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ABOUT SOME FUNCTIONAL INTEGRAL EQUATIONS IN SPACES WITH PERTURBATED METRIC

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Abstract In spaces with perturbated metric the following functional integral equation

$$u(x) = h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s, \cdots, \theta_m s))ds, \tag{1}$$

where

$$x, s \in D = \prod_{i=1}^{m} [0, b_i],\ m(D) \leq 1,\ \theta_i \in (0,1), (\forall) i = \overline{1, m}$$

is studied.

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1. INTRODUCTION

Let (X, d) be a metric space and $A : X \to X$ an operator. We shall use the following notations
$F_A := \{x \in X \mid A(x) = x\}$ the fixed points set of A;
$I(A) := \{Y \in P(X) \mid A(Y) \subset Y\}$ the family of the nonempty invariant subsets of A;
$A^{n+1} = A \circ A^n, A^0 = 1_X, A^1 = A, n \in N.$

Definition 1.1. *[4] An operator A is an weakly Picard operator (WPO) if the sequence*

$$(A^n(x))_{n \in N}$$

converges for all $x \in X$, and the limit (which depends on x) is a fixed point of A.

Definition 1.2. *[4] If the operator A is WPO and $F_A = \{x^*\}$ then A is called a Picard operator.*

Definition 1.3. *[4] If A is an WPO, then we define the operator*

$$A^\infty : X \to X,\ A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

We remark that $A^\infty(X) = F_A$.

Definition 1.4. *[4] Let be A an WPO and $c > 0$. The operator A is called a c-WPO if*

$$d(x, A^\infty(x)) \le c \cdot d(x, A(x)).$$

We have the following characterization of the WPOs

Theorem 1.1. *[4] Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is an WPO (c-WPO) if and only if there exists a partition of X,*

$$X = \bigcup_{\lambda\in\Lambda} X_\lambda$$

such that

(a) $X_\lambda \in I(A)$,

(b) $A \mid X_\lambda : X_\lambda \to X_\lambda$ is a Picard (c-Picard) operator, for all $\lambda \in \Lambda$.

2. MAIN RESULTS

Let (X, d) be a complete metric space. We denote by P the set of functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ which are strictly increasing, continuous and surjective. By Φ we denote the set of functions introduced by

Definition 2.1. *We say that the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the Φ class if the following conditions are met:*

(1) φ is increasing;

(2) $\varphi(t) < t$, for all $t \in \mathbb{R}_+$;

(3) φ is right continuous.

Example 2.1. *The function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = at$, $a < 1$ *belongs to the set* Φ.

Example 2.2. *The function* $g : \mathbb{R}_+ \to \mathbb{R}_+$, $g(t) = t^2$, *belongs to the set* P.

Proposition 2.1. *[3] Let* $f : X \to X$ *be an operator and* $\varphi \in \Phi$, $g \in P$ *such that:*

(i) $g(d(f(x), f(y))) \leq \varphi(g(d(x, y)))$, *for all* $x, y \in X$.

Then f *has a unique fixed point, which is the limit of successively approximations sequence.*

Proposition 2.2. *We suppose that:*

(i) $h \in C(D \times \mathbb{R}^n)$ *and* $K \in C(D \times D \times \mathbb{R}^n)$*;*

(ii) $h(0, \alpha) = \alpha$, $(\forall)\alpha \in \mathbb{R}^n$*;*

(iii) there exists $g \in P$, $\varphi \in \Phi$ *such that*

$$g(\|K(x, s, u_1) - K(x, s, u_2)\|_{\mathbb{R}^n}) \leq \varphi(g(\|u_1 - u_2\|_{\mathbb{R}^n})),$$

for all $x, s \in D$ *and* $u_1, u_2 \in \mathbb{R}^n$.

In these conditions the equation(1) has in $C(D, \mathbb{R})$ *an infinity of solutions.*

Proof: Consider the operator

$$A : (C(D, \mathbb{R}^n), |\cdot|) \to (C(D, \mathbb{R}^n), |\cdot|),$$

$$A(u)(x) := h(x, u(0)) + \int_0^{x_1} \cdots \int_0^{x_m} K(x, s, u(\theta_1 s, \cdots, \theta_m s))ds.$$

Here $|u| = \max_{x \in D} |u(x)|$.

Let $\lambda \in \mathbb{R}^n$ and $X_\lambda = \{u \in C(D, \mathbb{R}^n) \mid u(0) = \lambda\}$. Then

$$C(D, \mathbb{R}^n) = \bigcup_{\lambda \in \mathbb{R}^n} X_\lambda.$$

is a partition of $C(D, \mathbb{R}^n)$ and $X_\lambda \in I(A)$, for all $\lambda \in \mathbb{R}^n$.

For all $u, v \in X_\lambda$, we have

$$g(\|A(u)(x) - A(v)(x)\|_{\mathbb{R}^n}) \leq$$

$$g(\int_0^{x_1}\cdots\int_0^{x_m} g^{-1}(\varphi(g(\|K(x,s,u(\theta_1 s,\cdots,\theta_m s)) - K(x,s,v(\theta_1 s,\cdots,\theta_m s))\|)))))ds$$
$$\leq g(m(D)g^{-1}\varphi(g(|u-v|))) \leq \varphi(g(|u-v|))$$

Then, via Proposition 2.1, $A \mid X_\lambda$ is a Picard, while A is an weakly Picard operator.

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