AN ITERATIVE ALGORITHM FOR THE CAUCHY PROBLEM IN STEADY-STATE ANISOTROPIC HEAT CONDUCTION

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Abstract

An inverse problem in (an)isotropic heat conduction is characterised by the (partial) absence of at least one of the following: the geometry of the domain occupied by the solid, the boundary conditions and the (an)isotropic thermal conductivity tensor. Such a situation usually occurs for numerous real life problems in science and engineering that are related to the steady-state (an)isotropic heat conduction.

The Cauchy problem in (an)isotropic heat conduction is a classical example of an inverse problem which requires the reconstruction of the missing boundary conditions (boundary temperature and normal heat flux) on an inaccessible portion of the boundary of the domain occupied by a solid, as well as the primary field (temperature) in the domain, from prescribed Dirichlet (i.e. temperature) and Neumann conditions (i.e. normal heat flux) on the remaining and accessible part of the boundary, see e.g. Beck et al. (1985), and Özişik and Orlande (2000). Moreover, this is a well-known example of ill-posed problems in the sense of Hadamard (1923), i.e. the existence, uniqueness and stability of their solutions are not always guaranteed.

In this paper, we investigate the following problem:

Given a solid that occupies the bounded domain $\Omega \subset \mathbb{R}^d$, usually $d \in \big\{2, 3\big\}$, such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\varnothing \neq \Gamma_j \subsetneq \partial\Omega$, $j = 0, 1$, and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$, $u^* \in \mathrm{H}^{1/2}(\Gamma_0)$ and $q^* \in \mathrm{H}^{-1/2}(\Gamma_0)$, determine $u \in \mathrm{H}^1(\Omega)$, as well as the missing boundary conditions $u\big|_{\Gamma_1}$ and $q\big|_{\Gamma_1}$, satisfying the anisotropic heat conduction equation in the absence of heat sources (Özişik, 1993)

$$-\nabla \cdot \big(\mathbf{K}(\mathbf{x})\nabla u(\mathbf{x})\big) = 0\,, \quad \mathbf{x} \in \Omega\,, \tag{1}$$

together with the following over-prescribed boundary conditions

$$u(\mathbf{x}) = u^*(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma_0\,, \qquad \text{and} \qquad q(\mathbf{x}) = q^*(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma_0\,. \tag{2}$$

Here $\mathbf{K} := \big(K_{ij}\big)_{i,j=\overline{1,d}} : \overline{\Omega} \longrightarrow \mathbb{R}^{d\times d}$, with $K_{ij} \in \mathrm{L}^\infty(\Omega)$, $i, j = \overline{1, d}$, is the symmetric and positive definite thermal conductivity tensor characterising the anisotropic solid that occupies the domain Ω, $q(\mathbf{x}) := \boldsymbol{\nu}(\mathbf{x}) \cdot \big(\mathbf{K}(\mathbf{x})\nabla u(\mathbf{x})\big)$ is the normal heat flux at a point $\mathbf{x} \in \partial\Omega$ and $\boldsymbol{\nu}(\mathbf{x})$ is the unit outward normal vector at $\mathbf{x} \in \partial\Omega$.

The inverse Cauchy problem (1)–(2) is approached by transforming it into a control problem or, equivalently, a minimisation problem. The corresponding functional that measures the difference between the solutions of two direct problems with the control given by the unknown Dirichlet condition on

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Γ_1 (Lions, 1971), is minimised with respect to the control in the space $L^2(\Gamma_1)$ via a Newton-type method. It is proved that this functional is twice Fréchet differentiable and strictly convex, whilst a formula for its gradient is also obtained. Consequently, an iterative algorithm that consists of the resolution of two direct problems and four adjoint problems at each step and is robust with respect to the choice of the initial guess for the unknown Dirichlet condition on Γ_1, is obtained.

The numerical implementation of the aforementioned algorithm is realized for two-dimensional ($d = 2$) (non)homogeneous (an)isotropic solids by employing the finite-difference method (FDM), see e.g. Colaço et al. (2017). Both exact and noisy Cauchy data are considered on the over-prescribed and accessible boundary Γ_0 and a corresponding regularizing/stabilising stopping criterion, which is based on either the discrepancy principle (Morozov, 1966) or Hansen's L-curve method (Hansen, 1998), is also provided. The numerical results obtained for the proposed iterative algorithm, in conjunction with either the discrepancy principle or the L-curve method, show that the errors in the reconstructed temperature distribution in Ω, as well as the boundary temperature and the normal heat flux on Γ_1, decrease with respect to refining the FDM grid size and decreasing the amount of noise added to the Cauchy data on Γ_1, hence showing the convergence and the stability of the proposed algorithm, respectively.

Overall, it can be concluded that the alternating iterative algorithm presented, analysed and implemented in this study, provides us with very accurate, convergent and stable numerical solutions for the numerical reconstruction of the temperature distribution in the domain Ω, as well as the missing boundary temperature and normal heat flux on the under-prescribed and inaccessible boundary Γ_1, in the case of Cauchy problems in steady-state anisotropic heat conduction.

Keywords: inverse problem; anisotropic heat conduction; control problem; minimisation problem; regularization; finite-difference method (FDM).

Domain: Mathematics.
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References

J.V. Beck, B. Blackwell, C.R. St. Clair Jr., *Inverse Heat Conduction: Ill-Posed Problems*. John Wiley & Sons, New York, 1985.

M.J. Colaço, R.M. Cotta, H.R.B. Orlande, M.N. Özişik, *Inverse Heat Transfer: Fundamentals and Applications*. CRC Press, Boca Raton, 2017.

J. Hadamard, *Lectures on Cauchy Problem in Linear Partial Differential Equations*. Yale University Press, New Haven, 1923.

P.C. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion*. SIAM, Philadelphia, 1998.

J. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*. Springer–Verlag, Berlin, 1971.

V.A. Morozov, On the solution of functional equations by the method of regularization. *Doklady Mathematics* **7** 414–417, 1966.

M.N. Özişik, *Heat Conduction*, John Wiley & Sons, New York, 1993.

M.N. Özişik, H.R.B. Orlande, *Inverse Heat Transfer: Fundamentals and Applications*. CRC Press, Boca Raton, 2000.