = 0\).

**Proof** (a) Assume \(v(S) = 0\). Let \(R \subset S\), \(R \neq \emptyset\). Then \(v(R) = 0\) and since \(v(S\setminus R) > 0\) it follows
\[
e(R, v) = v(R) - v(R) = v(S) - v(S) + v(S\setminus R) > e(S, v).
\]

But this contradicts \(S \in \mathcal{D}_1(v)\).

(b) Let \(R \subset S\) be a coalition such that \(v(S) = v(R) = b(R) - C(R)\). Then \(R \neq \emptyset\) by part (a) of this lemma. By Lemma 3.4,
\[
e(R, v) = v(R) - v(R) = v(S) - v(S) + v(S\setminus R) > e(S, v).
\]

This contradicts \(S \in \mathcal{D}_1(v)\). \(\square\)
(c) If \( v([j]) > 0 \), then \( v([j]) = b_j - C(j) \), and therefore \( v([1,j]) = b_1 + b_j - C(j) \). Then
\[
e([1,j], v) = b_1 + b_j - C(j) - v_1 - v_j \geq e([j], v) + b_1 - v_1 > e([j], v).
\]
This contradicts \( \{j\} \in \mathcal{D}_1(v) \). \(\square\)

A coalition is said to be complete if
(a) \( |S| = 1 \), or
(b) \( |S| = n - 1 \), or
(c) there exists \( i_0 \neq n \) such that \( S = \{1, 2, \ldots, i_0\} \).

**Lemma 3.6** \( \mathcal{D}_1(v) \) only contains complete coalitions.

**Proof** Assume that \( S \in \mathcal{D}_1(v) \) is not complete. Then \( n - 1 > |S| > 1 \), and by Lemma 3.5(b) it follows that \( v(S) = b(S) - C(S) \). Let \( i \in N \) be a predecessor of some member of \( S \). Then \( v(S \cup \{i\}) = b(S) + b_i - C(S) \). Hence
\[
e(S \cup \{i\}, v) = b(S) + b_i - C(S) = b(S) - C(S) - v(S) - v_i
= e(S, v) + b_i - v_i > e(S, v),
\]
where the last inequality follows from Lemma 3.4. This contradicts \( S \in \mathcal{D}_1(v) \). \(\square\)

The following proposition identifies some partitions or antipartitions contained in \( \mathcal{D}_1(v) \) according to Theorem 1. Cases (A), (C), and (D) correspond to partitions, while cases (B) and (C) to antipartitions. (Case (C) corresponds simultaneously to a partition and an antipartition.)

**Proposition 3.1** At least one of the following statements is true.

(A) there exists \( i_0 \neq n \) such that
  (i) \( \{1,\ldots,i_0\}, \{i_0 + 1\}, \ldots, \{n\} \subseteq \mathcal{D}_1(v) \) and
  (ii) \( v([1,\ldots,i_0]) \neq 0 \) and \( v([i_0 + 1]) = \ldots = v([n]) = 0 \).

(B) there exists \( i_0 \neq n \) such that
  (i) \( \{1,\ldots,i_0\}, N\setminus\{1\}, \ldots, N\setminus\{i_0\} \subseteq \mathcal{D}_1(v) \) and
  (ii) \( v([1,\ldots,i_0]) \neq 0 \).

(C) there exists \( i_0 \neq n \) such that
  (i) \( \{i_0\}, N\setminus\{i_0\} \subseteq \mathcal{D}_1(v) \) and
  (ii) \( v([i_0]) = 0 \).

(D) \( (i) \{1\}, \{2\}, \ldots, \{n\} \subseteq \mathcal{D}_1(v) \) and
  (ii) \( v([1]) = \ldots = v([n]) = 0 \).

**Proof** We have two cases:
(a) \( \mathcal{D}_1(v) \) contains a partition \( \mathcal{P} \). Let \( S \in \mathcal{P} \) of maximal cardinality. We consider three subcases:
(i) $|S| = 1$. If $v(\{1\}) > 0$, we are in case (A) (with $i_0 = 1$). Otherwise $v(\{1\}) = 0$, and we are in case (D).

(ii) $|S| = n - 1$. Here, $S = N \setminus \{i_0\}$ for some $i_0 \in N$. If $i_0 = 1$ and $v(\{1\}) > 0$ we are in case (B). Otherwise, by Lemma 3.5, we are in case (C).

(iii) $1 < |S| < n - 1$. Here, since $S$ is complete, then $S = \{1, \ldots, i_0\}$ for some $i_0 \in N$, and by Lemma 3.5 (a), $v(S) > 0$. Moreover, if $T \in \mathcal{P}$, $T \neq S$, then by Lemma 3.6 coalition $T$ is complete, hence it has to be a singleton. So by Lemma 3.5(c), we are in case (A).

(b) $\mathcal{D}_1(\nu)$ contains an antipartition $\mathcal{A}$. Let $S \in \mathcal{A}$ of minimal cardinality. We consider three subcases:

(i) $|S| = 1$. Here, since $\mathcal{A}$ is an antipartition, $\mathcal{A} = \{S, N \setminus S\}$. Let $S = \{i_0\}$. Then, if $i_0 = 1$, and $v(\{1\}) > 0$, we are in case (B). Otherwise, we are in case (C).

(ii) $|S| = n - 1$. Here, since $\mathcal{A}$ is an antipartition, $\mathcal{A} = \{N \setminus \{1\}, \ldots, N \setminus \{n\}\}$. So we are in case (B) (with $i_0 = n - 1$).

(iii) $1 < |S| < n - 1$. Here, $S$ is complete, so $S = \{1, \ldots, i_0\}$ for some $i_0 \in N$, and by Lemma 3.5(a), $v(S) > 0$. Let $T \in \mathcal{A}$, $T \neq S$. Since $S$ has minimal cardinality, then $|T| > 1$. If $|T| < n - 1$, since $T$ must be complete, $(N \setminus T) \cap (N \setminus S) \neq \emptyset$, and $\mathcal{A}$ is not an antipartition. So $|T| = n - 1$ for every $T \in \mathcal{A}$, $T \neq S$, and since $\mathcal{A}$ is an antipartition, we are in case (B).

The following proposition permits to calculate very easily the nucleolus for some players in an airport profit game.

**Proposition 3.2** At least one of the following statements is true.

(A) Let $\mathcal{P} = \{\{1, \ldots, i_0\}, \{i_0 + 1\}, \ldots, \{n\}\}$, $i_0 \neq n$, such that $v(\{1, \ldots, i_0\}) \neq 0$ and $v(\{i_0 + 1\}) = \ldots = v(\{n\}) = 0$. Then

(a) $e(\mathcal{P}, \nu) \geq -\alpha_{i_0}$;

(b) If $\mathcal{P} \subseteq \mathcal{D}_1(\nu)$, then $e(\mathcal{P}, \nu) = -\alpha_{i_0}$.

(B) Let $\mathcal{A} = \{\{1, \ldots, i_0\}, N \setminus \{1\}, \ldots, N \setminus \{i_0\}\}$, and $v(\{1, \ldots, i_0\}) \neq 0$. Then

(a) $e(\mathcal{A}, \nu) \geq -\beta_{i_0}$;

(b) If $\mathcal{A} \subseteq \mathcal{D}_1(\nu)$, then $e(\mathcal{A}, \nu) = -\beta_{i_0}$.

(C) Let $\mathcal{P} = \{\{i_0\}, N \setminus \{i_0\}\}$, and $v(\{i_0\}) = 0$. Then

(a) $e(\mathcal{P}, \nu) \geq -\gamma_{i_0}$;

(b) If $\mathcal{P} \subseteq \mathcal{D}_1(\nu)$, then $e(\mathcal{P}, \nu) = -\gamma_{i_0}$.

(D) Let $\mathcal{P} = \{\{1\}, \{2\}, \ldots, \{n\}\}$, and $v(\{1\}) = \ldots = v(\{n\}) = 0$. Then

(i) $e(\mathcal{P}, \nu) \geq -\delta$;

(ii) If $\mathcal{P} \subseteq \mathcal{D}_1(\nu)$, then $e(\mathcal{P}, \nu) = -\delta$.

**Proof** We only prove case (A). The proofs of the remaining cases are similar.
For case (i)
\[
e(P, \nu) = \frac{v([1, \ldots, i_0]) + \sum_{k > i_0} v([k]) - v(N)}{n - i_0 + 1} \geq \frac{b([1, \ldots, i_0]) - C(i_0) - b(N) + C(N)}{n - i_0 + 1} = -\alpha_{i_0}.
\]

where the first equality follows from Remark 3.2.

For case (ii), notice that the inequality above becomes an equality by Lemma 3.5 parts (b) and (c).

Proposition 3.3
(A) If \( \lambda = \alpha_{i_0} \), then \( \nu_i = \lambda \) for every \( i > i_0 \).
(B) If \( \lambda = \beta_{i_0} \), then \( \nu_i = b_i - \lambda \), for every \( i \leq i_0 \).
(C) If \( \lambda = \gamma_{i_0} \), then \( \nu_{i_0} = \lambda \).
(D) If \( \lambda = \delta \), then \( \nu_i = \lambda \) for every \( i \in N \).

Proof We only prove case (A).

By Propositions 3.1 and 3.2, it follows that \( e(S, \nu) = -\lambda \) for every \( S \in D_1(\nu) \).

Now if \( \lambda = \alpha_{i_0} \), then by Proposition 3.2, for every \( i > i_0 \), we have \( \{i\} \in D_1(\nu) \) and \( v([i]) = 0 \). Hence \( \alpha_{i_0} = -e([i], \nu) = \nu_i \).

3.2 Case \( \lambda \leq 0 \).

Notice that if \( \lambda \leq 0 \), then necessarily \( \lambda \neq \gamma_i \) for all \( i = 1, \ldots, n \).

Proposition 3.4
(a) If \( \lambda = \alpha_{i_0} \leq 0 \) for some \( i_0 \in N \), then \( \nu_i = 0 \) for every \( i > i_0 \).
(b) If \( \lambda = \beta_{i_0} \) for some \( i_0 \in N \) then \( \nu_i = b_i \) for every \( i \leq i_0 \).
(c) If \( \lambda = \delta \leq 0 \) then \( \nu_i = 0 \) for every \( i \in N \).

Proof (a) Let \( i \in N \) such that \( i_0 < i \). We show that \( i \) is a null player.

First note that since \( \lambda = \alpha_{i_0} \leq \alpha_i \), and \( \alpha_{i_0} < 0 \), then for every \( j \geq i_0 \)
\[
b([k \in N : i_0 < k]) + C(i_0) \leq b([k \in N : j \leq k]) + C(j).
\]

Hence
\[
b([k \in N : i_0 < k \leq j]) \leq C(j) - C(i_0) \quad \text{for all} \; j \geq i_0.
\]

Let \( S \subseteq N \setminus \{i\} \), and \( Q \subseteq S \cup \{i\} \) be such that \( v(S \cup \{i\}) = b(Q) - C(Q) \). If \( Q = \emptyset \), then by monotonicity of \( v \), we have \( v(S \cup \{i\}) = v(S) \). So assume \( Q \neq \emptyset \), and denote \( \ell_Q = \max\{k : k \in \mathbb{Q}\} \). Then
\[ v(S \cup \{i\}) = b(Q) - C(Q) \]
\[ = b(\{k \in Q : k \leq i_0\}) - C(i_0) + b(\{k \in Q : i_0 < k\}) - (C(\ell_Q) - C(i_0)) \]
\[ \leq v(S) + b(\{k \in Q : i_0 < k\}) - (C(\ell_Q) - C(i_0)) \]
\[ \leq v(S) + b(\{k \in N : i_0 < k \leq \ell_Q\}) - (C(\ell_Q) - C(i_0)) \leq v(S), \]

where the first inequality follows since \(i_0 < i\) implies \(\{k \in Q : k \leq i_0\} \subseteq S\), and the last one from expression (3) since \(i_0 \leq \ell_Q\).

From Propositions 3.3 and 3.4 the following result easily follows.

4 The algorithm

From Propositions 3.3 and 3.4 the following result easily follows.

**Proposition 4.1** (a) If \(\lambda = \alpha_{i_0}\), then \(v_{i_0} = \lambda_+\) for every \(i > i_0\).
(b) If \(\lambda = \beta_{i_0}\), then \(v_{i_0} = b_i - \lambda_+\) for every \(i \leq i_0\).
(c) If \(\lambda = \gamma_{i_0}\), then \(v_{i_0} = \lambda_+\) for every \(i \in N\).
(d) If \(\lambda = \delta\), then \(v_i = \lambda_+\) for every \(i \in N\).

Propositions 2.2 and 4.1 permit us the design of an algorithm for calculating the nucleolus of the airport profit game \(v\) as follows.

We construct a finite sequence of airport problems \((N_m, C_m, b_m)\), with \(m = 1, \ldots, M\), where \((N_1, C_1, b) = (N, C, b)\). In each step \(m\) we calculate the nucleolus, \(\xi\), for a subgroup of players \(Z_m \subseteq N_m\) according to Proposition 4.1. If \(Z_m = N_m\) the algorithm stops. Otherwise, let \(N_{m+1} = N_m \setminus Z_m\), and \(C_{m+1} = C^\xi_m\), and consider the reduced problem \((N_{m+1}, C_{m+1}, b_{N_{m+1}})\).
To be precise, at each step $m$ we calculate the following numbers for every $i \in N_m \setminus \{\ell_m\}$, where $\ell_m = \max\{i : i \in N_m\}$:

\[
\alpha^m_i = \frac{b\left(\{k \in N_m : k > i\}\right) - (C_m(N_m) - C_m(i))}{\left|\{k \in N_m : k > i\}\right| + 1},
\]

\[
\beta^m_i = \frac{C_m(i)}{\left|\{k \in N_m : k \leq i\}\right| + 1},
\]

\[
\gamma^m_i = \frac{b_i}{2},
\]

\[
\delta^m = \frac{b(N_m) - C_m(N_m)}{|N_m|}.
\]

Further, we consider the minimum $\lambda^m$ of all these numbers, that is

\[
\lambda^m = \min \left(\{\delta^m\} \cup \{\alpha^m_i : i \in N_m \setminus \{\ell_m\}\} \cup \{\beta^m_i : i \in N_m \setminus \{\ell_m\}\} \cup \{\gamma^m_i : i \in N_m \setminus \{\ell_m\}\}\right).
\]

Now define $x^m_i$ according to:

(i) If $\lambda^m = \alpha^m_{k_m}$ then $x^m_i = \lambda^m_+$ for every $i \in N_m, i > k_m$.
(ii) If $\lambda^m = \beta^m_{k_m}$ then $x^m_i = b_i - \lambda^m$, for every $i \in N_m, i \leq k_m$.
(iii) If $\lambda^m = \gamma^m_{i_m}$ then $x^m_i = \lambda^m_+$.
(iv) If $\lambda^m = \delta^m$ then $x^m_i = \lambda^m_+$ for every $i \in N_m$.

Let $Z_m$ be the set of players, for which $x^m_i$ is defined according to these conditions (i)–(iv)

**Theorem 2** The nucleolus of the game $v$ is given by

\[
v_i = x^m_i, \quad \text{for every } i \in Z_m, \quad m = 1, \ldots, M.
\]

**Proof** Since the nucleolus of a convex game satisfies the reduced game property, the result immediately follows from Propositions 2.2 and 4.1.

**Remark 4.1** The procedure described above has at most $n$ steps. For each step $m$, given the special nature of the sets $N \setminus N_m$, the reduced cost function for every $i \in N_m$ is

\[
C^{N_m x}(i) = \left(\min \left\{C([i]), C(N \setminus N_m \cup [i]) - (b(N \setminus N_m) - x(N \setminus N_m))\right\}\right)_+.
\]

Thus in each step, $O(n)$ calculations are made, implying that this algorithm can be performed in $O(n^2)$ time.
References