1. The set of real numbers

We will suppose that the reader is acquainted with the algebraic properties of the sets \( \mathbb{N} \) (natural numbers), \( \mathbb{Z} \) (integers), \( \mathbb{Q} \) (rationals) and \( \mathbb{R} \) (real numbers). For his sake, we remind the definition of \( \mathbb{R} \):

**Definition.** A set of real numbers is a Dedekind-complete ordered field, i.e. a quadruplet \( (\mathbb{R}, +, \cdot, \leq) \) where \( \mathbb{R} \) is a set with at least two elements, + (addition) and \( \cdot \) (multiplication) are two algebraic operations on \( \mathbb{R} \) and \( \leq \) is a total order on \( \mathbb{R} \) such that:

1. Every property of real numbers can be proved starting from the above “axioms”. As usual, subtration and division can be introduced as:

   - \( x - y := x + (\cdot 0) \), \( x, y \in \mathbb{R} \);
   - \( x/y := x \cdot (y^{-1}) \), \( x, y \in \mathbb{R} \).

   It can be proven (but this is beyond the scope of this presentation) that there exists a unique (up to a homeomorphism of ordered fields) set of real numbers, so we will call \( \mathbb{R} \) the set of real numbers (or the real line) and its elements real numbers. In fact, one can construct \( \mathbb{R} \) starting with \( \mathbb{N} \) (constructed in the previous lecture as \( \omega \)), then continuing with \( \mathbb{Z} \), \( \mathbb{Q} \) and finally \( \mathbb{R} \).

   Reciprocally, if we already have set \( \mathbb{R} \), we can define \( \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \) as follows:

   - \( \mathbb{N} := \bigcap \{ \mathbb{N} \in \mathcal{P}(\mathbb{R}) \mid 0 \in \mathbb{N}, n \in \mathbb{N} \Rightarrow n + 1 \in \mathbb{N}, \forall n \in \mathbb{N} \} = \{ 0, 1, 1 + 1, 2 + 1, \ldots \} = \{ 0, 1, 2, 3, \ldots \} \);

   - \( \mathbb{Z} := \mathbb{N} \cup \{-n \mid n \in \mathbb{N} \} \);

   - \( \mathbb{Q} := \{ m/n \mid m \in \mathbb{Z}, n \in \mathbb{N}^+ \} \).

   We will adopt this point of view from now on.

   The absolute value of a number \( x \) is defined as

   \[
   |x| := \begin{cases} 
   x, & x \geq 0; \\
   -x, & x < 0.
   \end{cases}
   \]

**Proposition 1.1.** We have:

1. \( |x| \geq 0, \forall x \in \mathbb{R} \);
2. \( |x| = 0 \iff x = 0, \forall x \in \mathbb{R} \);
3. \( |xy| = |x| \cdot |y|, \forall x, y \in \mathbb{R} \);
4. \( |x + y| \leq |x| + |y|, \forall x, y \in \mathbb{R} \).

   Concerning the supremum and infimum of a non-empty subset of \( \mathbb{R} \), they can be characterized as follows:

**Proposition 1.2.** Let \( A \) be a non-empty subset of \( \mathbb{R} \).

1. An element \( \alpha \in \mathbb{R} \) is the supremum of \( A \) if and only if:
   - \( x \leq \alpha, \forall x \in A \);
   - \( \forall \varepsilon > 0, \exists x_\varepsilon \in A : \alpha - \varepsilon < x_\varepsilon \).

\textsuperscript{1} \( \mathbb{R}^* \) denotes \( \mathbb{R} \setminus \{0\} \); similar notations for \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{Q} \).
ii) An element $\beta \in \mathbb{R}$ is the infimum of $A$ if and only if:

(a) $x \geq \beta, \forall x \in A$;
(b) $\forall \varepsilon > 0, \exists x_\varepsilon \in A : \beta + \varepsilon > x_\varepsilon$.

Proof. We will show just the supremum property, the proof of the other being analogous.

$\Rightarrow$ Since $\alpha = \sup A$, $\alpha$ is an upper bound for $A$; therefore, $x \leq \alpha, \forall x \in A$. On the other hand, it is the smallest upper bound; hence for some $\varepsilon > 0, \alpha - \varepsilon$ cannot be an upper bound for $A$. Consequently, the order $\leq$ being total, there exists $x_\varepsilon \in A$ such that $\alpha - \varepsilon < x_\varepsilon$.

$\Leftarrow$ Let $\alpha \in \mathbb{R}$ satisfying (a) and (b). Then, from (a), $\alpha$ is an upper bound for $A$. Suppose now that there exists another upper bound for $A, \eta \in \mathbb{R}$, such that $\alpha > \eta$. Then, taking $\varepsilon := \alpha - \eta > 0$, by (b) there exists $x_\varepsilon \in A$ such that $\alpha - \varepsilon < x_\varepsilon$, i.e. $\eta < x_\varepsilon$. But this contradicts the fact that $\eta$ is an upper bound for $A$.

In conclusion, $\alpha$ is the least upper bound of $A$.

□

It is now easy to check that, for $a, b \in \mathbb{R}$ with $a < b$,

- $\inf\{a, b\} = \inf\{a, b\} = \inf\{a, b\} = \inf\{a, b\} = a$;
- $\sup\{a, b\} = \sup\{a, b\} = \sup\{a, b\} = \sup\{a, b\} = b$.

Having recalled a few algebraic and set theoretic properties of $\mathbb{R}$, let us focus on the so called "topological" properties of the real line.

2. Sequences of real numbers

A sequence of real numbers is simply a function $x : \mathbb{N} \to \mathbb{R}$. It is customary to denote $x_n$ instead of $x(n)$, for $n \in \mathbb{N}$; also, the function $x$ itself will be denoted $(x_n)_{n \in \mathbb{N}}$, $(x_n)_{n \geq 0}$ or even $(x_n)$. If $p \in \mathbb{N}^*$, by $(x_n)_{n \geq p}$ we understand the sequence $(x_n)_{n \in \mathbb{N}}$ or even a function $x : \{n \in \mathbb{N} | n \geq p\} \to \mathbb{R}$. By $(x_n)_{n \in \mathbb{N}}$ we denote the set $\{x_n | n \in \mathbb{N}\}$. Sometimes, if $A \subseteq \mathbb{R}$ is a set and $(x_n)_{n \in \mathbb{N}}$ is a sequence, we denote $(x_n)_{n \in \mathbb{N}} \subseteq A$ (abuse of language) instead of $(x_n)_{n \in \mathbb{N}} \subseteq A$, i.e. $x_n \in A, \forall n \in \mathbb{N}$.

Definition. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ is:

a) upper bounded if $(x_n)_{n \in \mathbb{N}}$ is upper bounded, i.e. there exists $M \in \mathbb{R}$ such that $x_n \leq M, \forall n \in \mathbb{N}$;
b) lower bounded if $(x_n)_{n \in \mathbb{N}}$ is lower bounded, i.e. there exists $m \in \mathbb{R}$ such that $x_n \geq m, \forall n \in \mathbb{N}$;
c) bounded if $(x_n)_{n \in \mathbb{N}}$ is bounded, i.e. there exists $m, M \in \mathbb{R}$ such that $m \leq x_n \leq M, \forall n \in \mathbb{N}$;
d) unbounded if $(x_n)_{n \in \mathbb{N}}$ is not bounded.

For instance, the sequence $((-1)^n)_{n \geq 1}$ is bounded (since $((-1)^n)_{n \geq 1} = \{-1, 1\}$), but $(2^n)_{n \in \mathbb{N}}$ is not bounded (it is lower bounded, but not upper bounded).

Definition. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ is:

a) increasing if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$;
b) decreasing if $x_n \geq x_{n+1}, \forall n \in \mathbb{N}$;
c) monotone if it is increasing or decreasing;
d) strictly increasing if $x_n < x_{n+1}, \forall n \in \mathbb{N}$;
e) strictly decreasing if $x_n > x_{n+1}, \forall n \in \mathbb{N}$;
f) strictly monotone if it is strictly increasing or strictly decreasing.

Again, the sequence $((-1)^n)_{n \geq 1}$ is not monotone, but $(2^n)_{n \in \mathbb{N}}$ and $\left(\frac{1}{n}\right)_{n \geq 1}$ are (the first is strictly increasing, the second strictly decreasing).

Definition. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ is convergent if there exists an element $x \in \mathbb{R}$, called the limit of the sequence $(x_n)_{n \in \mathbb{N}}$, such that for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that:

$$|x_n - x| < \varepsilon, \forall n \geq n_\varepsilon.$$ 

In this case we say that $(x_n)_{n \in \mathbb{N}}$ converges to $x$ and we write this $x_n \xrightarrow{n \to \infty} x$ (more simple, $x_n \to x$) or $\lim_{n \to \infty} x_n = x$.

This last notation is legitimated by the following result:

**Proposition 2.1.** The limit of a sequence of real numbers is unique.

Proof. Suppose that the sequence $(x_n)_{n \in \mathbb{N}}$ has two distinct limits $x$ and $y$. Let us take $\varepsilon := |x - y|/2 > 0$. Then, by the definition, there exist some $n_\varepsilon, m_\varepsilon \in \mathbb{N}$ such that:

$$|x_n - x| < \varepsilon, \forall n \geq n_\varepsilon;$$
$$|x_n - y| < \varepsilon, \forall n \geq m_\varepsilon.$$
If \( n \geq \max \{ n_x, m_x \} \), then
\[
|x - y| \leq |x - x_n| + |x_n - y| < 2\varepsilon = |x - y|,
\]
which is, of course, absurd. Therefore we cannot have \( x \neq y \).

\[\square\]

**Proposition 2.2.** Any convergent sequence is bounded.

**Proof.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence converging to \( x \in \mathbb{R} \). Then, letting \( \varepsilon := 1 \), there exists \( n_1 \in \mathbb{N} \) such that \( |x_n - x| < 1, \forall n \geq n_1 \). This implies that
\[
|x_n| \leq |x_n - x| + |x| < 1 + |x|, \forall n \geq n_1.
\]

Letting \( M := \max \{ |x_1|, \ldots, |x_{n_1-1}|, 1 + |x| \} \), we can see that
\[
|x_n| \leq M, \forall n \in \mathbb{N}.
\]
The sequence is therefore bounded. \[\square\]

**Examples.**
- Perhaps the most used example is the sequence \( \left\{ \frac{1}{n} \right\}_{n \geq 1} \). We have
  \[
  \lim_{n \to \infty} \frac{1}{n} = 0,
  \]
  because for every \( \varepsilon > 0 \) there exists \( n_\varepsilon := \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1 \) such\(^2\) that \( n \geq n_\varepsilon \) implies
  \[
  -\varepsilon < 0 < 1 - \frac{1}{n} < \varepsilon.
  \]
- The constant sequence \( (c)_{n \in \mathbb{N}} \) (for \( c \in \mathbb{R} \)) is increasing and decreasing in the same time; it is convergent to \( c \).
- The sequence \( (a^n)_{n \geq 1} \) is convergent for \( a \in (-1, 1) \) and divergent for \( a \in \mathbb{R} \setminus (-1, 1) \); we have
  \[
  \lim_{n \to \infty} a^n = \begin{cases} 0, & -1 < a < 1; \\ 1, & a = 1. \end{cases}
  \]
- Another well known limit is
  \[
  \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = e,
  \]
  where \( e \) is Euler’s number (in fact, the sequence \( \left( \left( 1 + \frac{1}{n} \right)^n \right)_{n \geq 1} \) is increasing and bounded, so \( e \) can be defined as the limit of this sequence – see Theorem 2.6 below).

Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of real numbers. A **subsequence** of \( (x_n)_{n \in \mathbb{N}} \) is a sequence \( (x_{n_k})_{k \in \mathbb{N}} \) where \( (n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N} \) is a strictly increasing sequence of natural numbers.

**Proposition 2.3.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence converging to \( x \in \mathbb{R} \). If \( (x_{n_k})_{k \in \mathbb{N}} \) is a subsequence of \( (x_n)_{n \in \mathbb{N}} \), then \( (x_{n_k})_{k \in \mathbb{N}} \) converges to \( x \).

**Proof.** Let \( \varepsilon > 0 \). Since \( x_n \to x \), there exists \( n_\varepsilon \in \mathbb{N} \) such that \( |x_n - x| < \varepsilon, \forall n \geq n_\varepsilon \). On the other hand, the sequence \( (n_k)_{k \in \mathbb{N}} \) is strictly increasing, so there exists \( k_\varepsilon \) such that
\[
n_k \geq n_\varepsilon, \forall k \geq k_\varepsilon
\]
(otherwise, we would have \( (n_k)_{k \in \mathbb{N}} \subseteq \{0, 1, \ldots, n_\varepsilon - 1\} \)). Therefore,
\[
|x_{n_k} - x| < \varepsilon, \forall k \geq k_\varepsilon.
\]
This proves that \( \lim_{k \to \infty} x_{n_k} = x \). \[\square\]

Therefore, a simple way for proving that a sequence \( (x_n)_{n \in \mathbb{N}} \) does not converge is to put our hands on two subsequences \( (x_{n_k})_{k \in \mathbb{N}} \) and \( (x_{m_k})_{k \in \mathbb{N}} \) converging to different limits. For instance, the sequence \( (\left( -1 \right)^n)_{n \geq 1} \) is not convergent, because the subsequence \( (\left( -1 \right)^{2k})_{k \geq 1} \) is the constant sequence converging to 1, while the subsequence \( (\left( -1 \right)^{2k+1})_{k \geq 0} \) is the constant sequence converging to \(-1\).

Concerning how operations on \( \mathbb{R} \) and sequences relate, we have the following properties:

**Proposition 2.4.** Let \( (x_n) \) and \( (y_n) \) be sequences of real numbers, convergent to \( x \in \mathbb{R} \), respectively to \( y \in \mathbb{R} \). Then:
\[
\begin{align*}
i & \lim_{n \to \infty} (x_n + y_n) = x + y; \\
ii & \lim_{n \to \infty} (x_n y_n) = xy; \\
iii & \lim_{n \to \infty} (x_n - y_n) = x - y;
\end{align*}
\]

\( ^2 \)for \( x \in \mathbb{R}, |x| := \sup \{ n \in \mathbb{Z} \mid n \leq x \} \) is called the floor or the integer part of \( x \)
iv) if $y \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that $y_n \neq 0$, $\forall n \geq n_0$ and $\lim_{n \to \infty} \left( \frac{x_n}{y_n} \right) = \frac{x}{y}$.

v) if $x_n \leq y_n$, $\forall n \in \mathbb{N}$, then $x \leq y$.

vi) (squeeze theorem) if the sequence $(z_n)$ is such that $x_n \leq z_n \leq y_n$, $\forall n \in \mathbb{N}$ and $x = y$, then $(z_n)$ is convergent and $\lim_{n \to \infty} z_n = x$.

A simple consequence of the last property is the following criterium of convergence:

**Proposition 2.5.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers and $x \in \mathbb{R}$. If there exists a sequence $(\alpha_n)_{n \in \mathbb{N}}$ converging to 0 such that

$$|x_n - x| \leq \alpha_n, \forall n \in \mathbb{N},$$

then $(x_n)$ converges to $x$.

The following result is also known as the Monotone convergence theorem.

**Theorem 2.6 (Weierstrass).** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers.

i) If $(x_n)_{n \in \mathbb{N}}$ is increasing and upper bounded, then it converges to $\sup \{x_n\}_{n \in \mathbb{N}}$.

ii) If $(x_n)_{n \in \mathbb{N}}$ is decreasing and lower bounded, then it converges to $\inf \{x_n\}_{n \in \mathbb{N}}$.

**Proof.** We will prove only the first part; we can treat the second in the same manner.

Since the non-empty set $\{x_n\}_{n \in \mathbb{N}}$ is upper bounded, it has a least upper bound; let $\alpha := \sup \{x_n\}_{n \in \mathbb{N}}$. By Proposition 1.2,

$$x_n \leq \alpha, \forall n \in \mathbb{N}$$

and

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} : \alpha - \varepsilon < x_{n_\varepsilon}.$$ 

Since $(x_n)$ is increasing, we have that $x_n \geq x_{n_\varepsilon}$, $\forall n \geq n_\varepsilon$. Combining the two inequalities, we get for $\varepsilon > 0$,

$$\alpha - \varepsilon < x_n, \forall n \geq n_\varepsilon.$$ 

Therefore,

$$|x_n - \alpha| = \alpha - x_n < \varepsilon, \forall n \geq n_\varepsilon.$$ 

This relation, $\varepsilon$ being choosen in an arbitrary way, expresses the fact that $(x_n)$ is convergent to $\alpha$. \qed

**Lema 2.7.** Any sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers has a monotone subsequence.

**Proof.** For $n \in \mathbb{N}$, let the set $A_n \subseteq \mathbb{N}$ be defined by

$$A_n := \{ k > n \mid x_k \geq x_n \}.$$ 

I. Suppose first that there exists some $n_0 \in \mathbb{N}$ such that $A_n$ is an infinite set for every $n \geq n_0$. Then, starting with $n_0 := 0$, one can construct a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that $n_{k+1} \in A_{n_k}$, $\forall k \in \mathbb{N}$.

II. The other possibility is that the set

$$A := \{ n \in \mathbb{N} \mid A_n \text{ is finite} \}$$

is infinite. Let $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence such that $A = \{ n_k \}_{k \in \mathbb{N}}$ (such a sequence can be constructed based on the existence of a bijective function between $\mathbb{N}$ and $A$).

Therefore, for every $k \in \mathbb{N}$, $A_{n_k}$ is finite, which implies that for any $k \in \mathbb{N}$ we can choose $k' > k$ such that $x_{n_{k'}} < x_{n_k}$ (because $A \cap \{ n \geq n_k \mid x_n < x_{n_k} \}$ is infinite, hence non-empty).

With the aid of this property we can easily construct a strictly decreasing subsequence $(x_{n_{k_l}})_{l \in \mathbb{N}}$ of $(x_{n_k})_{k \in \mathbb{N}}$. But $(x_{n_{k_l}})_{l \in \mathbb{N}}$ is also a subsequence of $(x_n)_{n \in \mathbb{N}}$; this ends the proof. \qed

The following result is now an easy consequence of Theorem 2.6 and of the above Lemma.

**Theorem 2.8 (Bolzano–Weierstrass).** If a sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, then it possesses a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$.

**Definition.** We say that a sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy (or fundamental) if for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that:

$$|x_n - x_m| < \varepsilon, \forall n, m \geq n_\varepsilon.$$ 

The following result is an important criterium for determining if a sequence is convergent, without even guessing its supposed limit.

**Theorem 2.9.** A sequence of real numbers is convergent if and only if it is Cauchy.

The “if” part of this result, applied to general spaces where convergence can be defined, is the definition of a **complete space**; thus this theorem tells us that $\mathbb{R}$ is a complete space.

**Proof.**
\( \Rightarrow \) If \((x_n)_{n \in \mathbb{N}}\) is a sequence convergent to some \(x \in \mathbb{R}\), then, according to the definition of convergence, for any \(\varepsilon > 0\), there exists \(n_\varepsilon \in \mathbb{N}\) such that
\[
|x_n - x| < \varepsilon/2, \quad \forall n \geq n_\varepsilon.
\]
Hence, if \(m, n \geq n_\varepsilon\), we have
\[
|x_m - x_n| \leq |x_m - x| + |x - x_n| < \varepsilon.
\]
This proves that \((x_n)\) is a Cauchy sequence.

\(\Leftarrow\) Suppose that \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence. Then \((x_n)_{n \in \mathbb{N}}\) is bounded (the proof goes approximately the same as for 2.2: if \(\varepsilon = 1\), then \(|x_n| \leq |x_n| + |x_n - x_n| \leq 1 + |x_n|, \forall n \geq n_1\)). By Bolzano-Weierstrass theorem, it has a convergent subsequence \((x_{n_k})_{k \in \mathbb{N}}\). Let \(x \in \mathbb{R}\) be the limit of this subsequence. Our aim is to show that the whole sequence \((x_n)_{n \in \mathbb{N}}\) converges to \(x\).

Let now \(\varepsilon > 0\). Since \(x_{n_k} \to x\), there exists some \(k_\varepsilon \in \mathbb{N}\) such that
\[
|x_{n_k} - x| < \varepsilon/2, \quad \forall k \geq k_\varepsilon.
\]
On the other hand, since \((x_n)\) is Cauchy, there exists some \(n_\varepsilon \in \mathbb{N}\) such that
\[
|x_m - x_n| < \varepsilon/2, \quad \forall m, n \geq n_\varepsilon.
\]
Let \(n_\varepsilon' := \max\{n_\varepsilon, n_{k_\varepsilon}\}\) and \(k_\varepsilon'\) such that \(n_{k_\varepsilon'} \geq n_\varepsilon'\). If \(n \geq n_{k_\varepsilon'}\), then
\[
|x_n - x| \leq |x_n - x_{n_{k_\varepsilon'}| + |x_{n_{k_\varepsilon'}} - x| < \varepsilon,
\]
because \(n \geq n_{k_\varepsilon'} \geq n_\varepsilon\) and \(k_\varepsilon' \geq k_\varepsilon\).

\[\square\]

3. The extended real line. Limit points

Obviously, not every subset of \(\mathbb{R}\) possesses a supremum or an infimum (for instance, \(\mathbb{R}\) itself); a necessary condition is to be bounded. If we want to get rid of this restriction, we should add the "bounds" of \(\mathbb{R}\): an upper bound and a lower bound.

Let \(+\infty\) and \(-\infty\) be two distinct points, neither elements of \(\mathbb{R}\) \((-\infty, +\infty \notin \mathbb{R}\), which we call “plus infinity”, respectively “minus infinity”. We denote \(\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}\), the extended real line. We extend the natural order \(\leq\) on \(\mathbb{R}\) to \(\bar{\mathbb{R}}\) in the following manner:

- \(-\infty \leq +\infty\);
- \(-\infty \leq x, x \leq +\infty, \forall x \in \mathbb{R}\) (in fact, the inequalities are strict, since the related elements are distinct).

With this extended order, every subset of \(\mathbb{R}\) has a supremum and an infimum. In fact, for \(A \subseteq \mathbb{R}\), we have:

- \(\sup A = +\infty\) if and only if \(A\) is not upper bounded;
- \(\inf A = -\infty\) if and only if \(A\) is not lower bounded;
- \(\sup A = -\infty \iff \inf A = +\infty \iff A = \emptyset\).

Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers. We write \(\lim_{n \to \infty} x_n = +\infty\) \((x_n \to +\infty)\) or \(\lim_{n \to \infty} x_n = -\infty\) \((x_n \to -\infty)\) if for every \(a \in \mathbb{R}\), there exists \(n_a \in \mathbb{N}\) such that:
\[
x_n > a, \quad \forall n \geq n_a,
\]
respectively
\[
x_n < a, \quad \forall n \geq n_a.
\]

**Proposition 3.1.** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers.

i) If \((x_n)_{n \in \mathbb{N}}\) is increasing and unbounded, then \(\lim_{n \to \infty} x_n = +\infty\).

ii) If \((x_n)_{n \in \mathbb{N}}\) is decreasing and unbounded, then \(\lim_{n \to \infty} x_n = -\infty\).

Therefore, any monotone sequence in \(\mathbb{R}\) has a limit in \(\bar{\mathbb{R}}\).

Concerning the usual operations, they can be extended partially to \(\bar{\mathbb{R}}\) in order to obtain an analogous result to Proposition 2.4 (exercise: state and prove the result!). In this regard, we introduce:

- \((-\infty) + a = a + (-\infty) := -\infty, \text{ for } -\infty \leq a < +\infty; (+\infty) + a = a + (+\infty) := +\infty, \text{ for } -\infty < a \leq +\infty;\)
- \((-\infty) \cdot a = a \cdot (-\infty) := -\infty, (+\infty) \cdot a = a \cdot (+\infty) := +\infty, \text{ for } 0 < a \leq +\infty;\)
- \((-\infty) \cdot a = a \cdot (-\infty) := +\infty, (+\infty) \cdot a = a \cdot (+\infty) := -\infty, \text{ for } -\infty \leq a < 0;\)
- \((-\infty) := +\infty, (-\infty) := -\infty, 1/(+\infty) = 1/(-\infty) = 0.\)

The cases \((-\infty) + (+\infty), (+\infty) + (-\infty), (-\infty) - (+\infty), (+\infty) - (-\infty), 0 \cdot (-\infty), 0 \cdot (+\infty), \frac{\pm\infty}{\pm\infty}, \frac{\pm\infty}{0}\) (partially) remain not defined.

**Definition.** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers.

a) We call \(x \in \mathbb{R}\) a limit point of the sequence \((x_n)_{n \in \mathbb{N}}\) if there exists a subsequence \((x_{n_k})_{k \in \mathbb{N}}\) such that \(x_{n_k} \to x\).
b) The set of the limit points of the sequence \((x_n)_{n \in \mathbb{N}}\) is denoted \(L(x_n)\).

An immediate application of Lemma 2.7 shows that \(L(x_n) \neq \emptyset\) for any sequence \((x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}.

**Definition.** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers.

a) We call the **inferior limit** of \((x_n)_{n \in \mathbb{N}}\) the number (in \(\bar{\mathbb{R}}\)):

\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf L(x_n).
\]

b) We call the **superior limit** of \((x_n)_{n \in \mathbb{N}}\) the number (in \(\mathbb{R}\)):

\[
\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup L(x_n).
\]

**Remarks.** Let \((x_n)_{n \in \mathbb{N}}\) be a sequence of real numbers.

- We have
  \[
  \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n.
  \]
- If \(x \in \bar{\mathbb{R}}\), then \(\lim_{n \to \infty} x_n = x\) if and only if \(L(x_n) = \{x\}\), i.e.
  \[
  \liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = x.
  \]
- It can be shown (see the proof of Lemma 2.7) that there exists a monotone subsequence of \((x_n)_{n \in \mathbb{N}}\) which has the limit \(\lim_{n \to \infty} x_n\) and there exists a monotone subsequence of \((x_n)_{n \in \mathbb{N}}\) which has the limit \(\lim_{n \to \infty} x_n\).

### 4. Sequences of functions

Sometimes, the terms of a sequence depend on a parameter, as for instance, the sequence \(\left((1 + \frac{x}{n})^n\right)_{n \geq 1}\), where \(x \in \mathbb{R}\) (we already know that this sequence converges to \(e^x\)). We can regard such a sequence as a sequence of functions (in this example, the functions would be \(x \mapsto (1 + \frac{x}{n})^n\)).

In general, let \(E\) be a set and, for every \(n \in \mathbb{N}\), \(f_n : E \to \mathbb{R}\) be a function. By analogy with sequences of real numbers, we say that the function \(F : \mathbb{N} \to \mathcal{F}(E, \mathbb{R})\) defined by \(F(n) := f_n\), is a **sequence of functions**. Also, we denote \((f_n)_{n \in \mathbb{N}}\) instead of \(F\).

In most applications, \(E\) will be a subset of \(\mathbb{R}\) (or, later during the course, of \(\mathbb{R}^m\)).

In contrast with sequences of real numbers, we will have here several types of convergence:

**Definition.** Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of functions from a set \(E\) to \(\mathbb{R}\) and \(f : E \to \mathbb{R}\). We say that:

a) \((f_n)_{n \in \mathbb{N}}\) **converges pointwise to** \(f\) if \(f_n(x) \to f(x), \forall x \in E\) (we note \(f_n \xrightarrow{p} f\) or \(f_n \xrightarrow{p,E} f\), if we want to specify on which set the pointwise convergence holds);

b) \((f_n)_{n \in \mathbb{N}}\) **converges uniformly to** \(f\) if for any \(\varepsilon > 0\) there exists \(n_\varepsilon \in \mathbb{N}\) such that:

\[
|f_n(x) - f(x)| < \varepsilon, \forall n \geq n_\varepsilon, \forall x \in E.
\]

In this case we note \(f_n \xrightarrow{u} f\) or \(f_n \xrightarrow{u,E} f\).

Of course, if \((f_n)_{n \in \mathbb{N}}\) converges uniformly to \(f\), it will also converge pointwise to \(f\).

Also, \(f_n \xrightarrow{u} f\) if and only if \(\sup_{x \in E} |f_n(x) - f(x)| \in \mathbb{R}\) for \(n\) sufficiently large and

\[
\lim_{n \to \infty} \sup_{x \in E} |f_n(x) - f(x)| = 0.
\]

**Example.** Let \(f_n : [0,1] \to \mathbb{R}\) be defined as \(f_n(x) := x^n\), for \(n \geq 1\). It is clear that

\[
\lim_{n \to \infty} x^n = \begin{cases} 0, & x \in (0,1); \\ 1, & x = 1. \end{cases}
\]

Therefore, \((f_n)_{n \in \mathbb{N}}\) converges pointwise to \(f\), where \(f : [0,1] \to \mathbb{R}\) is defined as \(f(x) := \begin{cases} 0, & x \in (0,1); \\ 1, & x = 1. \end{cases}\)

On the other hand,

\[
\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} x^n = 1,
\]

hence \(\sup_{x \in [0,1]} |f_n(x) - f(x)| \neq 0\). By (*) this means suppose that \((f_n)_{n \in \mathbb{N}}\) doesn’t converge uniformly to \(f\).

The following results assert some criteria for the uniform convergence of functions:
Proposition 4.1. Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of functions from a set \(E\) to \(\mathbb{R}\) and \(f : E \to \mathbb{R}\). If there exists a sequence \((a_n)_{n \in \mathbb{N}}\) converging to 0 such that
\[
|f_n(x) - f(x)| \leq a_n, \quad \forall n \in \mathbb{N}, \quad \forall x \in E
\]
then \((f_n)\) converges uniformly to \(f\).

Theorem 4.2. Let \((f_n)_{n \in \mathbb{N}}\) be a sequence of functions from a set \(E\) to \(\mathbb{R}\). Then there exists a function \(f : E \to \mathbb{R}\) such that
\[
f_n \xrightarrow{\text{u}} f \text{ if and only if } (f_n)_{n \in \mathbb{N}} \text{ is a uniform Cauchy sequence, i.e. for any } \varepsilon > 0, \text{ there exists } n_\varepsilon \in \mathbb{N} \text{ such that:}
\]
\[
|f_n(x) - f(x)| < \varepsilon, \quad \forall m, n \geq n_\varepsilon, \quad \forall x \in E.
\]

As we will see later, uniform convergence is closed to (keeps) properties as boundedness, continuity or integrability.

5. Remarkable inequalities

Proposition 5.1 (Hölder inequality). Let \(n \in \mathbb{N}^+, \ a_1, \ldots, a_n, \ b_1, \ldots, b_n \in \mathbb{R}_+\) and \(p, q \in \mathbb{R}^+_0\) such that \(\frac{1}{p} + \frac{1}{q} = 1\). Then:
\[
\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} b_i^q \right)^{\frac{1}{q}}.
\]

It is easy to prove (exercice!) a variant of this inequality, the weighted Hölder inequality:
\[
\sum_{i=1}^{n} \lambda_i a_i b_i \leq \left( \sum_{i=1}^{n} \lambda_i a_i^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^{n} \lambda_i b_i^q \right)^{\frac{1}{q}},
\]
where \(n \in \mathbb{N}^+, \lambda_1, \ldots, \lambda_n, \ a_1, \ldots, a_n, \ b_1, \ldots, b_n \in \mathbb{R}_+\) and \(p, q \in \mathbb{R}^+_0\) such that \(\frac{1}{p} + \frac{1}{q} = 1\).

In the case \(p = q = 2\) we obtain the Cauchy-Buniakowski-Schwarz inequality:
\[
\sum_{i=1}^{n} a_i b_i \leq \left( \sum_{i=1}^{n} a_i^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{i=1}^{n} b_i^2 \right)^{\frac{1}{2}}.
\]
The equality holds if and only if there exist \(u, v \in \mathbb{R}\) with \(u^2 + v^2 \neq 0\), such that \(ua_i + vb_i = 0, \forall i \in \{1, 2, \ldots, n\}\).

Proposition 5.2 (Minkowski inequality). Let \(n \in \mathbb{N}^+, \ a_1, \ldots, a_n, \ b_1, \ldots, b_n \in \mathbb{R}_+\) and \(p \in \mathbb{R}^+_0\).

i) If \(p \geq 1\), then
\[
\left( \sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}}.
\]

ii) If \(0 < p < 1\), then
\[
\left( \sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{p}} \geq \left( \sum_{i=1}^{n} a_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} b_i^p \right)^{\frac{1}{p}}.
\]

In both cases, if \(p \neq 1\), the equality holds if the \(n\)-tuples \((a_0, a_1, \ldots, a_n)\) and \((b_0, b_1, \ldots, b_n)\) are proportional.

Proposition 5.3 (Carleman inequality). For any \(n \in \mathbb{N}^+\) and \(a_1, a_2, \ldots, a_n \in \mathbb{R}_+\), it holds
\[
\sum_{k=1}^{n} (a_1 a_2 \ldots a_k)^{\frac{1}{k}} \leq e \sum_{k=1}^{n} a_k.
\]
The equality holds if and only if \(a_1 = a_2 = \ldots = a_n = 0\).

Selective bibliography


\(^3\) Recall that \(\mathbb{R}_+ := [0, +\infty)\) and \(\mathbb{R}_+^* := \mathbb{R}_+ \setminus \{0\} = (0, +\infty)\)