BEM-FADING REGULARIZATION METHOD FOR THE CAUCHY PROBLEM IN ANISOTROPIC HEAT CONDUCTION

Andreea–Paula Voinea–Marinescu[1,2,*], Liviu Marin[1,2,3,†]

[1]Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest, 14 Academiei, 010014 Bucharest, Romania
[2]Research Institute of the University of Bucharest (ICUB), University of Bucharest, 90–92 Şos. Panduri, 050663 Bucharest, Romania
[3]"Gheorghe Mihoc – Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 13 Calea 13 Septembrie, 050711 Bucharest, Romania

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For many real life problems in science and engineering which are associated with the steady-state (an)isotropic heat conduction, the geometry of the domain occupied by the solid, the boundary conditions or the (an)isotropic thermal conductivity tensor are either incomplete or even entirely missing. Such a situation gives rise to the so-called inverse problems which represent a well-known example of ill-posed problems in the sense of Hadamard (1923), i.e. the existence, uniqueness and stability of their solutions are not always guaranteed. A classical example of an inverse problem in (an)isotropic heat conduction is the Cauchy problem which is characterised by the absence of boundary conditions on a part of the boundary of the domain, whilst the remaining boundary is over-specified by prescribing both the Dirichlet and the Neumann conditions.

We consider that $\Omega \subset \mathbb{R}^d$, where $d \in \{2,3\}$, is a bounded domain with the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, such that $\varnothing \neq \Gamma_j \subsetneq \partial\Omega$, $j = 0,1$, and $\Gamma_0 \cap \Gamma_1 = \varnothing$, which is occupied by an anisotropic solid characterised by the symmetric and positive definite thermal conductivity tensor $\mathbf{K} := \big(K_{ij}\big)_{i,j=\overline{1,d}} : \overline{\Omega} \longrightarrow \mathbb{R}^{d\times d}$, with $K_{ij} \in \mathrm{L}^\infty(\Omega)$, $i,j = \overline{1,d}$. By further assuming the absence of heat sources, the temperature distribution u satisfies the anisotropic heat conduction equation (Özişik, 1993):

$$-\nabla \cdot \big(\mathbf{K}(\mathbf{x})\nabla u(\mathbf{x})\big) = 0\,, \quad \mathbf{x} \in \Omega\,, \tag{1}$$

along with the following over-prescribed boundary conditions:

$$u(\mathbf{x}) = u^*(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma_0\,, \qquad \text{and} \qquad q\big(u(\mathbf{x})\big) = q^*(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma_0\,, \tag{2}$$

where $q\big(u(\mathbf{x})\big) := \boldsymbol{\nu}(\mathbf{x}) \cdot \big(\mathbf{K}(\mathbf{x})\nabla u(\mathbf{x})\big)$ is the normal heat flux at a point $\mathbf{x} \in \partial\Omega$, $\boldsymbol{\nu}(\mathbf{x})$ is the unit outward normal vector at $\mathbf{x} \in \partial\Omega$, whilst $u^* \in \mathrm{H}^{1/2}(\Gamma_0)$ and $q^* \in \mathrm{H}^{-1/2}(\Gamma_0)$ are the prescribed temperature and normal heat flux on Γ_0.

Herein we investigate the numerical reconstruction of the missing boundary conditions $u\big|_{\Gamma_1}$ and $q\big|_{\Gamma_1}$, as well as $u\big|_{\Omega}$, by considering the inverse Cauchy problem (1)–(2) with both exact and perturbed data. This is achieved by adapting the fading regularization method, originally proposed by Cimetière et al. (2000, 2001) for Laplace's equation (i.e. steady-state isotropic heat conduction), to the anisotropic case. To do so, the Cauchy problem (1)–(2) is reformulated as an equivalent optimisation problem for a functional that contains a control term which may be regarded as a regularization term. The solution

*E-mails: andreea.paula.marinescu@drd.unibuc.ro; andreeapaula.marinescu@gmail.com
†E-mails: liviu.marin@fmi.unibuc.ro; marin.liviu@gmail.com

to this optimisation problem is retrieved iteratively as the limit of a sequence of well-posed optimisation problems, with the mention that the convergence of this sequence to the unique solution of the Cauchy problem (1)–(2) is also proved.

The numerical implementation of the fading regularization method for anisotropic heat conduction is realized for $d = 2$ and a constant thermal conductivity tensor by employing a boundary element method (BEM) with constant elements (Aliabadi, 2002). Both exact and noisy Cauchy data are considered on the over-prescribed and accessible boundary Γ_0 and a corresponding regularizing/stabilising stopping criterion, which is based on Hansen's L-curve (Hansen, 1998), is also provided for each type of data. The accuracy, convergence, stability and robustness are thoroughly analysed for the Cauchy problem in the case of a homogeneous anisotropic solid occupying either a simply or a doubly connected bounded domain with a smooth boundary.

From the numerical results obtained, it can be concluded that the BEM-fading regularization method presented, analysed and implemented in this study provides us with very accurate, convergent and stable numerical solutions for boundary data reconstruction in the case of Cauchy problems in 2D steady-state anisotropic heat conduction, for a wide range of values of the admissible parameter associated with the control term in the minimisation functional and, at the same time, is very robust and versatile.

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