ROMAI J., **4**, 1(2008), 107–111

GOOD AND SPECIAL WPO PROPERTIES FOR BERNSTEIN OPERATORS IN P VARIABLES

Loredana-Florentina Galea

Faculty of Law and Economics, The Agora University of Oradea

loredana.galea@univagora.ro

Abstract In 1969, D. D. Stancu have introduced the Bernstein operators with arguments real functions in p variables. Using weakly Picard operators and contraction principle, C. Bacotiu studied the convergence of these operators' iterates. In the present paper good and special convergence for Bernstein operators in p variables are investigated.

Keywords: weakly Picard operators, Bernstein operators, good and special weakly Picard operators.

2000 MSC: 47H10, 41A36.

1. INTRODUCTION

In [8], D.D. Stancu has introduced the Bernstein operators in p variables, like as:

Definition 1.1. *[8]. Consider the set*

$$\overline{\Delta} = \{(x_1, x_2, ..., x_{p+}) \in \mathbb{R}^p : x_1 \geqslant 0,\ x_2 \geqslant 0, ..., x_p \geqslant 0;\ x_1 + x_2 + ... + x_p \leqslant 1\}$$

The operators $B_n : C\left(\overline{\Delta}\right) \to C\left(\overline{\Delta}\right)$ *defined by*

$$B_n\left(f\right)\left(x_1, ..., x_p\right) = \sum_{0 \leqslant i_1 + ... + i_p \leqslant n} p_{n;i_1,...,i_p}\left(x_1, ..., x_p\right) f\left(\frac{i_1}{n}, ..., \frac{i_p}{n}\right)$$

for any $f \in C\left(\overline{\Delta}\right)$ *and* $(x_1, ..., x_p) \in \overline{\Delta}$ *are called Bernstein operators with arguments real functions in p variables.*

The polynomials $p_{n;i_1,...,i_p}$ are generalizations of Bernstein fundamental polynomials and are defined by

$$p_{n;i_1,...,i_p}\left(x_1, ..., x_p\right) = \frac{n!}{i_1! i_2! ... i_p!} x_1^{i_1} ... x_p^{i_p} \left(1 - x_1 - x_2 - ... - x_p\right)^{n - i_1 - i_2 - ... - i_p}$$

for any $(x_1, ..., x_p) \in \overline{\Delta}$.

The set $\nu_{\overline{\Delta}} = \{(0, 0, ..., 0), (0, 1, ..., 0), ..., (0, 0, ..., 1)\} \subset \overline{\Delta}$ represents the set of knots and if $\alpha_0 = (0, 0, ..., 0), \alpha_1 = (0, 1, ..., 0), ..., \alpha_p = (0, 0, ..., 1)$, then $\nu_{\overline{\Delta}} = \{\alpha_k : k = \overline{0, p}\}$.

In the following, we present some properties of Bernstein operators in p variables, which were studied in [8]:

(P1) B_n are linear and positive;

(P2) for any $k = \overline{0, p}$, $e_{\alpha k}$ is a fixed point for B_n, so

$$B_n (e_{\alpha k}) (x_1, ..., x_p) = e_{\alpha k} (x_1, ..., x_p), \ \forall \ (x_1, ..., x_p) \in \overline{\Delta};$$

(P3) B_n is an interpolation for any $f \in C\left(\overline{\Delta}\right)$ and for the knots of $\nu_{\overline{\Delta}}$, so

$$B_n (f) (\alpha_k) = f (\alpha_k), \forall k = \overline{0, p}.$$

Definition 1.2. *[5], [6], [7]. Let (X, d) be a metric space.*
1) An operator $A : X \to X$ is weakly Picard operator (WPO) if the sequence of successive approximations $(A^m (x_0))_{m \in \mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.
2) If the operator $A : X \to X$ is an WPO and $F_A = \{x^*\}$, then the operator A is called a Picard operator (PO).
3) If the operator $A : X \to X$ is an WPO, then the operator A^∞ is defined by $A^\infty : X \to X, \ A^\infty (x) := \lim\limits_{m \to \infty} A^m (x)$.

The basic result in the WPO's theory is the following characterization theorem

Theorem 1.1. *[5], [6], [7]. An operator $A : X \to X$ is WPO if and only if there exists a partition of X, $X = \bigcup\limits_{\lambda \in \Lambda} X_\lambda$, such that:*
(a) $X_\lambda \in I (A), \ \forall \lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is PO, $\forall \lambda \in \Lambda$.

In [7], I.A. Rus has introduced the notions of good and special WPO

Definition 1.3. *. Let (X, d) be a metric space and $A : X \to X$ an WPO.*
1) $A : X \to X$ is a good WPO, if the series $\sum\limits_{m=1}^{\infty} d\left(A^{m-1} (x), A^m (x)\right)$ converges, for all $x \in X$ [7]. If the sequence $\left(d\left(A^{m-1} (x), A^m (x)\right)\right)_{m \in \mathbb{N}^*}$ is

strictly decreasing for all $x \in X$, the operator A is a good WPO of type M [2].

2) $A : X \to X$ is a special WPO, if the series $\sum\limits_{m=1}^{\infty} d\left(A^m(x), A^\infty(x)\right)$ converges, for all $x \in X$ [7]. If the sequence $\left(d\left(A^m(x), A^\infty(x)\right)\right)_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, A is a special WPO of type M [2].

Theorem 1.2. *[4]. Let (X, d) be a metric space and $A : X \to X$ a WPO. If A is a special WPO then A is a good WPO.*

Theorem 1.3. *[3]. The Bernstein operators in p variables $B_n : C\left(\overline{\Delta}\right) \to C\left(\overline{\Delta}\right)$ are weakly Pixard and $B_n^\infty(f) = \varphi_f^*, \forall f \in C\left(\overline{\Delta}\right)$, where the function $f \in C\left(\overline{\Delta}\right)$ are given by*

$$\varphi_f^*(x_1, ..., x_p) = f(\alpha_0) + [f(\alpha_1) - f(\alpha_0)]\, x_1 + [f(\alpha_2) - f(\alpha_0)]\, x_2 + ...+$$

$$+ [f(\alpha_p) - f(\alpha_0)]\, x_p,\ \forall\ f \in C\left(\overline{\Delta}\right),\ \forall\ (x_1, ..., x_p) \in \overline{\Delta}.$$

The convergence is in the space $C\left(\overline{\Delta}\right)$.

In order to apply the characterization theorem of weakly Picard operators, the partition of space $C\left(\overline{\Delta}\right)$

$$C\left(\overline{\Delta}\right) := \bigcup_{\Lambda\in\mathbb{R}^{p+1}} X_\Lambda$$

was considered, where $X_\Lambda := \left\{f \in C\left(\overline{\Delta}\right) : f(\alpha_k) = \lambda_k,\ \text{for } k = \overline{0, p}\right\}$, for any $\Lambda = (\lambda_0, \lambda_1, ..., \lambda_p) \in \mathbb{R}^{p+1}$.

Proposition 1.1. *[3]. The Bernstein operators in p variables satisfy the following contraction property*

$$\left\|B_n(f) - B_n(g)\right\|_C \leqslant \left(1 - \frac{1}{p^{n-1}}\right) \|f - g\|_C, \forall f, g \in X_\Lambda. \tag{1}$$

2. MAIN RESULTS

In this section, we study the good and special weakly Picard operators convergence for Bernstein operators in p variables.

Using the inequality (1), we obtain the estimation

$$\left|B_n^1(f)(x_1, ..., x_p) - B_n^\infty(f)(x_1, ..., x_p)\right| =$$

$$= \left|B_n^1(f)(x_1, ..., x_p) - B_n^1(B_n^\infty(f))(x_1, ..., x_p)\right| \leqslant$$

$$\leqslant \left(1-\frac{1}{p^{n-1}}\right)\left|f\left(x_1,...,x_p\right)-B_n^{\infty}\left(f\right)\left(x_1,...,x_p\right)\right| =$$

$$= \left(1-\frac{1}{p^{n-1}}\right)\left|f\left(x_1,...,x_p\right)-\left\{f\left(\alpha_0\right)+\left[f\left(\alpha_1\right)-f\left(\alpha_0\right)\right]x_1+...+\left[f\left(\alpha_p\right)-f\left(\alpha_0\right)\right]x_p\right\}\right| \leqslant$$

$$\leqslant \left(1-\frac{1}{p^{n-1}}\right)\left(p+1\right)C,\ \forall\ \left(x_1,...,x_p\right)\in\overline{\Delta},$$

where $C = diam\left(\operatorname{Im} f\right) = diam\left(f\left(\overline{\Delta}\right)\right) =$

$$= \max\left\{\left|f\left(x_1,...,x_p\right)-f\left(y_1,...,y_p\right)\right| : \left(x_1,...,x_p\right),\left(y_1,...,y_p\right)\in\overline{\Delta}\right\}.$$

$$\left|B_n^2\left(f\right)\left(x_1,...,x_p\right)-B_n^{\infty}\left(f\right)\left(x_1,...,x_p\right)\right| =$$

$$= \left|B_n^1\left(B_n^1\left(f\right)\right)\left(x_1,...,x_p\right)-B_n^1\left(B_n^{\infty}\left(f\right)\right)\left(x_1,...,x_p\right)\right| \leqslant$$

$\leqslant \left(1-\frac{1}{p^{n-1}}\right)\left\|B_n^1\left(f\right)-B_n^{\infty}\left(f\right)\right\| = \left(1-\frac{1}{p^{n-1}}\right)^2\left(p+1\right)C,\ \forall\ \left(x_1,...,x_p\right)\in \overline{\Delta}$.

By induction, for $m\in\mathbb{N}^*$, we have

$$\left|B_n^m\left(f\right)\left(x_1,...,x_p\right)-B_n^{\infty}\left(f\right)\left(x_1,...,x_p\right)\right| =$$

$$= \left|B_n^1\left(B_n^{m-1}\left(f\right)\right)\left(x_1,...,x_p\right)-B_n^1\left(B_n^{\infty}\left(f\right)\right)\left(x_1,...,x_p\right)\right| \leqslant$$

$$\leqslant \left(1-\frac{1}{p^{n-1}}\right)^m\left(p+1\right)C,\ \forall\ \left(x_1,...,x_p\right)\in\overline{\Delta}.$$

So, $\sum\limits_{m=1}^{\infty}\left|B_n^m\left(f\right)\left(x_1,...,x_p\right)-B_n^{\infty}\left(f\right)\left(x_1,...,x_p\right)\right| \leqslant C\left(p+1\right)\left(p^{n-1}-1\right),$
$\forall\ \left(x_1,...,x_p\right)\in\overline{\Delta}$.

On the other hand, we have

$$\left|B_n^1\left(f\right)\left(x_1,...,x_p\right)-B_n^0\left(f\right)\left(x_1,...,x_p\right)\right| =$$

$$= \left|\sum_{0\leqslant i_1+...+i_p\leqslant n} p_{n;i_1,i_2,...,i_p}\left(x_1,...,x_p\right) f\left(\frac{i_1}{n},...,\frac{i_p}{n}\right)-f\left(x_1,...,x_p\right)\right| =$$

$$= C\cdot\sum_{0\leqslant i_1+...+i_p\leqslant n} p_{n;i_1,i_2,...,i_p}\left(x_1,...,x_p\right) = C,\ \forall\ \left(x_1,...,x_p\right)\in\overline{\Delta}\ .$$

By induction, we have

$$\left|B_n^m\left(f\right)\left(x_1,...,x_p\right)-B_n^{m-1}\left(f\right)\left(x_1,...,x_p\right)\right| =$$

$$= \left|B_n^1\left(B_n^{m-1}\left(f\right)\right)\left(x_1,...,x_p\right)-B_n^1\left(B_n^{m-2}\left(f\right)\right)\left(x_1,...,x_p\right)\right| \leqslant$$

$$\leqslant \left(1-\frac{1}{p^{n-1}}\right)^{m-1}\cdot C,\ \forall\ (x_1,...,x_p)\in\overline{\Delta}.$$

So, $\sum\limits_{m=1}^{\infty}\left|B_n^m(f)(x_1,...,x_p)-B_n^{m-1}(f)(x_1,...,x_p)\right|\leqslant C\cdot p^{n-1}$,
$\forall\ (x_1,...,x_p)\in\overline{\Delta},\ \forall\ f\in C\left(\overline{\Delta}\right)$.

From above results, we have the following

Proposition 2.1. *The Bernstein operators in p variables are special weakly Picard and good weakly Picard on* $C\left(\overline{\Delta}\right)$.

References

[1] L. D'Apuzzo, *On the convergence of the method of succesive approximations in metric spaces*, Ann. Istit. Univ. Navale Napoli, **45/46**(1976/1977). (in Italian).

[2] L. D'Apuzzo, *On the notion of good and special convergence of the method of successive approximations*, Ann. Istit. Univ. Navale Napoli, **45/46**(1976/1977), 123-138. (in Italian).

[3] C. Bacotiu, *Picard operators and application*, PhD Thesis, Univ. Babes-Bolyai, Cluj-Napoca, 2004. (in Romanian).

[4] S. Mureşan, L. Galea, *On good and special weakly Picard operators* (submitted).

[5] I. A. Rus, *Generalized contractions and applications*, Cluj University Press, 2001. (in Romanian).

[6] I. A. Rus, *Weakly Picard operators and applications*, Seminar on Fixed Point Theory, Cluj-Napoca, **2**(2000), 41-58.

[7] I. A. Rus, *Picard operators and applications*, Sci. Math.Japon., **58**(1)(2003), 191-219.

[8] D. D. Stancu, *A new class of uniform approximating polynomial operators in two and several variables*, Proc. Conf. on Constructive Theory of functions, Budapest, (1969), 443-455.