

Mathematics *for*
3D Game Programming
***and* Computer Graphics**

Second Edition

Eric Lengyel



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