A generalized approach on the stability and bifurcations in a differential equation with a general distributed delay

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We consider and analyze a linear scalar differential equation containing a general distributed delay. Our motivation consists in obtaining and proving results in the most general case of such equations, without imposing unwanted conditions on the parameters, variable or distribution and without assuming a specific distribution in the equation. To the best of our knowledge such ample analysis has not been done in such a general framework, most of the papers containing an analysis of particular versions of (1). Another motivation consists of the applicability in domains such as biology, engineering, medicine and neural department, because in real world phenomena the information transmission, processing and analysing is not without time delay.

The following linear differential equation with a distributed delay is analyzed:

$$\dot{x}(t) = -\alpha x(t) - \beta \int_0^\infty x(t-s)h(s)ds \tag{1}$$

For the above equation we determine the associated characteristic equation by means of the Laplace transforms method and obtain conditions for the stability of the trivial solution. We obtain parametrically the stability region and prove the obtained results by means of the transversality condition method. We conclude with a discussion on the types of bifurcation that are possible to appear in such equations as (1) and we apply our results on a concrete example to substantiate our theoretical findings.

Equation (1) is usually considered with a fixed delay distribution, or with a finite distribution, or has imposed some specific restrictions on the RHS part and then analysed [1, 2, 3]. In [2, 4], the authors portray a comparison of equation (1) incorporating a discrete delay and its equivalent form containing a Gamma distributed delay, showing that arbitrarily many stability switches are not possible in the first case, but in the second case a switch from stability to instability is followed by only one switch back to stability when the mean delay is increased. We need to point out that in the literature, no necessary conditions for stability has yet been given and the theory of existence & uniqueness of differential equations with infinite delay such as (1), as well as stability results in terms of Lyapunov functionals, can be found in [5, 6, 7, 8, 9] and in their respective references. Nonetheless a more important result, the principle of linearized stability has just been proven rigorously in the recent years, in the case of infinite delay [10]. A proof of the Hopf bifurcation theorem for equations such as (1) can be found in [11]. The following papers [5, 7, 12, 13] constitute an appropriate introduction in the field of DDEs and bifurcation analysis.

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