Linear, bilinear and quadratic forms

Lecture 8

Mathematics - 1\textsuperscript{st} year, English

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Outline of the lecture

1. Linear forms
2. Bilinear forms
3. Quadratic forms
Linear forms

Definition

Let \((V, +, \cdot)\) be a linear space.

- A linear mapping \(f : V \to \mathbb{R}\) is called a \textit{linear form} or a \textit{linear functional}.
- The linear space \(L(V; \mathbb{R})\) of all linear forms is called the \textit{dual} of \(V\) and is denoted \(V^*\).

Proposition

Let \((V, +, \cdot)\) be a finite-dimensional linear space. Then \(V^*\) is also finite-dimensional and \(\dim V^* = \dim V\).

Proposition

Let \((V, +, \cdot)\) be a finite-dimensional linear space. If \(v \in V \setminus \{0\}\) then there exists \(f \in V^*\) such that \(f(v) \neq 0\).

Consequence. If \(u, v \in V\) and \(u \neq v\) then there exists \(f \in V^*\) such that \(f(u) \neq f(v)\).
Bidual and evaluation map

Definition

Let \((V, +, \cdot)\) be a linear space.

- The dual of \(V^*\), denoted by \(V^{**}\), is called the **bidual** of \(V\).
- The function \(\psi: V \to V^{**}\) defined by

\[
\psi(v)(f) := f(v), \quad v \in V, \ f \in V^*
\]

is called the **evaluation map**.

The evaluation map is well-defined and it is linear:

1. It is clear that \(\psi(v): V^* \to \mathbb{R}\). If \(\alpha, \beta \in \mathbb{R}\) and \(f, g \in V^*\), then

\[
\psi(v)(\alpha f + \beta g) = (\alpha f + \beta g)(v) = \alpha f(v) + \beta g(v) = \alpha \psi(v)(f) + \beta \psi(v)(g).
\]

Hence \(\psi(v)\) is linear, i.e. \(\psi(v) \in V^{**}\). Therefore, \(\psi\) is well-defined.
2. If $\alpha, \beta \in \mathbb{R}$ and $u, v \in V$, then

$$
\psi(\alpha u + \beta v)(f) = f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) = \alpha \psi(u)(f) + \beta \psi(v)(f), \quad \forall f \in V^*.
$$

This means that $\psi(\alpha u + \beta v) = \alpha \psi(u) + \beta \psi(v)$. In conclusion, $\psi$ is linear.

3. If $V$ is finite-dimensional, then $\psi$ is a linear isomorphism.

- Indeed, if $v \in \ker \psi$, then

  $$
f(v) = 0, \quad \forall f \in V^*.
  $$

  Supposing that $v \neq 0_V$ would contradict the existence of some $f \in V^*$ such that $f(v) \neq 0$. Therefore, $v$ should be equal to $0_V$. This implies that $\ker \psi = \{0_V\}$, i.e. $\psi$ is injective.

- On the other hand, $\dim V^{**} = \dim V^* = \dim V$. By the dimension theorem, $\rank \psi = \dim V = \dim V^{**}$, so $\psi$ is surjective, too.

In conclusion, $\psi$ is a linear isomorphism. In this case, $\psi$ is also called the canonical isomorphism between $V$ and $V^{**}$. 
Vector hyperplanes

Definition

Let \((V, +, \cdot)\) be a linear space. A linear subspace \(W \subseteq V\) is called a (vector) hyperplane if there exists \(f \in V^* \setminus \{0_{V^*}\}\) such that \(\ker f = W\).

Proposition

If \((V, +, \cdot)\) is a finite-dimensional linear space with \(\dim V = n \in \mathbb{N}^*\), then a linear subspace \(W \subseteq V\) is a hyperplane if and only if \(\dim W = n - 1\).

Proof.

[Proof: “⇒”] If \(W = \ker f\) for some \(f \in V^* \setminus \{0_{V^*}\}\), then by the dimension theorem,

\[
\dim W = \dim(\ker f) = \dim V - \dim(\text{Im } f) = n - 1,
\]

because \(f \neq 0_{V^*}\) and thus \(\text{Im } f = \mathbb{R}\).
Proof.

[Proof: “⇐”] Conversely, if \( \dim W = n - 1 \), there exists a basis \( B = \{b_1, \ldots, b_{n-1}, b_n\} \) of \( V \) such that \( \text{Lin}\{b_1, \ldots, b_{n-1}\} = W \). Taking \( f : V \to \mathbb{R} \) defined by
\[
f(\alpha_1 b_1 + \cdots + \alpha_n b_n) := \alpha_n
\]
for \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), we have \( f \neq 0_{V^*} \) and
\[
f(b_1) = \cdots = f(b_{n-1}) = 0,
\]
implying that \( W \subseteq \ker f \) (i.e., \( f(v) = 0, \forall v \in W \)). On the other hand, by the direct implication, \( \dim(\ker f) = n - 1 \) and consequently \( W = \ker f \). \qed
Let \( V \) be a finite-dimensional linear space and \( B = \{ b_1, \ldots, b_n \} \) a basis of \( V \).

- If \( W \) is a hyperplane with \( W = \ker f \), where \( f \in V^* \setminus \{ 0_{V^*} \} \), let
  \[ \beta_1 := f(b_1), \ldots, \beta_n := f(b_n). \]
  Then \( v = x_1 b_1 + \cdots + x_n b_n \in \ker f \) is characterized by the equation
  \[
  \beta_1 x_1 + \cdots + \beta_n x_n = 0.
  \]
  Hence
  \[
  W = \left\{ x_1 b_1 + \cdots + x_n b_n \in V \mid \beta_1 x_1 + \cdots + \beta_n x_n = 0 \right\}.
  \]

- Conversely, having \( \beta_1, \ldots, \beta_n \in \mathbb{R} \), not all 0, the subset of \( V \) defined by the above relation is a hyperplane of \( V \).

- One can show that any linear subspace of \( V \) (not only hyperplanes) can be characterized by systems of equations of form (1).

- If \( V = \mathbb{R}^n \) and \( B \) is the canonical basis, relation (2) can be written as
  \[
  W = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \beta_1 x_1 + \cdots + \beta_n x_n = 0 \right\}.
  \]

- In the particular cases \( n = 2 \) and \( n = 3 \), equation (1) becomes the equation of a line, respectively a plane passing through the origin.
Affine functionals

The following notion allows us to characterize all the lines (when $n = 2$) and planes (when $n = 3$), not necessarily those passing through the origin.

**Definition**

Let $(V, +, ·)$ be a linear space. A function $f : V \rightarrow \mathbb{R}$ is called an **affine functional** if there exist a linear functional $f_0 \in V^*$ and a constant $c \in \mathbb{R}$ such that $f(v) = f_0(v) + c$, $\forall v \in V$.

For an affine functional $f : V \rightarrow \mathbb{R}$ one can define its *kernel* in the same way as for linear functionals, *i.e.* $\ker f := \{ v \in V \mid f(v) = 0 \}$.

**Definition**

Let $(V, +, ·)$ be a linear space. A subset $U \subseteq V$ is called an **affine hyperplane** if there exists a non-constant affine functional $f : V \rightarrow \mathbb{R}$ such that $\ker f = U$.

- In other words, $U$ is affine hyperplane if there exist a vector hyperplane $W$ and a vector $v_0 \in V$ such that

  $$U = W + v_0 := \{ v + v_0 \mid v \in W \}.$$
If $V$ is finite-dimensional with a basis $B = \{b_1, b_2, \ldots, b_n\}$, then affine hyperplanes are given by subsets of the form

$$U = \{x_1b_1 + \cdots + x_nb_n \in V \mid \beta_1x_1 + \cdots + \beta_nx_n + c = 0\},$$

where $c, \beta_1, \ldots, \beta_n \in \mathbb{R}$.

In the cases $n = 2$ and $n = 3$, the affine hyperplanes are the lines, respectively the planes.
Bilinear forms

Definition

Let \((V, +, \cdot)\) and \((W, +, \cdot)\) two linear spaces. A function \(g : V \times W \rightarrow \mathbb{R}\) is called a bilinear form (bilinear map/mapping) on \(V \times W\) if the following conditions are fulfilled:

1. \(g(\alpha u + \beta v, w) = \alpha g(u, w) + \beta g(v, w), \forall \alpha, \beta \in \mathbb{R}, \forall u, v \in V, \forall w \in W;\)
2. \(g(v, \lambda w + \mu z) = \lambda g(v, w) + \mu g(v, z), \forall \lambda, \mu \in \mathbb{R}, \forall v \in V, \forall w, z \in W.\)

In the case \(W = V\), a bilinear form on \(V \times V\) is also called bilinear form (functional, map/mapping) on \(V\).

1. Suppose now that \(V\) and \(W\) are finite-dimensional, with bases \(B = \{b_1, \ldots, b_n\}\) and \(\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_m\}\) on \(V\), respectively \(W\).

   If \(v \in V\) and \(w \in W\) having \(\alpha_1, \ldots, \alpha_n \in \mathbb{R}\) and \(\beta_1, \ldots, \beta_m \in \mathbb{R}\) as coordinates with respect to the bases \(B\), respectively \(\bar{B}\), then

   \[
g(v, w) = g \left( \sum_{i=1}^{n} \alpha_i b_i, \sum_{j=1}^{m} \beta_j \bar{b}_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j g(b_i, \bar{b}_j).\]
The scalars $a_{ij} := g(b_i, \bar{b}_j)$, $1 \leq i \leq n$, $1 \leq j \leq m$ are called the coefficients of the bilinear form $g$ with respect to the bases $B$ and $\bar{B}$;

the matrix $A^g_{B,\bar{B}} := (a_{ij})_{1 \leq i \leq n}$ in $\mathcal{M}_{nm}$ is called the matrix of the bilinear form $g$ with respect to the bases $B$, $\bar{B}$.

2. If $B' = \{b'_1, \ldots, b'_n\}$ is another basis of $V$ and $\bar{B}' = \{\bar{b}'_1, \ldots, \bar{b}'_m\}$ is another basis of $W$, let us denote $S = (s_{ij})_{1 \leq i, j \leq n} \in \mathcal{M}_n$ the transition matrix from $B$ to $B'$ and $\bar{S} = (\bar{s}_{ij})_{1 \leq i, j \leq m} \in \mathcal{M}_m$ the transition matrix from $\bar{B}$ to $\bar{B}'$.

Then the matrix of $g$ with respect to the bases $B'$ and $\bar{B}'$ can be written as

$$A^g_{B',\bar{B}'} = S \cdot A^g_{B,\bar{B}} \cdot \bar{S}^T.$$ 

It can be proven that $\text{rank} A^g_{B',\bar{B}'} = \text{rank} A^g_{B,\bar{B}}$, so the rank of the matrix of the bilinear form doesn't depend on the bases of reference. This common value is called the rank of $g$ and is denoted by $\text{rank} g$. 
Kernel of a bilinear form

- Fixing \( w \in W \), the bilinear form \( g : V \times W \to \mathbb{R} \) defines a linear functional \( f_w : V \to \mathbb{R} \), by
  \[
  f_w(v) := g(v, w), \quad v \in V.
  \]

Allowing now \( w \) to variate, the mapping \( w \mapsto f_w \) defines a linear operator \( g' : W \to V^* \).

- In a similar way, one can define a linear operator \( g'' : V \to W^* \) by
  \[
  g''(v) := h_v, \quad h_v \in W^*.
  \]
  where the linear functional \( h_v \in W^* \) is introduced by
  \[
  h_v(w) := g(v, w), \quad w \in V.
  \]

Definition

Let \( g : V \times W \to \mathbb{R} \) be a bilinear form and the associated linear operators
\( g' : W \to V^* \) and \( g'' : V \to W^* \) introduced above. The linear subspace
\( \ker g' \subseteq W \) is called the right kernel of \( g \), while the linear subspace \( \ker g'' \subseteq V \) is called
the left kernel of \( g \).

If \( \text{Ker}(g') = \{0_W\} \) and \( \text{Ker}(g'') = \{0_V\} \), then the bilinear form \( g \) is called
non-degenerate.
Definition

A bilinear form $g : V \times V \rightarrow \mathbb{R}$ is called *symmetric* if

$$g(u, v) = g(v, u), \forall u, v \in V,$$

respectively *antisymmetric* if

$$g(u, v) = -g(v, u), \forall u, v \in V.$$

Proposition

Let $g : V \times V \rightarrow \mathbb{R}$ be a symmetric bilinear form or an antisymmetric linear form. Then its right kernel coincides with its left kernel.

For such a bilinear form, the left kernel (which coincides with the right kernel) is called the *kernel* of $g$ and is denoted by $\text{ker} \ g$. 
Dimension theorem for bilinear forms

Proposition

Let \((V, +, \cdot)\) be a finite-dimensional linear space and \(g : V \times V \to \mathbb{R}\) a symmetric bilinear form. Then

\[
\text{rank } g + \dim (\ker g) = \dim V.
\]

Remark. By the above result, a necessary and sufficient condition for a symmetric bilinear form to be non-degenerate is that \(\text{rank } g = \dim V\).

Definition

Let \(g : V \times V \to \mathbb{R}\) be a symmetric bilinear form.

- Two vectors \(u, v \in V\) are called orthogonal with respect to \(g\) if \(g(u, v) = 0\).
- If \(U\) is a non-empty subset of \(V\), we say that \(U\) is orthogonal with respect to \(g\) (or \(g\)-orthogonal) if \(g(u, v) = 0\) for any distinct \(u, v \in U\).
- If \(U\) is a non-empty subset of \(V\), the set \(\{v \in V \mid g(u, v) = 0, \forall u \in U\}\) is a linear subspace of \(V\), called the orthogonal complement of \(U\) with respect to \(g\), denoted \(U^\perp_g\).
Sylvester’s law of inertia

Theorem

Let \((V, +, \cdot)\) be a finite-dimensional linear space and \(g : V \times V \to \mathbb{R}\) a symmetric bilinear form. If \(\{b_1, \ldots, b_n\}\) is a basis of \(V\) which is \(g\)-orthogonal, then \(\text{rank } g\) is precisely the number of elements among 
\[g(b_1, b_1), g(b_2, b_2), \ldots, g(b_n, b_n)\] which are non-zero.

Theorem (Sylvester’s law of inertia)

Let \((V, +, \cdot)\) be a finite-dimensional linear space and \(g : V \times V \to \mathbb{R}\) a symmetric bilinear form. Then there exist \(p, q, r \in \mathbb{N}\) such that for every \(g\)-orthogonal basis \(\{b_1, \ldots, b_n\}\) of \(V\), \(p, q\) and \(r\) represent the number of positive, negative, respectively null elements among 
\[g(b_1, b_1), g(b_2, b_2), \ldots, g(b_n, b_n)\].

- The numbers \(p\) and \(q\) are called the positive, respectively the negative index of inertia.
- The triple \((p, q, r)\) is called the signature of \(g\).
- Of course, \(p + q + r = n\) (\(n = \text{dim } V\)); moreover, \(\text{rank } g = p + q\).
Quadratic forms

Definition

Let $(V, +, \cdot)$ be a linear space and $g : V \times V \to \mathbb{R}$ a symmetric bilinear form. The function $h : V \to \mathbb{R}$, defined by

$$h(v) := g(v, v), \; v \in V$$

is called the quadratic form (functional) associated to $g$.

Remark. Since

$$h(u + v) = g(u + v, u + v) = g(u, u) + g(u, v) + g(v, u) + g(v, v)$$

and $g(u, v) = g(v, u)$, we have

$$h(u + v) = h(u) + 2g(u, v) + h(v), \; \forall u, v \in V.$$  

From this formula we can retrieve $g$ from $h$:

$$g(u, v) = \frac{1}{2} \left[ h(u + v) - h(u) - h(v) \right], \; \forall u, v \in V$$

or

$$g(u, v) = \frac{1}{4} \left[ h(u + v) - h(u - v) \right], \; \forall u, v \in V.$$
Suppose now that $V$ is a finite-dimensional space and $B = \{b_1, \ldots, b_n\}$ is a basis of $V$.

Let $A^g_{B,B} = (a_{ij})_{1 \leq i, j \leq n}$ be the matrix of $g$ with respect to $B$. If $x_1, \ldots, x_n \in \mathbb{R}$ are the coefficients of a vector $v \in V$ with respect to $B$, then

$$h(v) = h(x_1 b_1 + \cdots + x_n b_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$  

The right-hand side of this relation is a homogeneous polynomial of degree 2, called the \textit{quadratic polynomial} associated to the quadratic form $h$ and the basis $B$.

The determinant of the symmetric matrix $A^g_{B,B}$ is called the \textit{discriminant} of $h$ with respect to the basis $B$. Its sign does not depend on the basis $B$.

We say that $h$ is a \textit{non-degenerate quadratic form} if $g$ is a non-degenerate bilinear functional form, \textit{i.e.} the discriminant of $h$ (in any basis) is not zero ($\text{rank } A^g_{B,B} = \text{rank } g = n$). Otherwise, we say that $h$ is a \textit{degenerate quadratic form}.

If $(p, q, r)$ is the signature of $g$, we also call it the \textit{signature} of the quadratic form $h$. 

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Reduced form of a bilinear form

Definition

Let \((V, +, \cdot)\) be a finite-dimensional linear space and \(h : V \to V\) a quadratic form associated to some symmetric bilinear form \(g : V \times V \to \mathbb{R}\).

- If \(B\) is a basis of \(V\) such that the matrix of \(g\) is diagonal, we call **canonical (reduced) form** of \(h\) the quadratic polynomial associated to \(h\) and \(B\).
- A canonical form of \(h\) is called **normal** if the diagonal matrix associated to \(g\) has on its diagonal only the elements 1, \(-1\) and 0.

If \(B = \{b_1, \ldots, b_n\}\) is a basis of \(V\) giving a canonical form \(\omega_1 x_1^2 + \omega_2 x_2^2 + \cdots + \omega_n x_n^2\) of \(h\), then \(B' = \{c_1 b_1, \ldots, c_n b_n\}\) gives a normal form of \(h\), where \(c_i = 1\) if \(\omega_i = 0\), while \(c_i = \frac{1}{\sqrt{|\omega_i|}}\) if \(\omega_i \neq 0\), for \(1 \leq i \leq n\).
Theorem (Gauss method of reducing a quadratic form)

Let \((V, +, \cdot)\) be an \(n\)-dimensional linear space and \(h : V \to \mathbb{R}\) a quadratic form. Then there exists a basis \(\{b_1, \ldots, b_n\}\) of \(V\) and \(\omega_1, \ldots, \omega_n \in \mathbb{R}\) such that for any \(x_1, \ldots, x_n \in \mathbb{R}\) we have

\[
h(x_1 b_1 + \cdots + x_n b_n) = \omega_1 x_1^2 + \omega_2 x_2^2 + \cdots + \omega_n x_n^2.
\]

Remarks.

- The quadratic polynomial \(\omega_1 x_1^2 + \omega_2 x_2^2 + \cdots + \omega_n x_n^2\) is then a reduced form of \(h\) (the matrix of \(g\) with respect to \(\{b_1, \ldots, b_n\}\) is a diagonal matrix with entries \(\omega_1, \ldots, \omega_n\)).
- If \((p, q, r)\) is the signature of \(h\), then among the coefficients \(\omega_1, \ldots, \omega_n, p\) are positive, \(q\) are negative and \(r\) are equal to 0.
Jacobi method

Theorem (Jacobi method of reducing a quadratic form)

Let \((V, +, \cdot)\) be an \(n\)-dimensional linear space and \(h : V \to \mathbb{R}\) a quadratic form. Let \(\Delta_i, 1 \leq i \leq n\) the principal minors of the associated matrix \((a_{ij})_{1 \leq i, j \leq n}\) with respect to a basis of \(V\), i.e.

\[
\Delta_i = \begin{vmatrix}
  a_{11} & \cdots & a_{1i} \\
  a_{21} & \cdots & a_{2i} \\
  \vdots & \ddots & \vdots \\
  a_{i1} & \cdots & a_{ii}
\end{vmatrix}, \quad 1 \leq i \leq n.
\]

If \(\Delta_i \neq 0, \ \forall \ i \in \{1, \ldots, n\}\), then \(h\) can be reduced to the canonical form

\[
\mu_1 x_1^2 + \mu_2 x_2^2 + \cdots + \mu_n x_n^2,
\]

where \(\mu_j = \frac{\Delta_{j-1}}{\Delta_j}, \ \forall \ j = \{1, \ldots, n\}\), with \(\Delta_0 = 1\).
Definition

Let \((V, +, \cdot)\) be an \(n\)-dimensional linear space and \(h : V \to \mathbb{R}\) a quadratic form with signature \((p, q, r)\).

- If \(p = n\), \(h\) is called a \textit{positive-definite} quadratic form.
- If \(q = 0\), the quadratic form \(h\) is called \textit{positive semidefinite}.
- If \(q = n\), \(h\) is called a \textit{negative-definite} quadratic form.
- If \(p = 0\), the quadratic form \(h\) is called \textit{negative semidefinite}.
- The quadratic form \(h\) is called \textit{undefined} if \(p > 0\) and \(q > 0\).

Let \(\Delta_i, 1 \leq i \leq n\) be the principal minors of the associated matrix with respect to an arbitrary basis. Then \(h\) is positive-definite if and only if

\[\Delta_i > 0, \quad \forall i \in \{1, \ldots, n\}\]

and \(h\) is negative-definite if and only if

\[(-1)^i \Delta_i > 0, \quad \forall i \in \{1, \ldots, n\}\].
Theorem (Eigenvalues method of reducing a quadratic form)

Let \((V, \langle \cdot, \cdot \rangle)\) be a finite-dimensional prehilbertian space with \(\text{dim } V = n\). Then there exists an orthonormal basis with respect to which \(h\) has the canonical form

\[
\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2, \ x_1, x_2, \ldots, x_n \in \mathbb{R},
\]

where \(\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}\).

- In fact, \(\lambda_1, \ldots, \lambda_n\) are the eigenvalues of the associated matrix with respect to any basis of \(V\).
- The method of the proof is similar to the diagonalization algorithm for linear operators.
Non-homogeneous quadratic functionals

Definition

Let \((V, +, \cdot)\) be a linear space, \(h : V \to \mathbb{R}\) a quadratic form and \(f : V \to \mathbb{R}\) an affine functional. The sum \(h + f\) is called a **non-homogeneous quadratic functional** on \(V\).

- If \(V\) is finite-dimensional and \(B = \{b_1, \ldots, b_n\}\) of basis of \(V\), then for any \(x_1, \ldots, x_n \in \mathbb{R}\)

\[
(h + f)(x_1 b_1 + \cdots + x_n b_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i + c,
\]

where \(A = (a_{ij})_{1 \leq i, j \leq n}\) is the matrix associated to \(h\) and \(b_1, \ldots, b_n, c \in \mathbb{R}\).

- The right-hand side of this equality is called the **quadratic polynomial** associated to \(h + f\) (which is a polynomial of degree 2).

- If \(V = \mathbb{R}^n\) and \(B\) is its canonical basis, then (3) can be written as

\[
(h + f)(x) = \rho(x) := \langle Ax, x \rangle + \langle b, x \rangle + c, \quad \forall x \in \mathbb{R}^n,
\]

where \(b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n\) and the vectors \(x \in \mathbb{R}^n\) are interpreted as column matrices.
Conversely, for arbitrary symmetric matrix $A \in \mathcal{M}_n$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$, the function $\rho : V \to \mathbb{R}$ defined by (4), i.e.

$$\rho(x) := \langle Ax, x \rangle + \langle b, x \rangle + c, \quad \forall x \in \mathbb{R}^n$$

defines a non-homogeneous quadratic functional on $V$.

Moreover, $A$ can be taken not necessarily symmetric, since

$$\langle Ax, x \rangle = \frac{1}{2} \langle Ax, x \rangle + \frac{1}{2} \langle x, Ax \rangle$$

$$= \frac{1}{2} \langle Ax, x \rangle + \frac{1}{2} \langle A^T x, x \rangle = \left\langle \frac{1}{2} (A + A^T) x, x \right\rangle,$$

so the matrix $A$ can be replaced by the symmetric matrix $\frac{1}{2} (A + A^T)$. 


Normal form of non-homogeneous quadratic functionals

Let us now consider an affine change of coordinates, i.e. a transformation of the form

\[ x' = Sx + x_0, \]

where \( S \in \mathbb{M}_n \) is a non-singular matrix and \( x_0 \in \mathbb{R}^n \). Then

\[
\rho(x) = \langle AS^{-1}(x' - x_0), S^{-1}(x' - x_0) \rangle + \langle b, S^{-1}(x' - x_0) \rangle + c
\]

\[ = \left\langle \left(S^{-1}\right)^T AS^{-1}x', x'\right\rangle - \left\langle 2 \left(S^{-1}\right)^T AS^{-1}x_0 + \left(S^{-1}\right)^T b, x'\right\rangle + \left(c - \left\langle b, S^{-1}x_0 \right\rangle \right) . \]

Suppose now that \( S \) is the transition matrix from the canonical basis to an orthonormal basis giving the canonical form in eigenvalues method of reduction. Therefore, \( S \) is an orthonormal matrix (\( S^{-1} = S^T \)) and

\[ SAS^T = D := \text{diag}(\lambda_1, \ldots, \lambda_n) \], where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \).

Consequently, we have:

\[
\rho(x) = \langle Dx', x' \rangle - 2 \left\langle S \left(AS^T x_0 + \frac{1}{2}b\right), x' \right\rangle + \left(c - \left\langle b, S^{-1}x_0 \right\rangle \right) . \]
If $A$ is non-singular, we can take $x_0 := -\frac{1}{2}SA^{-1}b$, obtaining

$$\rho(x) = \langle Dx', x' \rangle + c_0,$$

where $c_0 := \langle Dx_0, x_0 \rangle - \langle Sb, x_0 \rangle + c$. Therefore, by the change of coordinates $x' = Sx - \frac{1}{2}SA^{-1}b$, we obtain

$$\rho(x) = \sum_{i=1}^{n} \lambda_i (x'_i)^2 + c_0, \quad \forall x \in \mathbb{R}^n,$$

where $x'_i$ are the coordinates of $x$ with respect to the new orthogonal basis.

If $\det A = 0$, then by letting $x_0 := 0$, we obtain

$$\rho(x) = \langle Dx', x' \rangle + \langle Sb, x' \rangle + c_0,$$

where $c_0 := -\langle Sb, x_0 \rangle + c$.

If $(p, q, r)$ is the signature of $h$, we have $r > 0$ and $n - r$ is the rank of $A$; one can further find an adequate basis $B''$ such that

$$\rho(x) = \sum_{i=1}^{n-r} \lambda_i(x''_i)^2 + \gamma x''_{n-r+1},$$

where $x''_1, \ldots, x''_n$ are the coordinates of $x$ with respect to this new basis and $\gamma \in \mathbb{R}$. 
Geometric classification

From the geometric point of view,

$$\ker \rho := \{ x \in \mathbb{R}^n \mid \rho(x) = 0 \}$$

is a conic in the case $n = 2$, a quadric if $n = 3$, a hyperquadric if $n \geq 4$.

1. Case $n = 1$: the normal forms of $\rho$ are:

   - $x^2 + 1$ (ker $\rho = \emptyset$: two “imaginary” points);
   - $x^2 - 1$ (ker $\rho = \{-1, 1\}$: two distinct points);
   - $x^2$ (ker $\rho = \{0\}$: two identical points).
2. Case $n = 2$: we have nine types of conics, according to the normal form of $\rho$:

- $x_1^2 + x_2^2 + 1 = 0$ (\(\emptyset\): “imaginary” ellipse);
- $x_1^2 - x_2^2 + 1 = 0$ (hyperbola);
- $x_1^2 + x_2^2 - 1 = 0$ (ellipse);
- $x_1^2 - 2x_2 = 0$ (parabola);
- $x_1^2 + x_2^2 = 0$ (a point: two “imaginary”, conjugate lines);
- $x_1^2 - x_2^2 = 0$ (two intersecting lines);
- $x_1^2 + 1 = 0$ (\(\emptyset\): two “imaginary” lines);
- $x_1^2 - 1 = 0$ (two parallel lines);
- $x_1^2 = 0$ (two identical lines).
1. Single point
2. Single line
3. Pair of lines
4. Parabola
5. Ellipse
6. Hyperbola
3. Case $n = 3$: we have 17 types of quadrics, characterized by the following normal forms:

- $x_1^2 + x_2^2 + x_3^2 + 1 = 0$ ("imaginary" ellipsoid);
- $x_1^2 + x_2^2 + x_3^2 - 1 = 0$ (ellipsoid);
- $x_1^2 + x_2^2 - x_3^2 - 1 = 0$ (hyperboloid of one sheet);
- $x_1^2 - x_2^2 - x_3^2 - 1 = 0$ (hyperboloid of two sheets);
- $x_1^2 + x_2^2 + x_3^2 = 0$ (a point: "imaginary" cone);
- $x_1^2 + x_2^2 - x_3^2 = 0$ (cone);
- $x_1^2 + x_2^2 - 2x_3 = 0$ (elliptic paraboloid);
- $x_1^2 - x_2^2 - 2x_3 = 0$ (hyperbolic paraboloid).

The remaining 9 normal forms are the same as those in the case $n = 2$, which in $\mathbb{R}^3$ represent cylinders of different types: elliptic, hyperbolic or parabolic. The first 6 quadrics are non-singular quadrics, while the others are singular quadrics.
### Graphs of quadric surfaces

<table>
<thead>
<tr>
<th>Surface</th>
<th>Equation</th>
<th>Surface</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Ellipsoid</strong></td>
<td>[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 ]</td>
<td><strong>Cone</strong></td>
<td>[ \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} ]</td>
</tr>
<tr>
<td></td>
<td>All traces are ellipses.</td>
<td></td>
<td>Horizontal traces are ellipses.</td>
</tr>
<tr>
<td></td>
<td>If ( a = b = c ), the ellipsoid is a sphere.</td>
<td></td>
<td>Vertical traces in the planes ( x = k ) and ( y = k ) are</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>hyperbolas if ( k \neq 0 ) but are</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>pairs of lines if ( k = 0 ).</td>
</tr>
<tr>
<td><strong>Elliptic Paraboloid</strong></td>
<td>[ \frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} ]</td>
<td><strong>Hyperboloid of One Sheet</strong></td>
<td>[ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 ]</td>
</tr>
<tr>
<td></td>
<td>Horizontal traces are ellipses.</td>
<td></td>
<td>Horizontal traces are ellipses.</td>
</tr>
<tr>
<td></td>
<td>Vertical traces are parabolas.</td>
<td></td>
<td>Vertical traces are hyperbolas.</td>
</tr>
<tr>
<td></td>
<td>The variable raised to the first power indicates the axis of the</td>
<td></td>
<td>The axis of symmetry corresponds to the variable whose coefficient is</td>
</tr>
<tr>
<td></td>
<td>paraboloid.</td>
<td></td>
<td>negative.</td>
</tr>
<tr>
<td><strong>Hyperbolic Paraboloid</strong></td>
<td>[ \frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2} ]</td>
<td><strong>Hyperboloid of Two Sheets</strong></td>
<td>[ -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 ]</td>
</tr>
<tr>
<td></td>
<td>Horizontal traces are hyperbolas.</td>
<td></td>
<td>Horizontal traces in ( z = k ) are</td>
</tr>
<tr>
<td></td>
<td>Vertical traces are parabolas.</td>
<td></td>
<td>ellipses if ( k &gt; c ) or ( k &lt; -c ).</td>
</tr>
<tr>
<td></td>
<td>The case where ( c &lt; 0 ) is illustrated.</td>
<td></td>
<td>Vertical traces are hyperbolas.</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>The two minus signs indicate two sheets.</td>
</tr>
</tbody>
</table>


