REGULAR ELEMENTS AND GENERALIZED INVERSES IN (MATRIX) RINGS OF RESIDUE CASSES

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Motivation

The concept of (von Neumann) regular ring was given by John von Neumann in his famous work [4]. A ring R is called regular if every element a in R is regular, in the sense that there exists an element b in R such that aba = a. In this case b is called a generalized inverse (also called {1}-inverse or inner inverse) of a. Generalized inverses have created a wide-field of applications, mainly towards solving linear systems of equations whose matrices are not necessarily invertible, but admit a generalized inverse. There has also been an interest in developing the theory of regular rings and generalized inverses in particular settings, such as rings of residue classes modulo n or matrices over finite fields [1, 3], sometimes with applications to cryptography [2].

Methodology of research

Every regular element a of a ring R has not only a generalized inverse, but also a {2}-inverse (or outer inverse), in the sense that there exists an element b in R such that bab = b. We generalize the known characterization of regular elements of the ring Zn of residue classes modulo n towards two directions. First, we describe the elements of Zn which admit a {2}-inverse. Secondly, we characterize regular elements of some matrix rings over Zn.

Results and comparison with State-of-the-Art

Theorem1: Let n = $p_{r1} . . . \ p_{rk}$ be a prime decomposition. Then a matrix A = ([a_{ij}]n) in M_2(Zn) is regular if and only if for every s in {1, . . . , k}, one of the following conditions holds:

(i) $([a_{ij}]\, p_s^{rs}) = O_2$,

(ii) $([a_{ij}]\, p_s^{rs})$ is invertible in $M_2(\mathrm{Z}p_s)$, det([aij] p_s^{rs}) = 0 and ([aij] p_s^{rs}) has an invertible entry in Zp_s^{rs}.

In this case, an inner inverse of A is given by the k-tuple consisting of inner inverses of $([a_{ij}]p_s^{rs})$ for s in {1, . . . , k}.

Conclusions

In the end, we generalized the results from regular elements in rings, to regular elements in rings of 2x2 matrices, with elements from residue classes modulo n. The generalizations from the article [3], were developed to get the generalizations for matrix rings.

This is based on a joint work with Septimiu Crivei (Babeș-Bolyai University, Cluj-Napoca, Romania).

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