Series of real numbers. Series with positive terms

Lecture 3

Mathematics - 1\textsuperscript{st} year, English

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Outline of the lecture

1. Definition and properties
   - Examples
   - Cauchy test
   - Operations with series

2. Series with positive terms
   - Comparison criteria
   - Cauchy’s criterion of condensation
   - Root test of Cauchy
   - Kummer criterion
   - Gauss criterion
What is an *infinite sum*

\[ x_1 + \cdots + x_n + \ldots \] ?

A natural approach is to study the behaviour of the partial sums \( x_1 + \cdots + x_n \) for \( n \) very large (going to \(+\infty\)).

**Definition**

Let \((x_n)_{n \geq 1} \subseteq \mathbb{R}\). The *series with general terms* \( x_n, x_n \in \mathbb{N} \) is the sequence

\[
S_n := \sum_{k=1}^{n} x_k = x_1 + \cdots + x_n, \quad n \in \mathbb{N}^*.
\]

We will denote this series by \( \sum_{n=1}^{\infty} x_n \), \( \sum_{n \geq 1} x_n \) or \( x_1 + \cdots + x_n + \ldots \).
For $n \in \mathbb{N}^*$, the term $S_n := x_1 + \cdots + x_n$ is called the *partial sum* of order $n$ of the series.

If $(S_n)$ is convergent, we say that the series is *convergent*; we denote this

$$\sum_{n=1}^{\infty} x_n \ (C).$$

If $(S_n)$ is divergent (it has no limit or has infinite limit), we say that the series is *divergent*; we denote this

$$\sum_{n=1}^{\infty} x_n \ (D).$$

If $S_n \to S \in \overline{\mathbb{R}}$, we write

$$\sum_{n=1}^{\infty} x_n = S.$$
Remainder of a series

If \( p \in \mathbb{N}^* \), the series \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=p+1}^{\infty} x_n \) have the same nature, i.e. they are either both convergent or both divergent.

Definition

If \( p \in \mathbb{N} \), we call the remainder of order \( p \) of the series \( \sum_{n=1}^{\infty} x_n \) the series \( \sum_{n=p+1}^{\infty} x_n \).

Proposition

Let \( \sum_{n=1}^{\infty} x_n \) be a convergent series of real numbers. Then, for any \( p \in \mathbb{N} \), the remainder of order \( p \) is convergent. Moreover, if we denote

\[
R_p := \sum_{n=p+1}^{\infty} x_n, \quad p \in \mathbb{N},
\]

then \( \lim_{p \to +\infty} R_p = 0 \).
1. The geometric series of ratio $r \in \mathbb{R}$: 
\[ \sum_{n=0}^{\infty} r^n. \]

Partial sums: 
\[ S_n = 1 + r + \cdots + r^n = \begin{cases} 
\frac{1 - r^n}{1 - r}, & r \neq 1, \\
n + 1, & r = 1.
\end{cases} \]

- $\sum_{n=0}^{\infty} r^n$ (C) for $r \in (-1, 1)$;
- $\sum_{n=0}^{\infty} r^n$ (D) for $r \in (-\infty, -1] \cup [1, +\infty)$.

We have also:
\[ \sum_{n=0}^{\infty} r^n = \begin{cases} 
\frac{1}{1-r}, & r \in (-1, 1), \\
+\infty, & r \geq 1.
\end{cases} \]

In the case $r = 1$, the series $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots$ is also known as Grandi series; it is a divergent series.
2. The series $\sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{n}\right)$ is divergent, because its partial sums go to $+\infty$:

$$S_n = \sum_{k=1}^{n} \ln \left(1 + \frac{1}{k}\right) = \sum_{k=1}^{n} \ln \frac{k+1}{k}$$

$$= \sum_{k=1}^{n} \left[ \ln(k+1) - \ln k \right] = \ln(n+1) - \ln 1 = \ln(n+1),$$

\[\text{telescopic sum}\]
3. The series \[ \sum_{n=2}^{\infty} \frac{n - \sqrt{n^2 - 1}}{\sqrt{n^2 - n}} \] is convergent, since we have

\[ S_n := \sum_{k=2}^{n} \frac{k - \sqrt{k^2 - 1}}{\sqrt{k^2 - k}} = \sum_{k=2}^{n} \left( \frac{k}{\sqrt{k^2 - k}} - \frac{\sqrt{k^2 - 1}}{\sqrt{k^2 - k}} \right) \]

\[ = \sum_{k=2}^{n} \left( \sqrt{\frac{k}{k^2 - k}} - \sqrt{\frac{k^2 - 1}{k^2 - k}} \right) = \sum_{k=2}^{n} \left( \sqrt{\frac{k}{k - 1}} - \sqrt{k + 1} \right) \]

\[ = \sqrt{2} - \sqrt{\frac{n + 1}{n}} \quad n \to +\infty \quad \sqrt{2} - 1. \]

Of course, we will have \[ \sum_{n=2}^{\infty} \frac{n - \sqrt{n^2 - 1}}{\sqrt{n^2 - n}} = \sqrt{2} - 1. \]
Necessary condition of convergence

Theorem

Let \( \sum_{n=1}^{\infty} x_n \) be a convergent series. Then \( \lim_{n \to \infty} x_n = 0 \).

Proof.

Let \( S_n := \sum_{k=1}^{n} x_k, n \in \mathbb{N}^* \) and \( S := \lim_{n \to \infty} S_n \in \mathbb{R} \). Then

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} (S_n - S_{n-1}) = S - S = 0.
\]

Remarks.

- If \( x_n \not\to 0 \), the series \( \sum_{n=1}^{\infty} x_n \) is clearly divergent;

- \( \lim_{n \to \infty} x_n = 0 \) is just a necessary condition for the convergence of \( \sum_{n=1}^{\infty} x_n \), but not a sufficient one.
Cauchy criterion

Theorem

The series \( \sum_{n=1}^{\infty} x_n \) is convergent if and only if

\[ \forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N}^*, \ \forall n \geq n_\varepsilon, \ \forall p \in \mathbb{N}^*: |x_{n+1} + \cdots + x_{n+p}| < \varepsilon. \]

Proof.

Let \( S_n := \sum_{k=1}^{n} x_k, \ n \in \mathbb{N}^* \). By Cauchy’s criterion for the convergence of the sequences, the sequence \((S_n)\) is convergent if and only if

\[ \forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N}^*, \ \forall n \geq n_\varepsilon, \ \forall p \in \mathbb{N}^*: |S_{n+p} - S_n| < \varepsilon. \]

But \( S_{n+p} - S_n = x_{n+1} + \cdots + x_{n+p} \) for any \( n, p \in \mathbb{N}^* \), which proves the assertion.
By negating the Cauchy condition of convergence for a series, we obtain:

**Proposition**

The series \( \sum_{n=1}^{\infty} x_n \) is divergent if and only if

\[ \exists \varepsilon > 0, \ \forall n \in \mathbb{N}^*, \ \exists k_n \geq n, \ \forall p_n \in \mathbb{N}^* : |x_{k_n+1} + \cdots + x_{k_n+p_n}| \geq \varepsilon. \]
The harmonic series

The series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is called the *harmonic* series. It is divergent, by the previous result. Indeed,

\[
\frac{1}{n+1} + \cdots + \frac{1}{n+p} \geq \frac{p}{n+p}, \quad \forall n, p \in \mathbb{N}^*.
\]

For \( \varepsilon := 1/2 \) and any \( n \in \mathbb{N}^* \), we can set \( k_n := n \) and \( p_n := n \); we will have

\[
\left| \frac{1}{k_n+1} + \cdots + \frac{1}{k_n+p_n} \right| \geq \frac{p_n}{k_n+p_n} = \frac{1}{2} \geq \varepsilon.
\]

Therefore \( \sum_{n=1}^{\infty} \frac{1}{n} \) (D).
Operations with series

Let now \( \lambda \in \mathbb{R} \) and \( \sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n \) two series of real numbers.

- The series \( \sum_{n=1}^{\infty} (x_n + y_n) \) is called the \textit{sum} of the series \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} y_n \).
- The series \( \sum_{n=1}^{\infty} (\lambda x_n) \) is called the \textit{product} of the series \( \sum_{n=1}^{\infty} x_n \) with the number (scalar) \( \lambda \).

Theorem

Let \( S := \sum_{n=1}^{\infty} x_n \) and \( S' := \sum_{n=1}^{\infty} y_n \). If \( S, S' \in \mathbb{R} \), then:

i) if \( x_n \leq y_n \), \( \forall n \in \mathbb{N}^* \), then \( S \leq S' \);

ii) the series \( \sum_{n=0}^{\infty} (x_n + y_n) \) is convergent and \( \sum_{n=0}^{\infty} (x_n + y_n) = S + S' \);

iii) the series \( \sum_{n=1}^{\infty} (\lambda x_n) \) is convergent and \( \sum_{n=1}^{\infty} (\lambda x_n) = \lambda S \).
Theorem

If we associate the terms of a convergent series in finite groups, by keeping the order of the terms, we still obtain a convergent series, with the same sum.

Associating the terms of \( \sum_{n=1}^{\infty} x_n \) means constructing a new series

\[
\sum_{k=1}^{\infty} \left( x_{n_k} + \cdots + x_{n_{k+1}-1} \right),
\]

where \( (n_k)_{k \in \mathbb{N}^*} \subseteq \mathbb{N}^* \) is a strictly increasing sequence with \( n_1 = 1 \).

Remark.

- Associating terms of a divergent series sometimes gives a convergent series.
- For instance, if we associate every two terms in Grandi series \( \sum_{n=0}^{\infty} (-1)^n \) we obtain the convergent series

\[
\sum_{k=0}^{\infty} \left( (-1)^{2k} + (-1)^{2k+1} \right) = (-1 + 1) + (-1 + 1) + \cdots + (-1 + 1) + \ldots
\]

\[
= 0 + 0 + \cdots + 0 + \ldots
\]
We say that a series $\sum_{n=1}^{\infty} x_n$ has \textit{positive terms} if $x_n \geq 0$, $\forall n \in \mathbb{N}^*$. 

A series with positive terms always has a sum (which may be finite or infinite).

Proposition

A series with positive terms is convergent if and only if the sequence of its partial terms is bounded.

This means $\sum_{n=1}^{\infty} x_n$ is convergent if and only if $(S_n)_{n\geq1}$ is bounded, where 

$$S_n := x_1 + \cdots + x_n, \; n \in \mathbb{N}^*.$$
Comparison criterion I

We will study convergence and divergence criteria for series with positive sums. The first we state are called *comparison criteria*; they specify the nature of a series(*i.e.*, it is convergent or divergent) by comparing it with another series whose nature is already known.

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be two series with positive terms.

**Theorem (CC1)**

Suppose that $x_n \leq y_n$, $\forall n \in \mathbb{N}^*$.

i) If $\sum_{n=1}^{\infty} y_n$ (C) then $\sum_{n=1}^{\infty} x_n$ (C).

ii) If $\sum_{n=1}^{\infty} x_n$ (D) then $\sum_{n=1}^{\infty} y_n$ (D).
Theorem (CC2)

Suppose that \( x_n > 0, y_n > 0 \) and \( \frac{x_{n+1}}{x_n} \leq \frac{y_{n+1}}{y_n} \), for every \( n \in \mathbb{N}^* \).

i) If \( \sum_{n=1}^{\infty} y_n \) (C) then \( \sum_{n=1}^{\infty} x_n \) (C).

ii) If \( \sum_{n=1}^{\infty} x_n \) (D) then \( \sum_{n=1}^{\infty} y_n \) (D).
Comparison criterion III

Theorem (CC3)

*Suppose that* $y_n > 0$, $\forall n \in \mathbb{N}^*$ *and there exists* $\lim_{n \to +\infty} \frac{x_n}{y_n} = \ell \in [0, +\infty]$. 

1. If $\ell \in (0, +\infty)$, the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ have the same nature.

2. If $\ell = 0$, we have:
   i) if $\sum_{n=1}^{\infty} y_n$ (C) then $\sum_{n=1}^{\infty} x_n$ (C);
   ii) if $\sum_{n=1}^{\infty} x_n$ (D) then $\sum_{n=1}^{\infty} y_n$ (D).

3. If $\ell = +\infty$, we have:
   i) if $\sum_{n=1}^{\infty} x_n$ (C) then $\sum_{n=1}^{\infty} y_n$ (C);
   ii) if $\sum_{n=1}^{\infty} y_n$ (D) then $\sum_{n=1}^{\infty} x_n$ (D).
Cauchy’s criterion of condensation

Theorem

Let \((x_n)_{n \geq 1} \subseteq \mathbb{R}_+\) a decreasing sequence. Then the series \(\sum_{n=1}^{\infty} x_n\) has the same nature as the series \(\sum_{n=1}^{\infty} (2^n x_{2^n})\).

- The above result holds also if we require only that \((x_n)_{n \geq 1} \subseteq \mathbb{R}_+\) is a monotone sequence.
- Indeed, if \((x_n)\) is not decreasing, we have that both sequences \((x_n)\) and \((2^n x_{2^n})\) do not converge to 0, which implies that the series \(\sum_{n=1}^{\infty} x_n\) and \(\sum_{n=1}^{\infty} (2^n x_{2^n})\) are divergent.
Generalized harmonic series

**Example.** The series \[ \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \] is called the generalized harmonic series (of parameter \( \alpha \in \mathbb{R} \)).

- In the case \( \alpha = 1 \), we recover the harmonic series.
- The series \( \sum_{n=1}^{\infty} \frac{1}{n^\alpha} \) has the same nature with the series

\[
\sum_{n=1}^{\infty} \frac{2^n}{(2^n)^\alpha} = \sum_{n=1}^{\infty} \frac{2^n}{2^n \alpha} = \sum_{n=1}^{\infty} \left(2^{1-\alpha}\right)^n.
\]

- This series is the geometric series with ratio \( 2^{1-\alpha} \) which converges if and only if \( 2^{1-\alpha} \in (-1, 1) \), i.e. \( \alpha > 1 \).
- As a conclusion,

\[
\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \ (C), \text{ if } \alpha > 1;
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \ (D), \text{ if } \alpha \leq 1.
\]
Root test of Cauchy

Theorem

Let $\sum_{n=1}^{\infty} x_n$ be a series with positive terms such that there exists

$$\lim_{n \to \infty} \sqrt[n]{x_n} = \ell \in [0, +\infty].$$

i) If $\ell < 1$, then $\sum_{n=1}^{\infty} x_n$ (C).

ii) If $\ell > 1$, then $\sum_{n=1}^{\infty} x_n$ (D).

In the case $\ell = 1$, we cannot say anything about the nature of the series $\sum_{n=1}^{\infty} x_n$ (take for example the generalized harmonic function).

In that case, we should apply further tests (criteria).
Kummer criterion

Theorem

Let $\sum_{n=1}^{\infty} x_n$ be a series with $x_n > 0$, $\forall n \in \mathbb{N}^*$ and a sequence $(a_n)_{n \geq 1} \subseteq \mathbb{R}_+^*$. Suppose that there exists

$$\lim_{n \to \infty} \left( a_n \cdot \frac{x_n}{x_{n+1}} - a_{n+1} \right) = \ell \in \overline{\mathbb{R}}.$$ 

i) If $\ell > 0$, then $\sum_{n=1}^{\infty} x_n$ (C).

ii) If $\ell < 0$ and $\sum_{n=1}^{\infty} \frac{1}{a_n}$ (D), then $\sum_{n=1}^{\infty} x_n$ (D).

Again, in the case $\ell = 0$, we cannot say anything about the nature of the series. The following criteria are applications of Kummer criterion for $a_n = 1$, $n$ or $n \ln n$. 

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Corollary

Let $\sum_{n=1}^{\infty} x_n$ be a series with $x_n > 0$, $\forall n \in \mathbb{N}^*$. Suppose that there exists

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = L \in [0, +\infty].$$

i) If $L < 1$, then $\sum_{n=1}^{\infty} x_n$ (C).

ii) If $L > 1$, then $\sum_{n=1}^{\infty} x_n$ (D).
Raabe-Duhamel criterion

Corollary

Let $\sum_{n=1}^{\infty} x_n$ be a series with $x_n > 0$, $\forall n \in \mathbb{N}^*$. Suppose that there exists

$$\lim_{n \to \infty} n \left( \frac{x_n}{x_{n+1}} - 1 \right) = \rho \in \overline{\mathbb{R}}.$$

i) If $\rho > 1$, then $\sum_{n=1}^{\infty} x_n \ (C)$.

ii) If $\rho < 1$, then $\sum_{n=1}^{\infty} x_n \ (D)$.

- In order to show the second part of this criterion we use the divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.
- We usually try to apply this criterion when the ratio test fails.
Bertrand criterion

Corollary

Let \( \sum_{n=1}^{\infty} x_n \) be a series with \( x_n > 0, \ \forall n \in \mathbb{N}^* \). Suppose that there exists

\[
\lim_{n \to \infty} \left( n \ln n \cdot \frac{x_n}{x_{n+1}} - (n + 1) \ln(n + 1) \right) = \mu \in \mathbb{R}.
\]

i) If \( \mu > 0 \), then \( \sum_{n=1}^{\infty} x_n \) (C).

ii) If \( \mu < 0 \), then \( \sum_{n=1}^{\infty} x_n \) (D).

We use the fact (for the 2nd part) that the series \( \sum_{n=1}^{\infty} \frac{1}{n \ln n} \) is divergent.

Indeed, by Cauchy's criterion of condensation, it has the same nature with

\[
\sum_{n=1}^{\infty} \frac{2^n}{2^n \ln(2^n)} = \sum_{n=1}^{\infty} \frac{1}{n \ln 2} = \frac{1}{\ln 2} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.
\]
Theorem

Let \( \sum_{n=1}^{\infty} x_n \) be a series with \( x_n > 0, \forall n \in \mathbb{N}^* \). Suppose that there exists \( \alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{R}_+^* \) and a bounded sequence \( (y_n)_{n \geq 1} \) such that

\[
\frac{x_n}{x_{n+1}} = \alpha + \frac{\beta}{n} + \frac{y_n}{n^{1+\gamma}}, \quad \forall n \in \mathbb{N}^*.
\]

i) If \( \alpha > 1 \), then \( \sum_{n=1}^{\infty} x_n \) (C).

ii) If \( \alpha < 1 \), then \( \sum_{n=1}^{\infty} x_n \) (D).

iii) If \( \alpha = 1 \) and \( \beta > 1 \), then \( \sum_{n=1}^{\infty} x_n \) (C).

iv) If \( \alpha = 1 \) and \( \beta \leq 1 \), then \( \sum_{n=1}^{\infty} x_n \) (D).


